Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2008, Article ID 149286, 12 pages
doi:10.1155/2008/149286

Research Article

On the Monotonicity and Log-Convexity of a Four-Parameter Homogeneous Mean

Zhen-Hang Yang

Electric Grid Planning and Research Center, Zhejiang Electric Power Test and Research Institute, Hangzhou 310014, China

Correspondence should be addressed to Zhen-Hang Yang, yzhkm@163.com

Received 13 April 2008; Accepted 29 July 2008

Recommended by Sever Dragomir

A four-parameter homogeneous mean $\mathbf{F}(p,q;r,s;a,b)$ is defined by another approach. The criterion of its monotonicity and logarithmically convexity is presented, and three refined chains of inequalities for two-parameter mean values are deduced which contain many new and classical inequalities for means.

1. Introduction

The so-called two-parameter mean or extended mean between two unequal positive numbers x and y was defined first by Stolarsky [1] as

$$E(r,s;x,y)=\begin{cases}\left(\dfrac{s(x^r-y^r)}{r(x^s-y^s)}\right)^{1/(r-s)}, & r\neq s,\ rs\neq 0,\\[2ex] \left(\dfrac{x^r-y^r}{r(\ln x-\ln y)}\right)^{1/r}, & r\neq 0,\ s=0,\\[2ex] \left(\dfrac{x^s-y^s}{s(\ln x-\ln y)}\right)^{1/s}, & r=0,\ s\neq 0,\\[2ex] \exp\left(\dfrac{x^r\ln x-y^r\ln y}{x^r-y^r}-\dfrac{1}{r}\right), & r=s\neq 0,\\[2ex] \sqrt{xy}, & r=s=0.\end{cases} \tag{1.1}$$

It contains many mean values, for instance,

$$E(1,0;x,y) = L(x,y) = \begin{cases} \dfrac{x-y}{\ln x - \ln y}, & x \neq y, \\ x, & x = y; \end{cases} \tag{1.2}$$

$$E(1,1;x,y) = I(x,y) = \begin{cases} e^{-1}\left(\dfrac{x^x}{y^y}\right)^{1/(x-y)}, & x \neq y, \\ x, & x = y; \end{cases} \tag{1.3}$$

$$E(2,1;x,y) = A(x,y) = \frac{x+y}{2}; \tag{1.4}$$

$$E\left(\frac{3}{2},\frac{1}{2};x,y\right) = h(x,y) = \frac{x+\sqrt{xy}+y}{3}. \tag{1.5}$$

The monotonicity of $E(r,s;x,y)$ has been researched by Stolarsky [1], Leach and Sholander [2], and others also in [3–5] using different ideas and simpler methods.

Qi studied the log-convexity of the extended mean with respect to parameters in [6], and pointed out that the two-parameter mean is a log-concave function with respect to either parameter r or s on interval $(0,+\infty)$ and is a log-convex function on interval $(-\infty,0)$.

In [7], Witkowski considered more general means defined by

$$R(u,v;r,s;x,y) = \left(\frac{E(u,v;x^r,y^r)}{E(u,v;x^s,y^s)}\right)^{1/(r-s)} \tag{1.6}$$

further and investigated the monotonicity of $\mathbb{R}$.

Denote $\mathbb{R}^+ := (0,\infty)$ and let $f(x,y)$ be defined on Ω. If for arbitrary $t \in \mathbb{R}^+$ with $(tx,ty) \in \Omega$, the following equation:

$$f(tx,ty) = t^n f(x,y) \tag{1.7}$$

is always true, then the function $f(x,y)$ is called an n-order homogeneous functions. It has many well properties [8–10]. Based on the conception and properties of homogeneous function, the extended mean was generalized to two-parameter homogeneous functions in [9], which is defined as follows.

Definition 1.1. Assume $f : \mathbb{U}(\subsetneqq \mathbb{R}^+ \times \mathbb{R}^+) \to \mathbb{R}^+$ is an n-order homogeneous function for variables x and y, continuous and first partial derivatives exist, $(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $a \neq b$, $(p,q) \in \mathbb{R} \times \mathbb{R}$.

If $(1,1) \notin \mathbb{U}$, then define that

$$\begin{aligned} \mathscr{H}_f(p,q;a,b) &= \left(\frac{f(a^p,b^p)}{f(a^q,b^q)}\right)^{1/(p-q)} \quad (p \neq q,\ pq \neq 0), \\ \mathscr{H}_f(p,p;a,b) &= \lim_{q\to p} \mathscr{H}_f(a,b;p,q) = G_{f,p} \quad (p = q \neq 0), \end{aligned} \tag{1.8}$$

where

$$G_{f,p} = G_f^{1/p}(a^p,b^p), \qquad G_f(x,y) = \exp\left(\frac{xf_x(x,y)\ln x + yf_y(x,y)\ln y}{f(x,y)}\right), \tag{1.9}$$

$f_x(x,y)$ and $f_y(x,y)$ denote partial derivatives with respect to first and second variable of $f(x,y)$, respectively.

