Research Article

Existence of Solutions for a Class of Weighted $p(t)$-Laplacian System Multipoint Boundary Value Problems

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This paper investigates the existence of solutions for weighted $p(t)$-Laplacian system multipoint boundary value problems. When the nonlinearity term $f(t, \cdot, \cdot)$ satisfies sub-$p^- - 1$ growth condition or general growth condition, we give the existence of solutions via Leray-Schauder degree.

1. Introduction

In this paper, we consider the existence of solutions for the following weighted $p(t)$-Laplacian system:

$$-\Delta_{p(t),w(t)} u + \delta f\left(t, u, (w(t))^{1/(p(t)-1)} u'\right) = 0, \quad t \in (0,1), \tag{1.1}$$

with the following multipoint boundary value condition:

$$u(0) = \sum_{i=1}^{m-2} \beta_i u(\eta_i) + e_0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i) + e_1, \tag{1.2}$$

where $p \in C([0,1], \mathbb{R})$ and $p(t) > 1$, $-\Delta_{p(t),w(t)} u = -(w(t)|u'|^{p(t)-2} u')'$ is called the weighted $p(t)$-Laplacian; $w \in C([0,1], \mathbb{R})$ satisfies $0 < w(t)$, for all $t \in (0,1)$, and $(w(t))^{-1/(p(t)-1)} \in L^1(0,1)$; $0 < \eta_1 < \cdots < \eta_{m-2} < 1$, $0 < \xi_1 < \cdots < \xi_{m-2} < 1$; $\alpha_i \geq 0$, $\beta_i \geq 0$ $(i = 1, \ldots, m-2)$, and $0 < \sum_{i=1}^{m-2} \alpha_i < 1$, $0 < \sum_{i=1}^{m-2} \beta_i < 1$; $e_0, e_1 \in \mathbb{R}^N$; δ is a positive parameter.

The study of differential equations and variational problems with variable exponent growth conditions is a new and interesting topic. Many results have been obtained on these problems, for example, [1–14]. We refer to [2, 15, 16] the applied background on these problems. If $w(t) \equiv 1$ and $p(t) \equiv p$ (a constant), $-\Delta_{p(t),w(t)}$ is the well-known p-Laplacian. If $p(t)$ is a general function, $-\Delta_{p(t),w(t)}$ represents a nonhomogeneity and possesses more nonlinearity, thus $-\Delta_{p(t),w(t)}$ is more complicated than $-\Delta_p$. We have the following examples.

(1) If $\Omega \subset \mathbb{R}^N$ is a bounded domain, the Rayleigh quotient

$$\lambda_{p(x)} = \inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega (1/p(x))|\nabla u|^{p(x)}\, dx}{\int_\Omega (1/p(x))|u|^{p(x)}\, dx} \tag{1.3}$$

is zero in general, and only under some special conditions $\lambda_{p(x)} > 0$ (see [6]), but the fact that $\lambda_p > 0$ is very important in the study of p-Laplacian problems.

(2) If $w(t) \equiv 1$ and $p(t) \equiv p$ (a constant) and $-\Delta_p u > 0$, then u is concave, this property is used extensively in the study of one-dimensional p-Laplacian problems, but it is invalid for $-\Delta_{p(t),1}$. It is another difference on $-\Delta_p$ and $-\Delta_{p(t),1}$.

(3) On the existence of solutions of the following typical $-\Delta_{p(x),1}$ problem:

$$-\left(|u'|^{p(x)-2}u'\right)' = |u|^{q(x)-2}u + C, \quad x \in \Omega \subset \mathbb{R}^N,$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

because of the nonhomogeneity of $-\Delta_{p(x),1}$, if $\max_{x\in\overline{\Omega}}q(x) < \min_{x\in\overline{\Omega}}p(x)$, then the corresponding functional is coercive; if $\max_{x\in\overline{\Omega}}p(x) < \min_{x\in\overline{\Omega}}q(x)$, then the corresponding functional satisfies Palais-Smale condition (see [4, 7, 12]). If $\min_{x\in\overline{\Omega}}p(x) \leq q(x) \leq \max_{x\in\overline{\Omega}}p(x)$, we can see that the corresponding functional is neither coercive nor satisfying Palais-Smale conditions, the results on this case are rare.

There are many results on the existence of solutions for p-Laplacian equation with multipoint boundary value conditions (see [17–20]). On the existence of solutions for $p(x)$-Laplacian systems boundary value problems, we refer to [5, 7, 10, 11]. But results on the existence of solutions for weighted $p(t)$-Laplacian systems with multipoint boundary value conditions are rare. In this paper, when $p(t)$ is a general function, we investigate the existence of solutions for weighted $p(t)$-Laplacian systems with multipoint boundary value conditions. Moreover, the case of $\min_{t\in[0,1]}p(t) \leq q(t) \leq \max_{t\in[0,1]}p(t)$ has been discussed.

Let $N \geq 1$ and $I = [0,1]$, the function $f = (f^1, \ldots, f^N) : I \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be Caratheodory, by this we mean the following:

(i) for almost every $t \in I$ the function $f(t, \cdot, \cdot)$ is continuous;

(ii) for each $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ the function $f(\cdot, x, y)$ is measurable on I;

(iii) for each $R > 0$ there is a $\beta_R \in L^1(I, \mathbb{R})$, such that for almost every $t \in I$ and every $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|x| \leq R$, $|y| \leq R$, one has

$$\left|f(t, x, y)\right| \leq \beta_R(t). \tag{1.5}$$

Throughout the paper, we denote

$$w(0)|u'|^{p(0)-2}u'(0) = \lim_{r \to 0^+} w(r)|u'|^{p(r)-2}u'(r),$$

$$w(1)|u'|^{p(0)-2}u'(1) = \lim_{r \to 1^-} w(r)|u'|^{p(r)-2}u'(r). \tag{1.6}$$

The inner product in $\mathbb{R}^N$ will be denoted by $\langle \cdot, \cdot \rangle$, $|\cdot|$ will denote the absolute value and the Euclidean norm on $\mathbb{R}^N$. For $N \geq 1$, we set $C = C(I, \mathbb{R}^N)$, $C^1 = \{u \in C \mid u' \in C((0,1), \mathbb{R}^N)$, $\lim_{t \to 0^+} w(t)|u'|^{p(t)-2}u'(t)$, and $\lim_{t \to 1^-} w(t)|u'|^{p(t)-2}u'(t)$ exist$\}$. For any $u(t) = (u^1(t), \ldots, u^N(t))$, we denote $|u^i|_0 = \sup_{t \in (0,1)} |u^i(t)|$, $\|u\|_0 = (\sum_{i=1}^N |u^i|_0^2)^{1/2}$, and $\|u\|_1 = \|u\|_0 + \|(w(t))^{1/(p(t)-1)}u'\|_0$. Spaces C and C^1 will be equipped with the norm $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively. Then $(C, \|\cdot\|_0)$ and $(C^1, \|\cdot\|_1)$ are Banach spaces.

