

Research Article

A Univalence Preserving Integral Operator

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We define an integral operator denoted by I , and we give sufficient conditions such that $F = I(f_1, f_2, \dots, f_m)$ is univalent.

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1. Introduction and preliminaries

Let U denote the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : |z| < 1\}. \quad (1.1)$$

Let $\mathcal{H}(U)$ denote the space of holomorphic functions in U and let

$$A_n = \{f \in \mathcal{H}(U), f(z) = z + a_{n+1}z^{n+1} + \dots, z \in U\} \quad (1.2)$$

with $A_1 = A$.

Let

$$S = \{f \in A : f \text{ is univalent in } U\}. \quad (1.3)$$

Definition 1.1 (Ruscheweyh [1]). For $f \in A$, $n \in \mathbb{N} \cup \{0\}$, let R^n be the operator defined by $R^n : A \rightarrow A$,

$$\begin{aligned} R^0 f(z) &= f(z), \\ (n+1)R^{n+1} f(z) &= z[R^n f(z)]' + nR^n f(z), \quad z \in U. \end{aligned} \quad (1.4)$$

Remark 1.2. If $f \in A$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad z \in U, \quad (1.5)$$

then

$$R^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^n a_j z^j, \quad z \in U. \quad (1.6)$$

In order to prove our main results, we shall use the following lemmas.

Lemma A (see [2]). Let α be a complex number, $\operatorname{Re} \alpha > 0$, and $f \in A$. If

$$\frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1, \quad \forall z \in U, \quad (1.7)$$

then the function

$$F_\alpha(z) = \left[\alpha \int_0^z t^{\alpha-1} f'(t) dt \right]^{1/\alpha} \quad (1.8)$$

is in the class S .

Lemma B (see [3]). Let α be a complex number, $\operatorname{Re} \alpha > 0$, and let $f(z) = z + a_2 z^2 + \dots$ be a regular function in U . If

$$\frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1, \quad \forall z \in U, \quad (1.9)$$

then, for any complex number β with $\operatorname{Re} \beta \geq \operatorname{Re} \alpha$, the function

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} f'(t) dt \right]^{1/\beta} \quad (1.10)$$

is in the class S .

2. Main results

By using the Ruscheweyh differential operator given by Definition 1.1, we introduce the following integral operator.

Definition 2.1. Let $n, m \in \mathbb{N} \cup \{0\}$, $i \in \{1, 2, 3, \dots, m\}$, $\alpha_i \in \mathbb{C}$, $\alpha \in \mathbb{C}$, with $\operatorname{Re} \alpha > 0$, $A^m = \underbrace{A \times A \times \dots \times A}_{m \text{ times}}$. We let $I : A^m \rightarrow A$ be the integral operator given by

$$I(f_1, f_2, \dots, f_m)(z) = F(z) = \left[\alpha \int_0^z t^{\alpha-1} \left(\frac{R^n f_1(t)}{t} \right)^{\alpha_1} \dots \left(\frac{R^n f_m(t)}{t} \right)^{\alpha_m} dt \right]^{1/\alpha}, \quad (2.1)$$

where $f_i \in A$, $i \in \{1, 2, 3, \dots, n\}$ and R^n is the Ruscheweyh differential operator.

Remark 2.2. (i) For $n = 0$, $m = 1$, $\alpha = 1$, $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = \dots = \alpha_m = 0$, and $f(z) \in A$, we obtain Alexander integral operator introduced in 1915 in [4]:

$$I(z) = \int_0^z \frac{f(t)}{t} dt, \quad z \in U. \quad (2.2)$$

(ii) For $n = 0$, $m = 1$, $\alpha = 1$, $\alpha_1 = \beta \in [0, 1]$, $\alpha_2 = \alpha_3 = \dots = \alpha_m = 0$, and $f(z) \in S$, we obtain the integral operator

$$I(z) = \int_0^z \left[\frac{f(t)}{t} \right]^\beta dt, \quad z \in U, \quad (2.3)$$

studied in [5].

For $\beta \in \mathbb{C}$ with $|\beta| \leq 1/4$, this integral operator was studied in [6, 7] and for $|\beta| \leq 1/3$, in [8].

(iii) For $n = 1$, $m = 1$, $\alpha = 1$, $\alpha_1 = \beta \in \mathbb{C}$, $|\beta| \leq 1/4$, $\alpha_2 = \dots = \alpha_m = 0$, $R^1 f(z) = z f'(z)$, $z \in U$, $f \in S$, we obtain the integral operator

$$I(z) = \int_0^z [f'(t)]^\beta dt, \quad z \in U \quad (2.4)$$

studied in [9].