If $(1,1) \in \mathbb{U}$, then define further

$$\mathscr{H}_f(p,0;a,b) = \left(\frac{f(a^p,b^p)}{f(1,1)}\right)^{1/p} \quad (p \neq 0,\ q = 0),$$
$$\mathscr{H}_f(0,q;a,b) = \left(\frac{f(a^q,b^q)}{f(1,1)}\right)^{1/q} \quad (p = 0,\ q \neq 0), \tag{1.10}$$
$$\mathscr{H}_f(0,0;a,b) = \lim_{p\to 0}\mathscr{H}_f(a,b;p,0) = a^{f_x(1,1)/f(1,1)} b^{f_y(1,1)/f(1,1)} \quad (p = q = 0).$$

Let $f(x,y) = L(x,y)$. We can get two-parameter logarithmic mean, which is just extended mean $E(p,q;a,b)$ defined by (1.1). In what follows we adopt our notations and denote by $\mathscr{H}_L(p,q;a,b)$ or $\mathscr{H}_L(p,q)$ or $\mathscr{H}_L$.

Concerning the monotonicity and log-convexity of the two-parameter homogeneous functions, there are the following results.

Theorem 1.2 (see [9]). *Let $f(x,y)$ be a positive n-order homogenous function defined on $\mathbb{U}(\subseteq \mathbb{R}^+ \times \mathbb{R}^+)$ and be second differentiable. If $\mathcal{I} = (\ln f)_{xy} < (>)0$, then $\mathscr{H}_f(p,q)$ is strictly increasing (decreasing) in either p or q on $(-\infty,0)$ and $(0,+\infty)$.*

Theorem 1.3 (see [10]). *Let $f(x,y)$ be a positive n-order homogenous function defined on $\mathbb{U}(\subseteq \mathbb{R}^+ \times \mathbb{R}^+)$ and be third-order differentiable. If*

$$\mathcal{J} = (x-y)(x\mathcal{I})_x < (>)0, \quad \text{where } \mathcal{I} = (\ln f)_{xy}, \tag{1.11}$$

then $\mathscr{H}_f(p,q)$ is strictly log-convex (log-concave) with respect to either p or q on $(0,+\infty)$ and log-concave (log-convex) on $(-\infty,0)$.

By the above theorems we have the following.

Corollary 1.4 (see [10]). *The conditions are the same as Theorem 1.3. If (1.11) holds, then $\mathscr{H}_f(p,1-p)$ is strictly decreasing (increasing) in p on $(0,1/2)$ and increasing (decreasing) on $(1/2,1)$.*

If $f(x,y)$ is symmetric with respect to x and y further, then the above monotone interval can be extended from $(0,1/2)$ to $(-\infty,0)$ and $(0,1/2)$, and from $(1/2,1)$ to $(1/2,1)$ and $(1,+\infty)$, respectively.

Corollary 1.5 (see [10]). *The conditions are the same as Theorem 1.3. If (1.11) holds, then for $p,q \in (0,+\infty)$ with $p \neq q$, the following inequalities:*

$$G_{f,(p+q)/2} < (>)\mathscr{H}_f(p,q) < (>)\sqrt{G_{f,p}G_{f,q}}. \tag{1.12}$$

hold. For $p,q \in (-\infty,0)$ with $p \neq q$, inequalities (1.12) are reversed.

If $f(x,y)$ is defined on $\mathbb{R}^+ \times \mathbb{R}^+$ and symmetric with respect to x and y further, then substituting $p+q > 0$ for $p,q \in (0,+\infty)$ and $p+q < 0$ for $p,q \in (-\infty,0)$, (1.12) are also true, respectively.

Let $f(x,y) = L(x,y)$, $A(x,y)$, $I(x,y)$, and $D(x,y)$ in Theorems 1.2 and 1.3, Corollaries 1.4 and 1.5, we can deduce some useful conclusions (see [9, 10]). These show the monotonicity and log-convexity of $L(x,y)$, $A(x,y)$, $I(x,y)$, and $D(x,y)$ depend on the signs of $\mathcal{I} = (\ln f)_{xy}$ and $\mathcal{J} = (x-y)(x\mathcal{I})_x$, respectively. Noting $\mathcal{H}_L(r,s;x,y)$ contains $L(x,y)$, $A(x,y)$, and $I(x,y)$, naturally, we could make conjecture on the similar conclusion is also true for $\mathcal{H}_f(p,q;a,b)$, where $f(x,y) = \mathcal{H}_L(r,s;x,y)$. Namely, the monotonicity and log-convexity of the function $\mathcal{H}_{\mathcal{H}_L}$ also depend on the signs of $\mathcal{I} = (\ln f)_{xy} < 0$ and $\mathcal{J} = (x-y)(x\mathcal{I})_x > 0$, respectively, which is just purpose of this paper.

2. Definition and main results

For stating the main results of this paper, let us introduce first the four-parameter mean as follows.

Definition 2.1. Assume $(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $a \neq b$, $(p,q), (r,s) \in \mathbb{R}\times\mathbb{R}$, then the four-parameter homogeneous mean denoted by $\mathbf{F}(p,q;r,s;a,b)$ is defined as follows:

$$\mathbf{F}(p,q;r,s;a,b) = \left(\frac{L(a^{pr},b^{pr})}{L(a^{ps},b^{ps})}\frac{L(a^{qs},b^{qs})}{L(a^{qr},b^{qr})}\right)^{1/(p-q)(r-s)}, \quad \text{if } pqrs(p-q)(r-s) \neq 0, \tag{2.1}$$

or

$$\mathbf{F}(p,q;r,s;a,b) = \left(\frac{a^{pr}-b^{pr}}{a^{ps}-b^{ps}}\frac{a^{qs}-b^{qs}}{a^{qr}-b^{qr}}\right)^{1/(p-q)(r-s)}, \quad \text{if } pqrs(p-q)(r-s) \neq 0; \tag{2.2}$$