We say a function $u : I \to \mathbb{R}^N$ is a solution of (1.1) if $u \in C^1$ with $w(t)|u'|^{p(t)-2}u'$ absolutely continuous on $(0, 1)$, which satisfies (1.1) *a.e.* on I.

In this paper, we always use C_i to denote positive constants, if it cannot lead to confusion. Denote

$$z^- = \min_{t \in I} z(t), \quad z^+ = \max_{t \in I} z(t), \quad \text{for any } z \in C(I, \mathbb{R}). \tag{1.7}$$

We say f satisfies sub-$p^- - 1$ growth condition, if f satisfies

$$\lim_{|u|+|v| \to +\infty} \frac{f(t, u, v)}{(|u| + |v|)^{q(t)-1}} = 0, \quad \text{for } t \in I \text{ uniformly}, \tag{1.8}$$

where $q(t) \in C(I, \mathbb{R})$ and $1 < q^- \leq q^+ < p^-$. We say that f satisfies general growth condition, if we do not know whether f satisfies sub-$p^- - 1$ growth condition or not.

We will discuss the existence of solutions of (1.1)-(1.2) in the following two cases:

(i) f satisfies sub-$p^- - 1$ growth condition;

(ii) f satisfies general growth condition.

This paper is divided into four sections. In the second section, we will do some preparation. In the third section, we will discuss the existence of solutions of (1.1)-(1.2), when f satisfies sub-$p^- - 1$ growth condition. Finally, in Section 4, we will discuss the existence of solutions of (1.1)-(1.2), when f satisfies general growth condition.

2. Preliminary

For any $(t, x) \in I \times \mathbb{R}^N$, denote $\varphi(t, x) = |x|^{p(t)-2}x$. Obviously, φ has the following properties.

Lemma 2.1 (see [4]). *φ is a continuous function and satisfies the following:*

(i) *for any $t \in [0, 1]$, $\varphi(t, \cdot)$ is strictly monotone, that is,*

$$\langle \varphi(t, x_1) - \varphi(t, x_2), x_1 - x_2 \rangle > 0, \quad \text{for any } x_1, x_2 \in \mathbb{R}^N, \ x_1 \neq x_2; \tag{2.1}$$

(ii) *there exists a function* $\rho : [0, +\infty) \rightarrow [0, +\infty)$, $\rho(s) \rightarrow +\infty$ *as* $s \rightarrow +\infty$, *such that*

$$\langle \varphi(t, x), x \rangle \geq \rho(|x|)|x|, \quad \forall x \in \mathbb{R}^N. \tag{2.2}$$

It is well known that $\varphi(t, \cdot)$ *is a homeomorphism from* $\mathbb{R}^N$ *to* $\mathbb{R}^N$ *for any fixed* $t \in [0, 1]$. *For any* $t \in I$, *denote by* $\varphi^{-1}(t, \cdot)$ *the inverse operator of* $\varphi(t, \cdot)$, *then*

$$\varphi^{-1}(t, x) = |x|^{(2 - p(t))/(p(t) - 1)} x, \quad for\ x \in \mathbb{R}^N \setminus \{0\},\ \varphi^{-1}(t, 0) = 0. \tag{2.3}$$

It is clear that $\varphi^{-1}(t, \cdot)$ is continuous and sends bounded sets to bounded sets. Let us now consider the following problem with boundary value condition (1.2):

$$(w(t)\varphi(t, u'(t)))' = g(t), \tag{2.4}$$

where $g \in L^1$. If u is a solution of (2.4) with (1.2), by integrating (2.4) from 0 to t, we find that

$$w(t)\varphi(t, u'(t)) = w(0)\varphi(0, u'(0)) + \int_0^t g(s)\, ds. \tag{2.5}$$

Denote $a = w(0)\varphi(0, u'(0))$. It is easy to see that a is dependent on $g(t)$. Define operator $F : L^1 \rightarrow C$ as $F(g)(t) = \int_0^t g(s)\, ds$. By solving for u' in (2.5) and integrating, we find

$$u(t) = u(0) + F\{\varphi^{-1}[t, (w(t))^{-1}(a + F(g))]\}(t). \tag{2.6}$$

From $u(0) = \sum_{i=1}^{m-2} \beta_i u(\eta_i) + e_0$, we have

$$u(0) = \frac{\sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} \varphi^{-1}(t, (w(t))^{-1}[a + F(g)(t)])\, dt + e_0}{1 - \sum_{i=1}^{m-2} \beta_i}. \tag{2.7}$$

From $u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i) + e_1$, we obtain

$$u(0) = \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \varphi^{-1}\{t, (w(t))^{-1}[a + F(g)(t)]\}\, dt - \int_0^1 \varphi^{-1}\{t, (w(t))^{-1}[a + F(g)(t)]\}\, dt + e_1}{1 - \sum_{i=1}^{m-2} \alpha_i}. \tag{2.8}$$

From (2.7) and (2.8), we have

$$\frac{\sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} \varphi^{-1}\{t, (w(t))^{-1}[a + F(g)(t)]\}\, dt + e_0}{1 - \sum_{i=1}^{m-2} \beta_i}$$

$$= \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \varphi^{-1}\{t, (w(t))^{-1}[a + F(g)(t)]\}\, dt - \int_0^1 \varphi^{-1}\{t, (w(t))^{-1}[a + F(g)(t)]\}\, dt + e_1}{1 - \sum_{i=1}^{m-2} \alpha_i}. \tag{2.9}$$

For fixed $h \in C$, we denote

$$
\Lambda_h(a) = \frac{\sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} \varphi^{-1} \{ t, (w(t))^{-1} [a + h(t)] \} \, dt + e_0}{1 - \sum_{i=1}^{m-2} \beta_i}
$$

$$
- \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \varphi^{-1} \{ t, (w(t))^{-1} [a + h(t)] \} \, dt - \int_0^1 \varphi^{-1} \{ t, (w(t))^{-1} [a + h(t)] \} \, dt + e_1}{1 - \sum_{i=1}^{m-2} \alpha_i}.
$$

(2.10)

Throughout the paper, we denote $E = \int_0^1 (w(t))^{-1/(p(t)-1)} \, dt$.