(iv) For $n = 0$, $m \in \mathbb{N} \cup \{0\}$, $\alpha = 1$, $\alpha_i > 0$, $i \in \{1, 2, \dots, m\}$, we obtain the integral operator

$$F(z) = \int_0^z \left[\frac{f_1(t)}{t} \right]^{\alpha_1} \dots \left[\frac{f_m(t)}{t} \right]^{\alpha_m} dt, \quad (2.5)$$

studied in [10].

(v) For $n = 0$, $m \in \mathbb{N} \cup \{0\}$, $\alpha \in \mathbb{R}$, $\operatorname{Re} \alpha > 0$, $\alpha_i \in \mathbb{C}$, $f_i \in S$, $i \in \{1, 2, \dots, n\}$, we obtain the integral operator introduced in [10] by D. Breaz and N. Breaz:

$$G(z) = \left[\alpha \int_0^z t^{\alpha-1} \left(\frac{f_1(t)}{t} \right)^{\alpha_1} \dots \left(\frac{f_n(t)}{t} \right)^{\alpha_m} dt \right]^{1/\alpha}. \quad (2.6)$$

(vi) For $n, m \in \mathbb{N} \cup \{0\}$, $\alpha = 1$, $\alpha_i \in \mathbb{R}$, $i \in \{1, 2, \dots, m\}$ with $\alpha_i \geq 0$, we obtain the integral operator studied in [11, 12]

$$I(f_1, f_2, \dots, f_m)(z) = \int_0^z \left[\frac{R^n f_1(t)}{t} \right]^{\alpha_1} \cdots \left[\frac{R^n f_m(t)}{t} \right]^{\alpha_m} dt. \quad (2.7)$$

(vii) For $n = 0$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = \cdots = \alpha_m = 0$, $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha \geq 3$, we obtain the integral operator studied in [7, 13]

$$G_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} \left(\frac{g(u)}{u} \right) du \right]^{1/\alpha}. \quad (2.8)$$

(viii) For $n = 1$, $m = 1$, $\alpha \in \mathbb{C}$, $\operatorname{Re} \alpha > 0$, $\alpha_1 = 1$, $\alpha_2 = \cdots = \alpha_m = 0$, we obtain the integral operator

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} f'(u) du \right]^{1/\alpha} \quad (2.9)$$

studied in [14, 15].

We study the conditions for the integral operator introduced in Definition 2.1 to be univalent.

Theorem 2.3. Let $n, m \in \mathbb{N} \cup \{0\}$, $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha > 0$, $f_i \in A$, $\alpha_i \in \mathbb{C}$, $i \in \{1, 2, \dots, m\}$ with $|\alpha_1| + |\alpha_2| + \cdots + |\alpha_m| \leq 1$.

If

$$\left| \frac{z(R^n f_i(z))'}{R^n f_i(z)} - 1 \right| \leq 1, \quad z \in U, i \in \{1, 2, \dots, m\}, \quad (2.10)$$

then $F(z)$ given by (2.1) belongs to class S .

Proof. Let

$$f(z) = \int_0^z \left[\frac{R^n f_1(t)}{t} \right]^{\alpha_1} \cdots \left[\frac{R^n f_m(t)}{t} \right]^{\alpha_m} dt, \quad z \in U. \quad (2.11)$$

By differentiating (2.11), we obtain

$$f'(z) = \left[\frac{R^n f_1(z)}{z} \right]^{\alpha_1} \cdots \left[\frac{R^n f_m(z)}{z} \right]^{\alpha_m} = 1 + A_2 z + A_3 z^2 + \cdots, \quad (2.12)$$

$z \in U$.

From (2.11), (2.12), and the condition in the theorem, we have that $f'(z) \neq 0$, $z \in U$.

Then using (2.12), we obtain

$$\log f'(z) = \alpha_1 [\log R^n f_1(z) - \log z] + \cdots + \alpha_m [\log R^n f_m(z) - \log z], \quad z \in U. \quad (2.13)$$

By differentiating (2.13), after a short calculation, we have

$$\frac{zf''(z)}{f'(z)} = \alpha_1 \left[\frac{z(R^n f_1(z))'}{R^n f_1(z)} - 1 \right] + \cdots + \alpha_m \left[\frac{z(R^n f_m(z))'}{R^n f_m(z)} - 1 \right], \quad z \in U. \quad (2.14)$$