if $pqrs(p-q)(r-s) = 0$, then the $\mathbf{F}(p,q;r,s;a,b)$ are defined as their corresponding limits, for example,

$$\begin{aligned}
\mathbf{F}(p,p;r,s;a,b) &= \lim_{q\to p}\mathbf{F}(p,q;r,s;a,b) = \left(\frac{I(a^{pr},b^{pr})}{I(a^{ps},b^{ps})}\right)^{1/p(r-s)}, \quad \text{if } prs(r-s) \neq 0,\ p = q;\\
\mathbf{F}(p,0;r,s;a,b) &= \lim_{q\to 0}\mathbf{F}(p,q;r,s;a,b) = \left(\frac{L(a^{pr},b^{pr})}{L(a^{ps},b^{ps})}\right)^{1/p(r-s)}, \quad \text{if } prs(r-s) \neq 0,\ q = 0;\\
\mathbf{F}(0,0;r,s;a,b) &= \lim_{p\to 0}\mathbf{F}(p,0;r,s;a,b) = G(a,b), \quad \text{if } rs(r-s) \neq 0,\ p = q = 0,
\end{aligned} \tag{2.3}$$

where $L(x,y), I(x,y)$ are defined by (1.2), (1.3) respectively, $G(a,b) = \sqrt{ab}$.

It is easy to verify that $\mathbf{F}(p,q;r,s;a,b)$ are symmetric with respect to a and b, p and q, r and s, (p,q) and (r,s), and then $\mathbf{F}(p,q;r,s;a,b)$ is also denoted by $\mathbf{F}(p,q)$ or $\mathbf{F}(r,s)$ or $\mathbf{F}(p,q;r,s)$ or $\mathbf{F}(a,b)$.

The four-parameter homogeneous mean $\mathbf{F}(p,q;r,s;a,b)$ contains many two-parameter means mentioned in [9], for example, (see Table 1).

In Table 1, $\mathbf{F}(2,1;r,s;a,b)$ is just the Gini mean (is also called two-parameter arithmetic mean), $\mathbf{F}(1,0;r,s;a,b)$ is just the two-parameter mean or extended mean or Stolarsky mean (is also called two-parameter logarithmic mean), $\mathbf{F}(1,1;r,s;a,b)$ is just the two-parameter exponential mean, and $\mathbf{F}(3/2,1/2;r,s;a,b)$ is just the two-parameter Heron mean.

Our main results can be stated as follows.

Theorem 2.2. *If* $r+s >\ (<)0$, *then* $\mathbf{F}(p,q;r,s;a,b)$ *are strictly increasing (decreasing) in either* p *or* q *on* $(-\infty,+\infty)$.

Table 1: Some familiar two-parameter mean values.

(p,q)	$\mathbf{F}(p,q;r,s;a,b)$	(p,q)	$\mathbf{F}(p,q;r,s;a,b)$
$(2,1)$	$\left(\frac{a^r+b^r}{a^s+b^s}\right)^{1/(r-s)}$	$\left(\frac{1}{2},\frac{1}{2}\right)$	$\left(\frac{I(a^{r/2},b^{r/2})}{I(a^{s/2},b^{s/2})}\right)^{2/(r-s)}$
$(1,1)$	$\left(\frac{I(a^r,b^r)}{I(a^s,b^s)}\right)^{1/(r-s)}$	$\left(\frac{2}{3},\frac{1}{3}\right)$	$\left(\frac{a^{r/3}+b^{r/3}}{a^{s/3}+b^{s/3}}\right)^{3/(r-s)}$
$\left(1,\frac{1}{2}\right)$	$\left(\frac{a^{r/2}+b^{r/2}}{a^{s/2}+b^{s/2}}\right)^{2/(r-s)}$	$\left(\frac{3}{4},\frac{1}{4}\right)$	$\left(\frac{a^{r/2}+(\sqrt{ab})^{r/2}+b^{r/2}}{a^{s/2}+(\sqrt{ab})^{s/2}+b^{s/2}}\right)^{2/(r-s)}$
$(1,0)$	$\left(\frac{s}{r}\frac{a^r-b^r}{a^s-b^s}\right)^{1/(r-s)}$	$\left(\frac{4}{3},-\frac{1}{3}\right)$	$\left(\frac{a^{r/3}+b^{r/3}}{a^{s/3}+b^{s/3}}\frac{a^{2r/3}+b^{2r/3}}{a^{2s/3}+b^{2s/3}}\right)^{3/5(r-s)}G^{2/5}$
$\left(1,-\frac{1}{2}\right)$	$\left(\frac{a^{r/2}+b^{r/2}}{a^{s/2}+b^{s/2}}\right)^{2/3(r-s)}G^{2/3}$	$\left(\frac{3}{2},-\frac{1}{2}\right)$	$\left(\frac{a^r+(\sqrt{ab})^r+b^r}{a^s+(\sqrt{ab})^s+b^s}\right)^{1/2(r-s)}(\sqrt{ab})^{1/2}$
$\left(\frac{3}{2},\frac{1}{2}\right)$	$\left(\frac{a^r+(\sqrt{ab})^r+b^r}{a^s+(\sqrt{ab})^s+b^s}\right)^{1/(r-s)}$	$(2,-1)$	$\left(\frac{a^r+b^r}{a^s+b^s}\right)^{1/3(r-s)}(\sqrt{ab})^{2/3}$

Theorem 2.3. *If $r+s>\ (<)0$, then $\mathbf{F}(p,q;r,s;a,b)$ are strictly log-concave (log-convex) in either p or q on $(0,+\infty)$ and log-convex (log-concave) on $(-\infty,0)$.*

By Corollary 1.4, we get Corollary 2.4.