Lemma 2.2. *The function $\Lambda_h(\cdot)$ has the following properties:*

(i) *for any fixed $h \in C$, the equation*

$$
\Lambda_h(a) = 0 \tag{2.11}
$$

has a unique solution $\tilde{a}(h) \in \mathbb{R}^N$;

(ii) *the function $\tilde{a} : C \to \mathbb{R}^N$, defined in (i), is continuous and sends bounded sets to bounded sets. Moreover,*

$$
|\tilde{a}(h)| \le 3N \left[\frac{(E+1)}{(1 - \sum_{i=1}^{m-2} \beta_i) E} + \frac{(E+1)}{(1 - \sum_{i=1}^{m-2} \alpha_i) E} + 1 \right]^{p^+ - 1} \cdot \left[\|h\|_0 + (2N)^{p^+} \left(|e_0| + |e_1| \right)^{p^\# - 1} \right],
$$

(2.12)

where the notation $M^{p^\# - 1}$ means

$$
M^{p^\# - 1} = \begin{cases} M^{p^+ - 1}, & M > 1 \\ M^{p^- - 1}, & M \le 1. \end{cases} \tag{2.13}
$$

Proof. (i) It is easy to see that

$$
\Lambda_h(a) = \frac{\sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} \varphi^{-1} \{ t, (w(t))^{-1} [a + h(t)] \} \, dt + e_0}{1 - \sum_{i=1}^{m-2} \beta_i}
$$

$$
+ \frac{\sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^1 \varphi^{-1} \{ t, (w(t))^{-1} [a + h(t)] \} \, dt - e_1}{1 - \sum_{i=1}^{m-2} \alpha_i}
$$

(2.14)

$$
+ \int_0^1 \varphi^{-1} \{ t, (w(t))^{-1} [a + h(t)] \} \, dt.
$$

From Lemma 2.1, it is immediate that

$$\langle \Lambda_h(a_1) - \Lambda_h(a_2), a_1 - a_2 \rangle > 0, \quad \text{for } a_1 \neq a_2, \tag{2.15}$$

and hence, if (2.11) has a solution, then it is unique.

Let

$$t_0 = 3N \left[\frac{(E+1)}{(1 - \sum_{i=1}^{m-2} \beta_i)E} + \frac{(E+1)}{(1 - \sum_{i=1}^{m-2} \alpha_i)E} + 1 \right]^{p^+ - 1} \cdot \left[\|h\|_0 + (2N)^{p^+} \left(|e_0| + |e_1| \right)^{p^\# - 1} \right]. \tag{2.16}$$

If $|a| \geq t_0$, since $(w(t))^{-1/(p(t)-1)} \in L^1(0,1)$ and $h \in C$, it is easy to see that there exists an $i \in \{1, \dots, N\}$ such that the ith component a^i of a satisfies

$$|a^i| \geq 3 \left[\frac{(E+1)}{(1 - \sum_{i=1}^{m-2} \beta_i)E} + \frac{(E+1)}{(1 - \sum_{i=1}^{m-2} \alpha_i)E} + 1 \right]^{p^+ - 1} \cdot \left[\|h\|_0 + (2N)^{p^+} \left(|e_0| + |e_1| \right)^{p^\# - 1} \right]. \tag{2.17}$$

Thus $(a^i + h^i(t))$ keeps sign on I and

$$|a^i + h^i(t)| \geq |a^i| - \|h\|_0$$

$$\geq 2 \left[\frac{(E+1)}{(1 - \sum_{i=1}^{m-2} \beta_i)E} + \frac{(E+1)}{(1 - \sum_{i=1}^{m-2} \alpha_i)E} + 1 \right]^{p^+ - 1} \cdot \left[\|h\|_0 + (2N)^{p^+} \left(|e_0| + |e_1| \right)^{p^\# - 1} \right], \quad \forall t \in I, \tag{2.18}$$

then

$$|a^i + h^i(t)|^{1/(p(t)-1)} \geq 2^{1/(p^+ - 1)} \left[\frac{(E+1)}{(1 - \sum_{i=1}^{m-2} \beta_i)E} + \frac{(E+1)}{(1 - \sum_{i=1}^{m-2} \alpha_i)E} + 1 \right] \left(|e_0| + |e_1| \right), \quad \forall t \in I. \tag{2.19}$$

Thus, when $|a|$ is large enough, the ith component $\Lambda_h^i(a)$ of $\Lambda_h(a)$ is nonzero, then we have

$$\Lambda_h(a) \neq 0. \tag{2.20}$$

Let us consider the equation

$$\lambda \Lambda_h(a) + (1 - \lambda)a = 0, \quad \lambda \in [0, 1]. \tag{2.21}$$

It is easy to see that all the solutions of (2.21) belong to $b(t_0) = \{x \in \mathbb{R}^N \mid |x| < t_0\}$. So, we have

$$d_B\left[\Lambda_h(a), b(t_0), 0\right] = d_B\left[I, b(t_0), 0\right] \neq 0, \tag{2.22}$$

it means the existence of solutions of $\Lambda_h(a) = 0$.

In this way, we define a function $\tilde{a}(h) : C[0, 1] \to \mathbb{R}^N$, which satisfies

$$\Lambda_h\left(\tilde{a}(h)\right) = 0. \tag{2.23}$$

(ii) By the proof of (i), we also obtain that $\tilde{a}$ sends bounded sets to bounded sets, and

$$\left|\tilde{a}(h)\right| \leq 3N \left[\frac{(E+1)}{\sum_{i=1}^{m-2}\beta_i E} + \frac{(E+1)}{\sum_{i=1}^{m-2}\alpha_i E} + 1\right]^{p^+ - 1} \cdot \left[\|h\|_0 + \left(|e_0| + |e_1|\right)^{p^\# - 1}\right]. \tag{2.24}$$

It only remains to prove the continuity of $\tilde{a}$. Let $\{u_n\}$ be a convergent sequence in C and $u_n \to u$ as $n \to +\infty$. Since $\{\tilde{a}(u_n)\}$ is a bounded sequence, then it contains a convergent subsequence $\{\tilde{a}(u_{n_j})\}$. Let $\tilde{a}(u_{n_j}) \to a_0$ as $j \to +\infty$. Since $\Lambda_{u_{n_j}}(\tilde{a}(u_{n_j})) = 0$, letting $j \to +\infty$, we have $\Lambda_u(a_0) = 0$. From (i), we get $a_0 = \tilde{a}(u)$, it means that $\tilde{a}$ is continuous. This completes the proof. □

Now, we define $a : L^1 \to \mathbb{R}^N$ as

$$a(u) = \tilde{a}\left(F(u)\right). \tag{2.25}$$

It is clear that $a(\cdot)$ is continuous and sends bounded sets of L^1 to bounded sets of $\mathbb{R}^N$, and hence it is a complete continuous mapping.