Using the conditions given by the hypothesis of Theorem 2.3, we obtain

$$\begin{aligned} \frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{zf''(z)}{f'(z)} \right| &= \frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left[|\alpha_1| \left| \frac{z(R^n f_1(z))'}{R^n f_1(z)} - 1 \right| + \cdots + |\alpha_m| \left| \frac{z(R^n f_m(z))'}{R^n f_m(z)} - 1 \right| \right] \\ &\leq \frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} [|\alpha_1| + |\alpha_2| + \cdots + |\alpha_m|] \\ &\leq \frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \leq \frac{1}{\operatorname{Re}\alpha} \leq 1. \end{aligned} \quad (2.15)$$

Using (2.12), the conditions in Lemma A are satisfied, hence $F(z)$ belongs to the class S . \square

Remark 2.4. For $\alpha_i \in \mathbb{R}$, $i \in \{1, 2, \dots, m\}$, then Theorem 2.3 can be rewritten as following.

Corollary 2.5. Let $n, m \in \mathbb{N} \cup \{0\}$, $\alpha \in \mathbb{C}$, with $\operatorname{Re}\alpha > 0$, let $\alpha_i \in \mathbb{R}$, $\alpha_i \geq 0$ with $\alpha_1 + \alpha_2 + \cdots + \alpha_m \leq 1$ and let $f_i \in A$, $i \in \{1, 2, \dots, m\}$.

If

$$\left| \frac{z(R^n f_i(z))'}{R^n f_i(z)} - 1 \right| \leq 1, \quad z \in U, i \in \{1, 2, \dots, m\}, \quad (2.16)$$

then $F(z)$ given by (2.1) belongs to the class S , where R^n is the Ruscheweyh differential operator.

Example 2.6. Let $n \in \mathbb{N} \cup \{0\}$, $m = 2$, $\alpha = 2 + 3i$, $\alpha_1 = 1/4 + i(\sqrt{3}/4)$, $\alpha_2 = 1/5 - i(2/5)$, $|\alpha_1| + |\alpha_2| = (5 + 2\sqrt{5})/10 < 1$, $f_1(z) = z + az^2$, $R^n f_1(z) = z + (n+1)az^2$, $f_2(z) = z + bz^2$, $R^n f_2(z) = z + (n+1)bz^2$, $a, b \in \mathbb{C}$, with $|a| \leq 1/2(n+1)$, $|b| \leq 1/2(n+1)$, $z \in U$. Then

$$\left| \frac{z[z + (n+1)bz^2]'}{z[1 + (n+1)bz]} - 1 \right| = \left| \frac{1 + 2(n+1)bz}{1 + (n+1)bz} - 1 \right| = \left| \frac{(n+1)bz}{1 + (n+1)bz} \right|, \quad z \in U. \quad (2.17)$$

But

$$\begin{aligned}
 \left| \frac{(n+1)bz}{1+(n+1)bz} \right|^2 &= \frac{|(n+1)bz|^2}{|1+(n+1)bz|^2} \\
 &= \frac{(n+1)bz(n+1)\bar{b}\bar{z}}{(1+(n+1)bz)(1+(n+1)\bar{b}\bar{z})} \\
 &= \frac{(n+1)^2|b|^2|z|^2}{1+(n+1)bz+(n+1)\bar{b}\bar{z}+(n+1)^2|b|^2|z|^2} \\
 &= \frac{(n+1)^2|b|^2|z|^2}{1+2(n+1)\operatorname{Re}bz+(n+1)^2|b|^2|z|^2} \tag{2.18} \\
 &\leq \frac{(n+1)^2|b|^2|z|^2}{1-2(n+1)|b||z|+(n+1)^2|b|^2|z|^2} = \frac{(n+1)^2|b|^2|z|^2}{(1-(n+1)|b||z|)^2} \\
 &\leq \frac{(n+1)|b|^2}{[1-(n+1)|b|]^2} \leq \frac{(n+1) \cdot 1/4(n+1)^2}{[1-(n+1) \cdot 1/2(n+1)]^2} \\
 &= \frac{1/4(n+1)^2}{1/4} = \frac{1}{(n+1)^2} \leq 1.
 \end{aligned}$$

It implies that

$$\left| \frac{(n+1)bz}{1+(n+1)bz} \right| \leq 1, \quad z \in U. \tag{2.19}$$

Similarly, we obtain

$$\left| \frac{z[z+(n+1)az^2]'}{z[1+(n+1)az]} - 1 \right| = \left| \frac{(n+1)az}{1+(n+1)az} \right| \leq 1, \quad z \in U. \tag{2.20}$$

Using Theorem 2.3, we have

$$I(f_1, f_2) = F(z) = \left[(2+3i) \int_0^z t^{1+3i} (1+at)^{1/4+i(\sqrt{3}/4)} (1+bt)^{1/5-i(2/5)} dt \right]^{1/(2+3i)} \in S, \tag{2.21}$$

for all $z \in U$.