Corollary 2.4. *If $r+s>\ (<)0$, then $\mathbf{F}(p,1-p;r,s;a,b)$ are strictly increasing (decreasing) in p on $(-\infty,1/2)$ and decreasing (increasing) on $(1/2,+\infty)$.*

Notice for $f(x,y)=\mathscr{L}_L(r,s;x,y)$,

$$\begin{aligned}G_f(x,y)&=\exp\left(\frac{xf_x(x,y)\ln x+yf_y(x,y)\ln y}{f(x,y)}\right)\\&=\exp\left(\frac{1}{r-s}\left(\frac{rx^r}{x^r-y^r}-\frac{sx^s}{x^s-y^s}\right)\ln x+\frac{1}{r-s}\left(-\frac{ry^r}{x^r-y^r}+\frac{sy^s}{x^s-y^s}\right)\ln y\right)\\&=\exp^{1/(r-s)}\left(\left(\frac{x^r}{x^r-y^r}\ln x^r-\frac{y^r}{x^r-y^r}\ln y^r\right)-\left(\frac{x^s}{x^s-y^s}\ln x^s-\frac{y^s}{x^s-y^s}\ln y^s\right)\right)\\&=\left(\frac{I(x^r,y^r)}{I(x^s,y^s)}\right)^{1/(r-s)},\end{aligned}\tag{2.4}$$

by Corollary 1.5, we get Corollary 2.5.

Corollary 2.5. *Let $p \neq q$. If $(p+q)(r+s) < 0$, then*

$$G_{\mathscr{L}_L,(p+q)/2} < \mathbf{F}(p,q;r,s;a,b) < \sqrt{G_{\mathscr{L}_L,p}G_{\mathscr{L}_L,q}}, \tag{2.5}$$

where $G_{\mathscr{L}_L,t} = G^{1/t}_{\mathscr{L}_L}(a^t,b^t)$, $G_{\mathscr{L}_L}(x,y) = (I(x^r,y^r)/I(x^s,y^s))^{1/(r-s)}$, $I(x,y)$ is defined by (1.3).

Inequalities (2.5) are reversed if $(p+q)(r+s) > 0$.

3. Lemmas

To prove our main results, we need the following three lemmas.

Lemma 3.1. *Suppose $x, y > 0$ with $x \neq y$, define*

$$U(t) := \begin{cases} x^t y^t \left(\dfrac{x^t - y^t}{t(x-y)}\right)^{-2}, & t \neq 0, \\ L^2(x,y), & t = 0, \end{cases} \tag{3.1}$$

then one has

(1) $U(-t) = U(t)$;

(2) $U(t)$ *is strictly increasing in* $(-\infty, 0)$ *and decreasing in* $(0, +\infty)$.

Proof. (1) A simple computation results in part (1) of the lemma, of which details are omitted.

(2) By directly calculations, we get

$$\begin{aligned} \frac{U'(t)}{U(t)} &= \ln x + \ln y - \frac{2(x^t \ln x - y^t \ln y)}{x^t - y^t} + \frac{2}{t} \\ &= \frac{2}{t}\left(\ln\sqrt{x^t y^t} - \left(\frac{x^t \ln x - y^t \ln y}{x^t - y^t} - 1\right)\right) \\ &= \frac{2}{t}\left(\ln G(x^t, y^t) - \ln I(x^t, y^t)\right). \end{aligned} \tag{3.2}$$

By the well-known inequality $I(a,b) > \sqrt{ab}$, we can get part two of the lemma immediately. □

The following lemma is a well-known inequality proved by Carlson (see [11]), which will be used in proof of Lemma 3.3.

Lemma 3.2. *For positive numbers a and b with $a \neq b$, the following inequality holds:*

$$L(a,b) < \frac{A + 2G}{3} = \frac{a + 4\sqrt{ab} + b}{6}. \tag{3.3}$$

Lemma 3.3. *Suppose $x, y > 0$ with $x \neq y$, define*

$$V(t) := \begin{cases} x^t y^t \dfrac{x^t + y^t}{2}\left(\dfrac{x^t - y^t}{t(x-y)}\right)^{-3}, & t \neq 0; \\ L^3(x,y), & t = 0, \end{cases} \tag{3.4}$$

then one has

(1) $V(-t) = V(t)$;

(2) $V(t)$ *is strictly increasing in* $(-\infty, 0)$ *and decreasing in* $(0, +\infty)$.

Proof. (1) A simple computation results in part one, of which details are omitted.