If u is a solution of (2.4) with (1.2), then

$$u(t) = u(0) + F\left\{\varphi^{-1}\left[t, \left(w(t)\right)^{-1}\left(a(g) + F(g)(t)\right)\right]\right\}(t), \quad \forall t \in [0, 1]. \tag{2.26}$$

The boundary condition (1.2) implies that

$$u(0) = \frac{\sum_{i=1}^{m-2}\beta_i \int_0^{\eta_i}\varphi^{-1}\left\{t, \left(w(t)\right)^{-1}\left[a(g) + F(g)(t)\right]\right\} dt + e_0}{1 - \sum_{i=1}^{m-2}\beta_i}. \tag{2.27}$$

We denote that

$$K_1(h)(t) := \left(K_1 \circ h\right)(t) = F\left\{\varphi^{-1}\left[t, \left(w(t)\right)^{-1}\left(a(h) + F(h)\right)\right]\right\}(t), \quad \forall t \in [0, 1]. \tag{2.28}$$

Lemma 2.3. *The operator K_1 is continuous and sends equi-integrable sets in L^1 to relatively compact sets in C^1.*

Proof. It is easy to check that $K_1(h)(t) \in C^1$. Since $(w(t))^{-1/(p(t)-1)} \in L^1$ and

$$K_1(h)'(t) = \varphi^{-1}[t, (w(t))^{-1}(a(h) + F(h))], \quad \forall t \in [0, 1], \tag{2.29}$$

it is easy to check that K_1 is a continuous operator from L^1 to C^1.

Let now U be an equi-integrable set in L^1, then there exists $\rho_* \in L^1$, such that

$$|u(t)| \le \rho_*(t) \quad \text{a.e. in } I, \text{ for any } u \in U. \tag{2.30}$$

We want to show that $\overline{K_1(U)} \subset C^1$ is a compact set.

Let $\{u_n\}$ be a sequence in $K_1(U)$, then there exists a sequence $\{h_n\} \in U$ such that $u_n = K_1(h_n)$. For any $t_1, t_2 \in I$, we have

$$|F(h_n)(t_1) - F(h_n)(t_2)| = \left| \int_0^{t_1} h_n(t)\, dt - \int_0^{t_2} h_n(t)\, dt \right| = \left| \int_{t_1}^{t_2} h_n(t)\, dt \right| \le \left| \int_{t_1}^{t_2} \rho_*(t)\, dt \right|. \tag{2.31}$$

Hence the sequence $\{F(h_n)\}$ is uniformly bounded and equicontinuous. By Ascoli-Arzela theorem, there exists a subsequence of $\{F(h_n)\}$ (which we rename the same) convergent in C. According to the bounded continuous operator a, we can choose a subsequence of $\{a(h_n) + F(h_n)\}$ (which we still denote $\{a(h_n) + F(h_n)\}$) which is convergent in C, then $w(t)\varphi(t, K_1(h_n)'(t)) = a(h_n) + F(h_n)$ is convergent in C.

Since

$$K_1(h_n)(t) = F\left\{ \varphi^{-1}[t, (w(t))^{-1}(a(h_n) + F(h_n))] \right\}(t), \quad \forall t \in [0, 1], \tag{2.32}$$

according to the continuity of φ^{-1} and the integrability of $w(t)^{-1/(p(t)-1)}$ in L^1, we can see that $K_1(h_n)$ is convergent in C. Thus $\{u_n\}$ is convergent in C^1. This completes the proof. □

Let us define $P : C^1 \to C^1$ as

$$P(h) = \frac{\sum_{i=1}^{m-2} \beta_i (K_1 \circ h)(\eta_i) + e_0}{1 - \sum_{i=1}^{m-2} \beta_i}. \tag{2.33}$$

It is easy to see that P is compact continuous.

We denote $N_f(u) : [0, 1] \times C^1 \to L^1$ the Nemytski operator associated to f defined by

$$N_f(u)(t) = f(t, u(t), (w(t))^{1/(p(t)-1)} u'(t)), \quad \text{a.e. on } I. \tag{2.34}$$

Lemma 2.4. *u is a solution of* (1.1)-(1.2) *if and only if u is a solution of the following abstract equation:*

$$u = P(\delta N_f(u)) + K_1(\delta N_f(u)). \tag{2.35}$$

Proof. If u is a solution of (1.1)-(1.2), by integrating (1.1) from 0 to t, we find that

$$w(t)\varphi(t, u'(t)) = a(\delta N_f(u)) + F(\delta N_f(u))(t). \tag{2.36}$$

From (2.36), we have

$$u(t) = u(0) + F\{\varphi^{-1}[r, (w(r))^{-1}(a(\delta N_f(u)) + F(\delta N_f(u)))]\}(t),$$

$$u(0) = \sum_{i=1}^{m-2}\beta_i[u(0) + F\{\varphi^{-1}[r, (w(r))^{-1}(a(\delta N_f(u)) + F(\delta N_f(u)))]\}(\eta_i)] + e_0, \tag{2.37}$$

then we have

$$u(0) = \frac{\sum_{i=1}^{m-2}\beta_i F\{\varphi^{-1}[r, (w(r))^{-1}(a(\delta N_f(u)) + F(\delta N_f(u)))]\}(\eta_i) + e_0}{1 - \sum_{i=1}^{m-2}\beta_i}$$

$$= \frac{\sum_{i=1}^{m-2}\beta_i K_1(\delta N_f(u))(\eta_i) + e_0}{1 - \sum_{i=1}^{m-2}\beta_i} = P(\delta N_f(u)). \tag{2.38}$$