Theorem 2.7. Let $n, m \in \mathbb{N} \cup \{0\}$, $\alpha, \beta \in \mathbb{C}$, with $\operatorname{Re} \beta \geq \operatorname{Re} \alpha > 0$, let $f_i \in A$, and let $\alpha_i \in \mathbb{C}$, $i \in \{1, 2, \dots, m\}$, with $|\alpha_1| + |\alpha_2| + \dots + |\alpha_m| \leq 1$.

If

$$\left| \frac{z(R^n f_i(z))'}{R^n f_i(z)} - 1 \right| \leq 1, \quad z \in U, \quad i \in \{1, 2, \dots, m\}, \tag{2.22}$$

where R^n is the Ruscheweyh differential operator, then the function given by

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} \left(\frac{R^n f_1(t)}{t} \right)^{\alpha_1} \cdots \left(\frac{R^n f_m(t)}{t} \right)^{\alpha_m} dt \right]^{1/\beta} \quad (2.23)$$

belongs to the class S .

Proof. Using (2.14) and (2.22), from the proof of Theorem 2.3, we obtain

$$\frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1, \quad z \in U. \quad (2.24)$$

Using (2.12), the conditions from Lemma B are satisfied and by applying it we have that the function $F_\beta(z)$ given by (2.23) belongs to the class S . \square

Example 2.8. Let $n \in \mathbb{N} \cup \{0\}$, $m = 2$, $\beta = 4 - i$, $\alpha = 2 + 3i$, $\operatorname{Re} \beta > \operatorname{Re} \alpha > 0$, $\alpha_1 = 1/4 + i(\sqrt{3}/4)$, $\alpha_2 = 1/5 - i(2/5)$, $|\alpha_1| + |\alpha_2| = (5 + 2\sqrt{5})/10 < 1$, $f_1(z) = z + az^2$, $f_2(z) = z + bz^2$, $R^n f_1(z) = z + (n+1)az^2$, $R^n f_2(z) = z + (n+1)bz^2$, $a, b \in \mathbb{C}$, $|a| \leq 1/2(n+1)$, $|b| \leq 1/2(n+1)$, $z \in U$. Then

$$\begin{aligned} \left| \frac{z(z + (n+1)az^2)'}{z + (n+1)az^2} - 1 \right| &= \left| \frac{(n+1)az}{1 + (n+1)az} \right| \leq 1, \\ &z \in U. \quad (2.25) \\ \left| \frac{z(z + (n+1)bz^2)'}{z + (n+1)bz^2} - 1 \right| &= \left| \frac{(n+1)bz}{1 + (n+1)bz} \right| \leq 1, \end{aligned}$$

Using Theorem 2.7, we have

$$F_\beta(z) = \left[(4-i) \int_0^z t^{3-i} (1+at)^{1/4+i(\sqrt{3}/4)} (1+bt)^{1/5-i(2/5)} dt \right]^{1/(4-i)} \in S, \quad (2.26)$$

for all $z \in U$.

Theorem 2.9. Let $n, m \in \mathbb{N} \cup \{0\}$, $\mu > 0$, $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha > 0$, let $f_i \in A$ and let $\alpha_i \in \mathbb{C}$, $i \in \{1, 2, \dots, m\}$ with $|\alpha_1| + |\alpha_2| + \cdots + |\alpha_m| \leq 1/(2\mu + 1)$.

If

$$\begin{aligned} \text{(i)} \quad &|R^n f_i(z)| \leq \mu, \\ \text{(ii)} \quad &\left| \frac{z^2 (R^n f_i(z))'}{[R^n f_i(z)]^2} - 1 \right| \leq 1, \quad z \in U, i \in \{1, 2, \dots, m\}, \end{aligned} \quad (2.27)$$

where R^n is the Ruscheweyh differential operator, then the function $F(z)$ given by (2.1) belongs to the class S .