(2) By direct calculations, we get

$$\begin{aligned}\frac{V'(t)}{V(t)} &= \ln x + \ln y + \frac{x^t \ln x + y^t \ln y}{x^t + y^t} - \frac{3(x^t \ln x - y^t \ln y)}{x^t - y^t} + \frac{3}{t}\\ &= \left(1 + \frac{x^t}{x^t + y^t} - \frac{3x^t}{x^t - y^t}\right)\ln x + \left(1 + \frac{y^t}{x^t + y^t} + \frac{3y^t}{x^t - y^t}\right)\ln y + \frac{3}{t}\\ &= -\frac{x^{2t} + 4x^t y^t + y^{2t}}{x^{2t} - y^{2t}}\ln x + \frac{x^{2t} + 4x^t y^t + y^{2t}}{x^{2t} - y^{2t}}\ln y + \frac{3}{t}\\ &= \frac{3}{t} - \frac{x^{2t} + 4x^t y^t + y^{2t}}{x^{2t} - y^{2t}}(\ln x - \ln y)\\ &= \frac{3}{t}\frac{2t(\ln x - \ln y)}{x^{2t} - y^{2t}}\left(\frac{x^{2t} - y^{2t}}{2t(\ln x - \ln y)} - \frac{x^{2t} + 4x^t y^t + y^{2t}}{6}\right).\end{aligned} \tag{3.5}$$

Substituting a, b for x^{2t}, y^{2t} in the above last one expression, then

$$\frac{V'(t)}{V(t)} = \frac{3}{t}L^{-1}(a,b)\left(L(a,b) - \frac{a + 4\sqrt{ab} + b}{6}\right), \tag{3.6}$$

in which $L(a,b) - (a + 4\sqrt{ab} + b)/6 < 0$ by Lemma 3.2, and $L^{-1}(a,b) > 0$. Consequently, $V'(t) > 0$ if $t < 0$ and $V'(t) < 0$ if $t > 0$.

The proof is completed. □

4. Proofs of main results

To prove our main results, it is enough to make certain the signs of $\mathcal{I} = (\ln \mathcal{H}_L)_{xy}$ and $\mathcal{J} = (x-y)(x\mathcal{I})_x$ because $\mathbf{F}(a,b;p,q;r,s) = \mathcal{H}_{\mathcal{H}_L}(a,b;p,q)$, where $\mathcal{H}_L = \mathcal{H}_L(r,s;x,y) = E(r,s;x,y)$ is defined by (1.1).

Proof of Theorem 2.2. Let us observe that

$$\ln \mathcal{H}_L = \frac{1}{r-s}\left(\ln|s| + \ln\left|x^r - y^r\right| - \ln|r| - \ln\left|x^s - y^s\right|\right). \tag{4.1}$$

Through straightforward computations, we have

$$\begin{aligned}\mathcal{I} &= \left(\ln \mathcal{H}_L\right)_{xy}\\ &= \frac{1}{xy(r-s)}\left(\frac{r^2 x^r y^r}{(x^r - y^r)^2} - \frac{s^2 x^s y^s}{(x^s - y^s)^2}\right)\\ &= \frac{1}{xy(r-s)}\left(\frac{r^2 x^r y^r}{(x^r - y^r)^2} - \frac{s^2 x^s y^s}{(x^s - y^s)^2}\right)\\ &= \frac{1}{xy(x-y)^2}\frac{U(r) - U(s)}{r-s}.\end{aligned} \tag{4.2}$$

By Lemma 3.1,

$$\frac{U(r)-U(s)}{r-s}=\frac{U(|r|)-U(|s|)}{|r|-|s|}\frac{r+s}{|r|+|s|}, \tag{4.3}$$

which shows that $\mathcal{I}<0$ if $r+s>0$ and $\mathcal{I}>0$ if $r+s<0$.

By Theorem 1.2, this proof is completed. □

Proof of Theorem 2.3. Let us consider that

$$\begin{aligned}\mathcal{J}&=(x-y)(x\mathcal{I})_x\\&=\frac{x-y}{xy(r-s)}\left(-\frac{r^3x^ry^r(x^r+y^r)}{(x^r-y^r)^3}+\frac{s^3x^sy^s(x^s+y^s)}{(x^s-y^s)^3}\right)\\&=\frac{-2}{xy(x-y)^2}\frac{V(r)-V(s)}{r-s}.\end{aligned} \tag{4.4}$$

By Lemma 3.3,

$$\frac{V(r)-V(s)}{r-s}=\frac{V(|r|)-V(|s|)}{|r|-|s|}\frac{r+s}{|r|+|s|}, \tag{4.5}$$

it follows that $\mathcal{J}>0$ if $r+s>0$ and $\mathcal{J}<0$ if $r+s<0$.

Using Theorem 1.3, this completes the proof. □

Proof of Corollary 2.4. By the proof of Theorem 2.3, there must be $\mathcal{J}<0$ if $r+s<0$. Note $f(x,y)=\mathcal{H}_L(r,s;x,y)$ is symmetric with respect to x and y, it follows from Corollary 1.4 that $\mathbf{F}(p,1-p;r,s;a,b)=\mathcal{H}_{\mathcal{H}_L}(a,b;p,1-p)$ is strictly decreasing in p on $(-\infty,0)$ and $(0,1/2)$. Because

$$\begin{aligned}\mathbf{F}(0,1;r,s;a,b)&=\lim_{p\to 0}\mathbf{F}(p,1-p;r,s;a,b)\\&=\left(\frac{L(a^r,b^r)}{L(a^s,b^s)}\right)^{1/(r-s)}\\&=\left(\frac{s}{r}\frac{a^r-b^r}{a^s-b^s}\right)^{1/(r-s)},\end{aligned} \tag{4.6}$$

thus $\mathbf{F}(p,1-p;r,s;a,b)$ is strictly decreasing in p on $(-\infty,1/2)$.

Likewise, $\mathbf{F}(p,1-p;r,s;a,b)$ is strictly increasing in p on $(1/2,\infty)$ if $r+s>0$.