So we have

$$u = P(\delta N_f(u)) + K_1(\delta N_f(u)). \tag{2.39}$$

Conversely, if u is a solution of (2.35), it is easy to see that

$$u(0) = P(\delta N_f(u)) = \frac{\sum_{i=1}^{m-2}\beta_i K_1(\delta N_f(u))(\eta_i) + e_0}{1 - \sum_{i=1}^{m-2}\beta_i},$$

$$u(0) = \sum_{i=1}^{m-2}\beta_i[u(0) + K_1(\delta N_f(u))(\eta_i)] + e_0 = \sum_{i=1}^{m-2}\beta_i u(\eta_i) + e_0, \tag{2.40}$$

$$u(1) = P(\delta N_f(u)) + K_1(\delta N_f(u))(1).$$

By the condition of the mapping a,

$$u(0) = \frac{\sum_{i=1}^{m-2}\beta_i K_1(\delta N_f(u))(\eta_i) + e_0}{1 - \sum_{i=1}^{m-2}\beta_i} = \frac{\sum_{i=1}^{m-2}\alpha_i K_1(\delta N_f(u))(\xi_i) - K_1(\delta N_f(u))(1) + e_1}{1 - \sum_{i=1}^{m-2}\alpha_i}, \tag{2.41}$$

then we have

$$u(1) = \frac{\sum_{i=1}^{m-2}\alpha_i K_1(\delta N_f(u))(\xi_i) - K_1(\delta N_f(u))(1) + e_1}{1 - \sum_{i=1}^{m-2}\alpha_i} + K_1(\delta N_f(u))(1), \tag{2.42}$$

thus

$$
\begin{aligned}
u(1) &= \sum_{i=1}^{m-2} \alpha_i \left[u(1) - K_1 (\delta N_f(u))(1) + K_1 (\delta N_f(u))(\xi_i) \right] + e_1 \\
&= \sum_{i=1}^{m-2} \alpha_i \left[P(\delta N_f(u)) + K_1 (\delta N_f(u))(\xi_i) \right] + e_1 = \sum_{i=1}^{m-2} \alpha_i u(\xi_i) + e_1,
\end{aligned}
\tag{2.43}
$$

from (2.40) and (2.43), we obtain (1.2).

From (2.35), we have

$$
\begin{aligned}
u'(t) &= \varphi^{-1} \left[t, (w(t))^{-1} (a + F(\delta N_f(u))) \right], \\
(w(t)\varphi(t, u'))' &= \delta N_f(u)(t).
\end{aligned}
\tag{2.44}
$$

Hence u is a solution of (1.1)-(1.2). This completes the proof. □

Lemma 2.5. *If u is a solution of (1.1)-(1.2), then for any $j = 1, \ldots, N$, there exists a $\varsigma^j \in (0,1)$, such that*

$$
\left| (u^j)'(\varsigma^j) \right| \le C_* := \left(\frac{|e_0|}{1 - \sum_{i=1}^{m-2} \beta_i} + \frac{|e_1|}{1 - \sum_{i=1}^{m-2} \alpha_i} \right).
\tag{2.45}
$$

Proof. For any $j = 1, \ldots, N$, if there exists $\varsigma^j \in (0,1)$ such that $(u^j)'(\varsigma^j) = 0$, then (2.45) is valid. If it is false, then u^j is strictly monotone.

(i) If u^j is strictly decreasing in $[0,1]$, then

$$
u^j(0) > u^j(\xi_i) > u^j(1), \quad u^j(0) > u^j(\eta_i) > u^j(1), \quad i = 1, \ldots, m - 2.
\tag{2.46}
$$

Thus

$$
\begin{aligned}
u^j(0) &= \sum_{i=1}^{m-2} \beta_i u^j(\eta_i) + e_0^j < \sum_{i=1}^{m-2} \beta_i u^j(0) + e_0^j, \\
u^j(1) &= \sum_{i=1}^{m-2} \alpha_i u^j(\xi_i) + e_1^j > \sum_{i=1}^{m-2} \alpha_i u^j(1) + e_1^j,
\end{aligned}
\tag{2.47}
$$

it means that

$$
\frac{e_0^j}{1 - \sum_{i=1}^{m-2} \beta_i} > u^j(0) > u^j(1) > \frac{e_1^j}{1 - \sum_{i=1}^{m-2} \alpha_i},
\tag{2.48}
$$

then there exists a $t^j \in (0,1)$ such that

$$0 > (u^j)'(t^j) = \frac{u^j(1) - u^j(0)}{1 - 0} > -\left(\frac{|e_1^j|}{1 - \sum_{i=1}^{m-2}\alpha_i} + \frac{|e_0^j|}{1 - \sum_{i=1}^{m-2}\beta_i}\right). \tag{2.49}$$

(ii) If u^j is strictly increasing in $[0,1]$, then

$$u^j(0) < u^j(\xi_i) < u^j(1), \quad u^j(0) < u^j(\eta_i) < u^j(1), \quad i = 1,\dots,m-2. \tag{2.50}$$

Thus

$$\begin{aligned}
u^j(0) &= \sum_{i=1}^{m-2}\beta_i u^j(\eta_i) + e_0^j > \sum_{i=1}^{m-2}\beta_i u^j(0) + e_0^j, \\
u^j(1) &= \sum_{i=1}^{m-2}\alpha_i u^j(\xi_i) + e_1^j < \sum_{i=1}^{m-2}\alpha_i u^j(1) + e_1^j,
\end{aligned} \tag{2.51}$$

it means that

$$\frac{e_0^j}{1 - \sum_{i=1}^{m-2}\beta_i} < u^j(0) < u^j(1) < \frac{e_1^j}{1 - \sum_{i=1}^{m-2}\alpha_i}, \tag{2.52}$$

then there exists a $r^j \in (0,1)$ such that

$$0 < (u^j)'(r^j) = \frac{u^j(1) - u^j(0)}{1 - 0} < \frac{|e_1^j|}{1 - \sum_{i=1}^{m-2}\alpha_i} + \frac{|e_0^j|}{1 - \sum_{i=1}^{m-2}\beta_i}. \tag{2.53}$$

Combining (2.49) and (2.53), then we obtain (2.45).
This completes the proof.   □

3. f satisfies sub-$p^- - 1$ growth condition

In this section, we will apply Leray-Schauder's degree to deal with the existence of solutions for (1.1)-(1.2), when f satisfies sub-$p^- - 1$ growth condition.