Proof. Using (2.14), we have

$$\left| \frac{zf''(z)}{f'(z)} \right| = |\alpha_1| \left| \frac{z(R^n f_1(z))'}{R^n f_1(z)} - 1 \right| + \cdots + |\alpha_m| \left| \frac{z(R^n f_m(z))'}{R^n f_m(z)} - 1 \right|, \quad z \in U. \quad (2.28)$$

Using the conditions from Theorem 2.9 and (2.28) we calculate

$$\begin{aligned} & \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \\ &= \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} |\alpha_1| \left| \frac{z(R^n f_1(z))'}{R^n f_1(z)} - 1 \right| + \cdots + \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} |\alpha_m| \left| \frac{z(R^n f_m(z))'}{R^n f_m(z)} - 1 \right| \\ &\leq \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} |\alpha_1| \left[\left| \frac{z(R^n f_1(z))'}{R^n f_1(z)} \right| + 1 \right] + \cdots + \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} |\alpha_m| \left[\left| \frac{z(R^n f_m(z))'}{R^n f_m(z)} \right| + 1 \right] \\ &= \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} |\alpha_1| \left| \frac{z(R^n f_1(z))'}{R^n f_1(z)} \right| + \cdots + \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} |\alpha_m| \left| \frac{z(R^n f_m(z))'}{R^n f_m(z)} \right| \\ &\quad + \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} (|\alpha_1| + \cdots + |\alpha_m|) \\ &= \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} |\alpha_1| \left| \frac{z^2(R^n f_1(z))'}{(R^n f_1(z))^2} \right| \frac{|R^n f_1(z)|}{|z|} + \cdots + \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} |\alpha_m| \left| \frac{z^2(R^n f_m(z))'}{(R^n f_m(z))^2} \right| \frac{|R^n f_m(z)|}{|z|} \\ &\quad + \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} (|\alpha_1| + \cdots + |\alpha_m|) \\ &\leq \mu \left[\frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{z^2(R^n f_1(z))'}{(R^n f_1(z))^2} \right| + \cdots + \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{z^2(R^n f_m(z))'}{(R^n f_m(z))^2} \right| \right] \\ &\quad + \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} (|\alpha_1| + \cdots + |\alpha_m|) \\ &= \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left[\left| \frac{z^2(R^n f_1(z))'}{(R^n f_1(z))^2} \right| - 1 + 1 \right] \cdot \mu + \cdots + \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left[\left| \frac{z^2(R^n f_m(z))'}{(R^n f_m(z))^2} \right| - 1 + 1 \right] \cdot \mu \\ &\quad + \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} (|\alpha_1| + \cdots + |\alpha_m|) \\ &\leq \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} |\alpha_1| \left| \frac{z^2(R^n f_1(z))'}{(R^n f_1(z))^2} - 1 \right| \cdot \mu + \cdots + \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} |\alpha_m| \left| \frac{z^2(R^n f_m(z))'}{(R^n f_m(z))^2} - 1 \right| \cdot \mu \\ &\quad + \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \cdot (|\alpha_1| + \cdots + |\alpha_m|) + \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \cdot \mu (|\alpha_1| + \cdots + |\alpha_m|) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} (|\alpha_1| + \cdots + |\alpha_m|) 2\mu + \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \cdot (|\alpha_1| + \cdots + |\alpha_m|) \\
&\leq \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} (|\alpha_1| + \cdots + |\alpha_m|) (2\mu + 1) \\
&\leq \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \leq \frac{1}{\operatorname{Re} \alpha} \leq 1, \quad z \in U.
\end{aligned} \tag{2.29}$$

By using (2.12) and (2.29) and by applying Lemma A, we obtain that the function $F(z)$ given by (2.1) belongs to the class S . \square

Theorem 2.10. Let $n, m \in \mathbb{N} \cup \{0\}$, $\mu > 0$, $\alpha, \beta \in \mathbb{C}$ with $\operatorname{Re} \beta \geq \operatorname{Re} \alpha > 0$, and let $\alpha_i \in \mathbb{C}$, $i \in \{1, 2, 3, \dots, m\}$, with $|\alpha_1| + |\alpha_2| + \cdots + |\alpha_m| \leq 1/(2\mu + 1)$.

If

$$\begin{aligned}
&\text{(i) } |R^n f_i(z)| \leq \mu, \\
&\text{(ii) } \left| \frac{z^2 R^n f_i(z)}{(R^n f_i(z))^2} - 1 \right| \leq 1, \quad z \in U, i \in \{1, 2, \dots, m\},
\end{aligned} \tag{2.30}$$

where R^n is the Ruscheweyh differential operator, then function $F_\beta(z)$ given by (2.23) belongs to the class S .

Proof. By using (2.30) and (2.12) and from the proof of Theorem 2.9, we have

$$\frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{z f''(z)}{f'(z)} \right| < 1, \quad z \in U, \tag{2.31}$$

and applying Lemma B we obtain that the $F_\beta(z)$ given by (2.23) belongs to the class S . \square

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