This proof is completed. □

Proof of Corollary 2.5. By the proof of Theorem 2.3, there must $\mathcal{J}<0$ if $r+s<0$. Notice $f(x,y)=\mathcal{H}_L(r,s;x,y)$ is defined on $\mathbb{R}^+\times\mathbb{R}^+$ and symmetric with respect to x and y, it follows from Corollary 1.5 that (2.5) holds for $p+q>0$. In this way, for $r+s<0$ and $p+q>0$ that (2.5) are also hold by Corollary 1.5. Hence, that (2.5) are always hold for $(p+q)(r+s)<0$.

Likewise, (2.5) are reversed for $(p+q)(r+s)>0$.

The proof ends. □

5. Chains of inequalities for two-parameter means

Let a and b be positive numbers. The p-order power mean, Heron mean, logarithmic mean, exponential (identic mean), power-exponential mean, and exponential-geometric mean are defined as

$$M_p := \begin{cases} M^{1/p}(a^p, b^p) & \text{if } p \neq 0, \\ G(a,b) & \text{if } p = 0, \end{cases} \quad M = A, h, L, I, Z \text{ and } Y, \tag{5.1}$$

where $L = L(a,b)$, $I = I(a,b)$, $A = A(a,b)$, and $h = h(a,b)$ are defined by (1.2)–(1.5), respectively; while the power-exponential mean and exponential-geometric mean are defined by $Z := a^{a/(a+b)}b^{b/(a+b)}$ and $Y := E\exp(1 - G^2/L^2)$, in which $G = G(a,b) = \sqrt{ab}$, respectively (see [9, Examples 2.2 and 2.3]).

Concerning the above means there are many useful and interesting results, such as $L < A_{1/3}$ (see [12]); $I > A_{2/3}$ (see [13]); $Z \geq A_2$ (see [5]); $h \leq I$ (see [14]); $L_2 \leq A_{2/3} \leq I$ (see [15]); $L(a,b) \leq h_p(a,b) \leq A_q(a,b)$ hold for $p \geq 1/2$, $q \geq 2p/3$ (see [16]).

Recently, Neuman applied the comparison theorem to obtain the following result. Let $p, q, r, s, t \in \mathbb{R}^+$. Then, the inequalities

$$L_p \leq h_r \leq A_s \leq I_t \tag{5.2}$$

hold true if and only if $p \leq 2r \leq 3s \leq 2t$ (see [17]).

It is worth mentioning that the author obtained the following chains of inequalities (see [9, 10]) by applying the monotonicity and log-convexity of two-parameter homogenous functions:

$$G < L < A_{1/2} < I < A, \tag{5.3}$$

$$G < I < Z_{1/2} < Y < Z, \tag{5.4}$$

$$L_2 < h < A_{2/3} < I < Z_{1/3} < Y_{1/2}. \tag{5.5}$$

Using our main results in this paper, the above chains of inequalities can be generalized in form of inequalities for two-parameter means, which contain many classical inequalities.

Example 5.1. By Theorem 2.2, for $r + s > 0$, we have

$$\begin{aligned} \mathbf{F}(1,-1;r,s;a,b) &< \mathbf{F}\left(1,-\frac{1}{2};r,s;a,b\right) < \mathbf{F}(1,0;r,s;a,b) \\ &< \mathbf{F}\left(1,\frac{1}{2};r,s;a,b\right) < \mathbf{F}(1,1;r,s;a,b) < \mathbf{F}(1,2;r,s;a,b), \end{aligned} \tag{5.6}$$

that is,

$$\begin{aligned} G &< \left(\frac{a^{r/2}+b^{r/2}}{a^{s/2}+b^{s/2}}\right)^{2/3(r-s)} G^{2/3} < \left(\frac{s}{r}\frac{a^r-b^r}{a^s-b^s}\right)^{1/(r-s)} \\ &< \left(\frac{a^{r/2}+b^{r/2}}{a^{s/2}+b^{s/2}}\right)^{2/(r-s)} < \left(\frac{I(a^r,b^r)}{I(a^s,b^s)}\right)^{1/(r-s)} < \left(\frac{a^r+b^r}{a^s+b^s}\right)^{1/(r-s)}, \end{aligned} \tag{5.7}$$

which can be concisely denoted by

$$G < \left(\frac{A(a^{r/2},b^{r/2})}{A(a^{s/2},b^{s/2})}\right)^{2/3(r-s)} G^{2/3} < \left(\frac{L(a^r,b^r)}{L(a^s,b^s)}\right)^{1/(r-s)}$$
$$< \left(\frac{A(a^{r/2},b^{r/2})}{A(a^{s/2},b^{s/2})}\right)^{2/(r-s)} < \left(\frac{I(a^r,b^r)}{I(a^s,b^s)}\right)^{1/(r-s)} < \left(\frac{A(a^r,b^r)}{A(a^s,b^s)}\right)^{1/(r-s)}, \tag{5.8}$$

where L, I, A are defined by (1.2)–(1.4).

In particular, putting $r = 1$, $s = 0$; $r = 2s = 2$; $r = s = 1$ in (5.7), respectively, we have the following inequalities:

$$G < A_{1/2}^{1/3} G^{2/3} < L < A_{1/2} < I < A, \tag{5.9}$$

$$G < A^{2/3} A_{1/2}^{-1/3} G^{2/3} < A < A^2 A_{1/2}^{-1} < Z < A_2 A^{-1}, \tag{5.10}$$

$$G < Z_{1/2}^{1/3} G^{2/3} < I < Z_{1/2} < Y < Z, \tag{5.11}$$

which contain (5.3) and (5.4). Here we have used the formula $I(a^2,b^2)/I(a,b) = Z(a,b)$ (see [9, Remark 3]).