Theorem 3.1. *If f satisfies sub-$p^- - 1$ growth condition, then for any fixed parameter δ, problem (1.1)-(1.2) has at least one solution.*

Proof. Denote $\Psi_f(u,\lambda) := P(\lambda\delta N_f(u)) + K_1(\lambda\delta N_f(u))$, where $N_f(u)$ is defined in (2.34). We know that (1.1)-(1.2) has the same solution of

$$u = \Psi_f(u,\lambda), \tag{3.1}$$

when $\lambda = 1$.

It is easy to see that the operator P is compact continuous. According to Lemmas 2.2 and 2.3, then we can see that $\Psi_f(\cdot, \lambda)$ is compact continuous from C^1 to C^1 for any $\lambda \in [0, 1]$.

We claim that all the solutions of (3.1) are uniformly bounded for $\lambda \in [0, 1]$. In fact, if it is false, we can find a sequence of solutions $\{(u_n, \lambda_n)\}$ for (3.1) such that $\|u_n\|_1 \to +\infty$ as $n \to +\infty$, and $\|u_n\|_1 > 1$ for any $n = 1, 2, \ldots$.

Let $t_n \in (0, 1)$ such that

$$\frac{1}{2} \sup_{t \in (0,1)} w(t) |u_n'(t)|^{p(t)-1} \leq w(t_n) |u_n'(t_n)|^{p(t_n)-1}, \quad n = 1, 2, \ldots . \tag{3.2}$$

For any fixed $n = 1, 2, \ldots$, there exists an $i_n \in \{1, \ldots, N\}$ such that

$$\left| \left(u_n^{i_n}\right)'(t_n) \right| \geq \frac{1}{N} |u_n'(t_n)|. \tag{3.3}$$

Thus, $\{u_n^{i_n}\}$ becomes a sequence with respect to n.

Since (u_n, λ_n) are solutions of (3.1), according to Lemma 2.5, for any $n = 1, 2, \ldots$, there exists $\xi_n^{i_n} \in (0, 1)$ such that $|(u_n^{i_n})'(\xi_n^{i_n})| \leq C_*$, then

$$w(t)|u_n'|^{p(t)-2}\left(u_n^{i_n}\right)'(t) = w\left(\xi_n^{i_n}\right)|u_n'\left(\xi_n^{i_n}\right)|^{p(\xi_n^{i_n})-2}\left(u_n^{i_n}\right)'\left(\xi_n^{i_n}\right)$$

$$+ \int_{\xi_n^{i_n}}^t \|u_n\|_1^{q(r)-1} \frac{\lambda_n \delta f^{i_n}\left[r, u_n, (w(r))^{1/(p(r)-1)} u_n'\right]}{\|u_n\|_1^{q(r)-1}} dr, \quad \forall t \in (0, 1). \tag{3.4}$$

For any $t \in (0, 1)$, we have

$$w(t)|u_n'|^{p(t)-2}\left|\left(u_n^{i_n}\right)'(t)\right| \leq w\left(\xi_n^{i_n}\right)|u_n'\left(\xi_n^{i_n}\right)|^{p(\xi_n^{i_n})-2}\left|\left(u_n^{i_n}\right)'\left(\xi_n^{i_n}\right)\right|$$

$$+ \left|\int_{\xi_n^{i_n}}^t \|u_n\|_1^{q(r)-1} \frac{\lambda_n \delta f^{i_n}\left[r, u_n, (w(r))^{1/(p(r)-1)} u_n'\right]}{\|u_n\|_1^{q(r)-1}} dr\right|. \tag{3.5}$$

Without loss of generality, we assume that $|u_n'(\xi_n^{i_n})| > 0$.

(1°) If $p(\xi_n^{i_n}) - 2 \leq 0$, then

$$w\left(\xi_n^{i_n}\right)|u_n'\left(\xi_n^{i_n}\right)|^{(p(\xi_n^{i_n})-2)}\left|\left(u_n^{i_n}\right)'\left(\xi_n^{i_n}\right)\right| = w\left(\xi_n^{i_n}\right)\frac{\left|\left(u_n^{i_n}\right)'\left(\xi_n^{i_n}\right)\right|}{|u_n'\left(\xi_n^{i_n}\right)|^{(2-p(\xi_n^{i_n}))}} \leq w\left(\xi_n^{i_n}\right)\frac{\left|\left(u_n^{i_n}\right)'\left(\xi_n^{i_n}\right)\right|}{\left|\left(u_n^{i_n}\right)'\left(\xi_n^{i_n}\right)\right|^{(2-p(\xi_n^{i_n}))}}$$

$$= w\left(\xi_n^{i_n}\right)\left|\left(u_n^{i_n}\right)'\left(\xi_n^{i_n}\right)\right|^{(p(\xi_n^{i_n})-1)} \leq w\left(\xi_n^{i_n}\right)C_*^{(p(\xi_n^{i_n})-1)}, \tag{3.6}$$

where C_* is defined in (2.45).

Combining (3.2), (3.3), (3.5), and (3.6), we have

$$
\frac{1}{2N} \sup_{t \in (0,1)} w(t) |u'_n|^{p(t)-1} \le \frac{1}{N} w(t_n) |u'_n(t_n)|^{p(t_n)-1} \le w(t_n) |u'_n(t_n)|^{p(t_n)-2} |(u_n^{i_n})'(t_n)|
$$

$$
\le w(\xi_n^{i_n}) C_*^{(p(\xi_n^{i_n})-1)} + \left| \int_{\xi_n^{i_n}}^{t_n} \|u_n\|_1^{q(r)-1} \frac{\lambda_n \delta f^{i_n} \left[r, u_n, (w(r))^{1/(p(r)-1)} u'_n \right]}{\|u_n\|_1^{q(r)-1}} dr \right|
$$

$$
\le C_1 + C_2 \|u_n\|_1^{q^+ - 1}.
$$

$$(3.7)$$

Then we have

$$
w(t) |u'_n(t)|^{p(t)-1} \le 2N \left(C_1 + C_2 \|u_n\|_1^{q^+ - 1} \right), \quad \forall t \in (0,1).
$$

$$(3.8)$$

Denoting $\alpha = (q^+ - 1)/(p^- - 1)$, we have

$$
\sup_{t \in (0,1)} \left| (w(t))^{1/(p(t)-1)} u'_n(t) \right| \le C_3 \|u_n\|_1^{\alpha}.
$$

$$(3.9)$$

Thus

$$
\left\| (w(t))^{1/(p(t)-1)} u'_n(t) \right\|_0 \le N C_3 \|u_n\|_1^{\alpha}.
$$

$$(3.10)$$

(2°) If $p(\xi_n^{i_n}) - 2 > 0$, since $|(u_n^{i_n})'(\xi_n^{i_n})| \le C_*$, we have

$$
w(\xi_n^{i_n}) |u'_n(\xi_n^{i_n})|^{(p(\xi_n^{i_n})-2)} |(u_n^{i_n})'(\xi_n^{i_n})|
$$

$$
= (w(\xi_n^{i_n}))^{1/(p(\xi_n^{i_n})-1)} |(u_n^{i_n})'(\xi_n^{i_n})| \left[w(\xi_n^{i_n}) |u'_n(\xi_n^{i_n})|^{(p(\xi_n^{i_n})-1)} \right]^{(p(\xi_n^{i_n})-2)/(p(\xi_n^{i_n})-1)}
$$

$$(3.11)$$

$$
\le C_4 \left[\left(\sup_{t \in (0,1)} w(t) |u'_n(t)|^{(p(t)-1)} \right)^{\epsilon} + 1 \right], \quad \text{where } \epsilon = \frac{p^+ - 2}{p^+ - 1}.
$$

According to (3.2), (3.3), (3.5), and (3.11), we have

$$
\frac{1}{2N} \sup_{t \in (0,1)} w(t) |u'_n(t)|^{p(t)-1} \le w(t_n) |u'_n(t_n)|^{(p(t_n)-2)} |(u_n^{i_n})'(t_n)|
$$

$$(3.12)$$

$$
\le C_4 \left[\left(\sup_{t \in (0,1)} w(t) |u'_n(t)|^{p(t)-1} \right)^{\epsilon} + 1 \right] + C_2 \|u_n\|_1^{q^+ - 1}.
$$

Since $\epsilon < 1$ is a positive constant, (3.12) means that

$$\sup_{t\in(0,1)} w(t)\big|u'_n(t)\big|^{p(t)-1} \leq C_5\|u_n\|_1^{q^+-1}. \tag{3.13}$$

Thus

$$\big\|(w(t))^{1/(p(t)-1)}u'_n(t)\big\|_0 \leq NC_6\|u_n\|_1^{\alpha}. \tag{3.14}$$

Summarizing this argument, we have

$$\big\|(w(t))^{1/(p(t)-1)}u'_n(t)\big\|_0 \leq C_7\|u_n\|_1^{\alpha}. \tag{3.15}$$

Since $|\tilde{a}(h)| \leq C_8[\|F(h)\|_0 + (|e_0| + |e_1|)^{p^\#-1}]$, then we have

$$\big|a(\delta N_f)\big| \leq C_8\big[\|F(N_f)\|_0 + (|e_0| + |e_1|)^{p^\#-1}\big] \leq C_9\|u\|_1^{q^+-1} + 1. \tag{3.16}$$

Thus

$$\big|a(\delta N_f(u_n)) + F(\delta N_f(u_n))\big| \leq \big|a(\delta N_f(u_n))\big| + \big|F(\delta N_f(u_n))\big| \leq C_{10}\|u_n\|_1^{q^+-1}. \tag{3.17}$$

Combining (2.38) and (3.17), we have

$$\big|u_n(0)\big| \leq C_{11}\|u_n\|_1^{\alpha}, \quad \text{where } \alpha = \frac{q^+-1}{p^--1}. \tag{3.18}$$

For any $j = 1,\ldots,N$, since

$$\begin{aligned}
\big|u_n^j(t)\big| &= \left|u_n^j(0) + \int_0^t (u_n^j)'(r)\,dr\right| \\
&\leq \big|u_n^j(0)\big| + \int_0^t (w(r))^{-1/(p(r)-1)} \sup_{t\in(0,1)}\big|(w(t))^{1/(p(t)-1)}(u_n^j)'(t)\big|\,dr \\
&\leq \|u_n\|_1^{\alpha}[C_{11} + C_7 E],
\end{aligned} \tag{3.19}$$

we have

$$\big|u_n^j\big|_0 \leq C_{12}\|u_n\|_1^{\alpha}, \quad j = 1,\ldots,N; \ n = 1,2,\ldots. \tag{3.20}$$

Thus

$$\|u_n\|_0 \leq NC_{12}\|u_n\|_1^{\alpha}, \quad n = 1,2,\ldots. \tag{3.21}$$

Combining (3.15) and (3.21), then we obtain that $\{\|u_n\|_1\}$ is bounded.

Thus, there exists a large enough $R_0 > 0$ such that all the solutions of (3.1) belong to $B(R_0) = \{u \in C^1 \mid \|u\|_1 < R_0\}$, then the Leray-Schauder degree $d_{\mathrm{LS}}[I - \Psi_f(\cdot, \lambda), B(R_0), 0]$ is well defined for $\lambda \in [0, 1]$, and

$$d_{\mathrm{LS}}[I - \Psi_f(\cdot, 1), B(R_0), 0] = d_{\mathrm{LS}}[I - \Psi_f(\cdot, 0), B(R_0), 0]. \tag{3.22}$$

Let

$$u_0 = \frac{\sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} \varphi^{-1}\left[t, (w(t))^{-1} a(0)\right] dt + e_0}{1 - \sum_{i=1}^{m-2} \beta_i} + \int_0^r \varphi^{-1}\left[t, (w(t))^{-1} a(0)\right] dt, \tag{3.23}$$

where $a(0)$ is defined in (2.25), thus u_0 is the unique solution of $u = \Psi_f(u, 0)$.

It is easy to see that u is a solution of $u = \Psi_f(u, 0)$ if and only if u is a solution of the following:

$$(I) \begin{cases} -\Delta_{p(t), w(t)} u = 0, & t \in (0, 1), \\ u(0) = \sum_{i=1}^{m-2} \beta_i u(\eta_i) + e_0, & u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i) + e_1. \end{cases} \tag{3.24}$$

Obviously, system (I) possesses only one solution u_0. Since $u_0 \in B(R_0)$, thus the Leray-Schauder degree

$$d_{\mathrm{LS}}[I - \Psi_f(\cdot, 1), B(R_0), 0] = d_{\mathrm{LS}}[I - \Psi_f(\cdot, 0), B(R_0), 0] = 1 \neq 0, \tag{3.25}$$

therefore, we obtain that (1.1)-(1.2) has at least one solution. This completes the proof. □

4. f satisfies general growth condition

In the following, we will deal with the existence of solutions for $p(t)$-Laplacian ordinary system, when f satisfies general growth condition.