Example 5.2. By Corollary 2.4, we can get another more refined inequalities. For $r + s > 0$, we have

$$\mathbf{F}\left(\frac{1}{2},\frac{1}{2};r,s;a,b\right) > \mathbf{F}\left(\frac{2}{3},\frac{1}{3};r,s;a,b\right) > \mathbf{F}\left(\frac{3}{4},\frac{1}{4};r,s;a,b\right) > \mathbf{F}(1,0;r,s;a,b)$$
$$> \mathbf{F}\left(\frac{4}{3},-\frac{1}{3};r,s;a,b\right) > \mathbf{F}\left(\frac{3}{2},-\frac{1}{2};r,s;a,b\right) > \mathbf{F}(2,-1;r,s;a,b), \tag{5.12}$$

that is,

$$\left(\frac{I(a^{r/2},b^{r/2})}{I(a^{s/2},b^{s/2})}\right)^{2/(r-s)} > \left(\frac{a^{r/3}+b^{r/3}}{a^{s/3}+b^{s/3}}\right)^{3/(r-s)} > \left(\frac{a^{r/2}+\sqrt{a^{r/2}b^{r/2}}+b^{r/2}}{a^{s/2}+\sqrt{a^{s/2}b^{s/2}}+b^{s/2}}\right)^{2/(r-s)}$$
$$> \left(\frac{s}{r}\frac{a^r-b^r}{a^s-b^s}\right)^{1/(r-s)} > \left(\frac{a^{r/3}+b^{r/3}}{a^{s/3}+b^{s/3}}\frac{a^{2r/3}+b^{2r/3}}{a^{2s/3}+b^{2s/3}}\right)^{3/5(r-s)} G^{2/5}$$
$$> \left(\frac{a^r+\sqrt{a^r b^r}+b^r}{a^s+\sqrt{a^s b^s}+b^s}\right)^{1/2(r-s)} \sqrt{G} > \left(\frac{a^r+b^r}{a^s+b^s}\right)^{1/3(r-s)} G^{2/3}, \tag{5.13}$$

which can be concisely denoted by

$$\left(\frac{I(a^{r/2},b^{r/2})}{I(a^{s/2},b^{s/2})}\right)^{2/(r-s)} > \left(\frac{A(a^{r/3},b^{r/3})}{A(a^{s/3},b^{s/3})}\right)^{3/(r-s)} > \left(\frac{h(a^{r/2},b^{r/2})}{h(a^{s/2},b^{s/2})}\right)^{2/(r-s)}$$
$$> \left(\frac{L(a^r,b^r)}{L(a^s,b^s)}\right)^{1/(r-s)} > \left(\frac{A(a^{r/3},b^{r/3})}{A(a^{s/3},b^{s/3})}\frac{A(a^{2r/3},b^{2r/3})}{A(a^{2s/3},b^{2s/3})}\right)^{3/5(r-s)} G^{2/5}$$
$$> \left(\frac{h(a^r,b^r)}{h(a^s,b^s)}\right)^{1/2(r-s)} \sqrt{G} > \left(\frac{A(a^r,b^r)}{A(a^s,b^s)}\right)^{1/3(r-s)} G^{2/3}, \tag{5.14}$$

where $L(x,y)$, $I(x,y)$, $A(x,y)$, and $h(x,y)$ are defined by (1.2)–(1.5), respectively.

In particular, put $r = 1$, $s = 0$; $r = 2$, $s = 1$; $r = 1$, $s \to 1$ in (5.14) and note

$$\lim_{r\to s}\left(\frac{A(a^r,b^r)}{A(a^s,b^s)}\right)^{1/(r-s)} = Z_s, \qquad \lim_{r\to s}\left(\frac{h(a^r,b^r)}{h(a^s,b^s)}\right)^{1/(r-s)} = I_{3s/2}^{3/2} I_{s/2}^{-1/2}, \tag{5.15}$$

we have

$$\begin{gathered} I_{1/2} > A_{1/3} > h_{1/2} > L > A_{1/3}^{1/5} A_{2/3}^{2/5} G^{2/5} > \sqrt{hG} > A^{1/3} G^{2/3}, \\ Z_{1/2} > A_{2/3}^{2} A_{1/3}^{-1} > h^2 h_{1/2}^{-1} > A > A_{4/3}^{4/5} A_{1/3}^{-1/5} G^{2/5} > h_2 h^{-1/2} G^{1/2} > A_2^{2/3} A^{-1/3} G^{2/3}, \\ Y_{1/2} > Z_{1/3} > I_{3/4}^{3/2} I_{1/4}^{-1/2} > I > Z_{1/3}^{1/5} Z_{2/3}^{2/5} G^{2/5} > I_{3/2}^{3/4} I_{1/2}^{-1/4} G^{1/2} > Z^{1/3} G^{2/3}, \end{gathered} \tag{5.16}$$

respectively. Here we have again used the formula $I(a^2, b^2)/I(a,b) = Z(a,b)$. This shows the inequalities (5.14) contain (5.11)–(5.13) in [10] and (5.5).

Example 5.3. Putting $r = 1$, $s = 0$; $r = 2$, $s = 1$; $r = 1$, $s \to 1$ in Corollary 2.5, we have the following inequalities:

$$\begin{aligned} I_{(p+q)/2} &> \left(\frac{q}{p}\frac{a^p - b^p}{a^q - b^q}\right)^{1/(p-q)} > \sqrt{I_p I_q}, \\ Z_{(p+q)/2} &> \left(\frac{a^p + b^p}{a^q + b^q}\right)^{1/(p-q)} > \sqrt{Z_p Z_q}, \\ Y_{(p+q)/2} &> \left(\frac{I(a^p,b^p)}{I(a^q,b^q)}\right)^{1/(p-q)} > \sqrt{Y_p Y_q}, \end{aligned} \tag{5.17}$$

for $p + q > 0$ with $p \neq q$.

On the other hand, putting $p = 1, q = 0$; $p = 2, q = 1$; $p = 3/2$, $q = 1/2$ in Corollary 2.5, we can get another inequalities

$$\begin{gathered} \left(\frac{I(a^{r/2},b^{r/2})}{I(a^{s/2},b^{s/2})}\right)^{2/(r-s)} > \left(\frac{s}{r}\frac{a^r - b^r}{a^s - b^s}\right)^{1/(r-s)} > \left(\frac{I(a^r,b^r)}{I(a^s,b^s)}\right)^{1/2(r-s)} G^{1/2}, \\ \left(\frac{I(a^{3r/2},b^{3r/2})}{I(a^{3s/2},b^{3s/2})}\right)^{2/3(r-s)} > \left(\frac{a^r + b^r}{a^s + b^s}\right)^{1/(r-s)} > \left(\frac{I(a^{2r},b^{2r})}{I(a^{2s},b2^s)}\right)^{1/4(r-s)} \left(\frac{I(a^r,b^r)}{I(a^s,b2^s)}\right)^{1/2(r-s)}, \\ \left(\frac{I(a^r,b^r)}{I(a^s,b2^s)}\right)^{1/(r-s)} > \left(\frac{a^r + \sqrt{a^r b^r} + b^r}{a^s + \sqrt{a^s b^s} + b^s}\right)^{1/(r-s)} \\ > \left(\frac{I(a^{3r/2},b^{3r/2})}{I(a^{3s/2},b^{3s/2})}\right)^{1/3(r-s)} \left(\frac{I(a^{r/2},b^{r/2})}{I(a^{s/2},b^{s/2})}\right)^{1/(r-s)} \end{gathered} \tag{5.18}$$

for $r + s > 0$.

References

[1] K. B. Stolarsky, "Generalizations of the logarithmic mean," *Mathematics Magazine*, vol. 48, pp. 87–92, 1975.

[2] E. B. Leach and M. C. Sholander, "Extended mean values," *The American Mathematical Monthly*, vol. 85, no. 2, pp. 84–90, 1978.

[3] B.-N. Guo, S. Q. Zhang, and F. Qi, "Elementary proofs of monotonicity for extended mean values of some functions with two parameters," *Mathematics in Practice and Theory*, vol. 29, no. 2, pp. 169–174, 1999 (Chinese).

[4] E. B. Leach and M. C. Sholander, "Extended mean values. II," *Journal of Mathematical Analysis and Applications*, vol. 92, no. 1, pp. 207–223, 1983.

[5] Zs. Páles, "Inequalities for differences of powers," *Journal of Mathematical Analysis and Applications*, vol. 131, no. 1, pp. 271–281, 1988.

[6] F. Qi, "Logarithmic convexity of extended mean values," *Proceedings of the American Mathematical Society*, vol. 130, no. 6, pp. 1787–1796, 2002.

[7] A. Witkowski, "Comparison theorem for two-parameter means," to appear in *Mathematical Inequalities & Applications*.

[8] Zh.-H. Yang, "Simple discriminances of convexity of homogeneous functions and applications," *Study in College Mathematics*, vol. 7, no. 4, pp. 14–19, 2004 (Chinese).

[9] Zh.-H. Yang, "On the homogeneous functions with two parameters and its monotonicity," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 6, no. 4, article 101, pp. 1–11, 2005.

[10] Zh.-H. Yang, "On the log-convexity of two-parameter homogeneous functions," *Mathematical Inequalities & Applications*, vol. 10, no. 3, pp. 499–516, 2007.

[11] B. C. Carlson, "The logarithmic mean," *The American Mathematical Monthly*, vol. 79, no. 6, pp. 615–618, 1972.

[12] T.-P. Lin, "The power mean and the logarithmic mean," *The American Mathematical Monthly*, vol. 81, no. 8, pp. 879–883, 1974.

[13] K. B. Stolarsky, "The power and generalized logarithmic means," *The American Mathematical Monthly*, vol. 87, no. 7, pp. 545–548, 1980.

[14] J. Sándor, "A note on some inequalities for means," *Archiv der Mathematik*, vol. 56, no. 5, pp. 471–473, 1991.

[15] E. Neuman and J. Sándor, "Inequalities involving Stolarsky and Gini means," *Mathematica Pannonica*, vol. 14, no. 1, pp. 29–44, 2003.

[16] G. Jia and J. Cao, "A new upper bound of the logarithmic mean," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 4, article 80, 4 pages, 2003.

[17] E. Neuman, "A generalization of an inequality of Jia and Cau," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 5, no. 1, article 15, pp. 1–4, 2004.