Denote

$$\Omega_\varepsilon = \left\{ u \in C^1 \mid \max_{1 \leq i \leq N} \left(|u^i|_0 + |(w(t))^{1/(p(t)-1)}(u^i)'|_0\right) < \varepsilon \right\}, \qquad \theta = \frac{\varepsilon}{2 + 1/E}. \tag{4.1}$$

Assumption 4.1. Let positive constant ε satisfy $u_0 \in \Omega_\varepsilon$, $|P(0)| < \theta$, and $|a(0)| < (1/N(2E + 1))\min_{t \in I}|\theta/E|^{p(t)-1}$, where u_0 is defined in (3.23), $a(\cdot)$ is defined in (2.25).

It is easy to see that Ω_ε is an open bounded domain in C^1.

Theorem 4.2. *Assume that Assumption 4.1 is satisfied. If positive parameter δ is small enough, then the problem (1.1)-(1.2) has at least one solution on $\overline{\Omega_\varepsilon}$.*

Proof. Denote $\Psi_f(u, \lambda) = P(\lambda \delta N_f(u)) + K_1(\lambda \delta N_f(u))$. According to Lemma 2.4, u is a solution of

$$-\Delta_{p(t), w(t)} u + \lambda \delta f\left(t, u, (w(t))^{1/(p(t)-1)} u'\right) = 0, \quad t \in (0, 1), \tag{4.2}$$

with (1.2) if and only if u is a solution of the following abstract equation:

$$u = \Psi_f(u, \lambda). \tag{4.3}$$

From Lemmas 2.2 and 2.3, then we can see that $\Psi_f(\cdot, \lambda)$ is compact continuous from C^1 to C^1 for any $\lambda \in [0, 1]$. According to Leray-Schauder degree theory, we only need to prove that

(1°) $u = \Psi_f(u, \lambda)$ has no solution on $\partial \Omega_\varepsilon$ for any $\lambda \in [0, 1)$,

(2°) $d_{\mathrm{LS}}[I - \Psi_f(\cdot, 0), \Omega_\varepsilon, 0] \neq 0$.

Then we can conclude that the system (1.1)-(1.2) has a solution on $\overline{\Omega_\varepsilon}$.

(1°) If it is false, then there exists a $\lambda \in [0, 1)$ and $u \in \partial \Omega_\varepsilon$ is a solution of (4.2) with (1.2). Thus (u, λ) satisfies

$$w(t)\varphi(t, u'(t)) = a(\lambda \delta N_f) + \lambda \delta F(N_f)(t). \tag{4.4}$$

Since $u \in \partial \Omega_\varepsilon$, then there exists an i such that $|u^i|_0 + |(w(t))^{1/(p(t)-1)}(u^i)'|_0 = \varepsilon$.

(i) Suppose that $|u^i|_0 \geq 2\theta$, then $|(w(t))^{1/(p(t)-1)}(u^i)'|_0 \leq \varepsilon - 2\theta = \theta/E$. On the other hand, for any $t, t' \in I$, we have

$$\left| u^i(t) - u^i(t') \right| = \left| \int_{t'}^{t} (u^i)'(r)\, dr \right| \leq \int_0^1 (w(r))^{-1/(p(r)-1)} \left| (w(r))^{1/(p(r)-1)} (u^i)'(r) \right| dr \leq \theta. \tag{4.5}$$

This implies that $|u^i(t)| \geq \theta$ for each $t \in I$.

Notice that $u \in \overline{\Omega_\varepsilon}$, then $|f(t, u, (w(t))^{1/(p(t)-1)} u')| \leq \beta_{N\varepsilon}(t)$, holding $|F(N_f)| \leq \int_0^1 \beta_{N\varepsilon}(t)\, dt$. Since $P(\cdot)$ is continuous, when $0 < \delta$ is small enough, from Assumption 4.1, we have

$$\left| u(0) \right| = \left| P(\lambda \delta N_f(u)) \right| < \theta. \tag{4.6}$$

It is a contradiction to $|u^i(t)| \geq \theta$ for each $t \in I$.

(ii) Suppose that $|u^i|_0 < 2\theta$, then $\theta/E < |(w(r))^{1/(p(r)-1)}(u^i)'|_0 \leq \varepsilon$. This implies that $|(w(t_2))^{1/(p(t_2)-1)}(u^i)'(t_2)| \geq \theta/E$ for some $t_2 \in I$. Since $u \in \overline{\Omega_\varepsilon}$, it is easy to see that

$$\left| (w(t_2))^{1/(p(t_2)-1)}(u^i)'(t_2) \right| \geq \frac{\theta}{E} = \frac{N\varepsilon}{N(2E+1)} \geq \frac{\left| (w(t_2))^{1/(p(t_2)-1)} u'(t_2) \right|}{N(2E+1)}. \tag{4.7}$$

Combining (4.4) and (4.7), we have

$$
\begin{aligned}
\frac{|\theta/E|^{p(t_2)-1}}{N(2E+1)} &\le \frac{1}{N(2E+1)}w(t_2)\big|(u^i)'(t_2)\big|^{p(t_2)-1} \\[2mm]
&\le \frac{1}{N(2E+1)}w(t_2)\big|u'(t_2)\big|^{p(t_2)-1} \\[2mm]
&\le w(t_2)\big|u'(t_2)\big|^{p(t_2)-1}\big|(u^i)'(t_2)\big| \\[2mm]
&\le \big|a(\lambda\delta N_f)\big| + \lambda\big|\delta F(N_f)(t)\big|.
\end{aligned}
\tag{4.8}
$$

Since $u \in \overline{\Omega_\varepsilon}$ and f is Caratheodory, it is easy to see that $|f(t,u,(w(t))^{1/(p(t)-1)}u')| \le \beta_{N\varepsilon}(t)$, thus $|\delta F(N_f)| \le \delta\int_0^1\beta_{N\varepsilon}(t)\,dt$. According to Lemma 2.2, $a(\cdot)$ is continuous, we have

$$
\big|a(\lambda\delta N_f)\big| \longrightarrow \big|a(0)\big| \quad \text{as } \delta \longrightarrow 0.
\tag{4.9}
$$

Thus, when $0 < \delta$ is small enough, from Assumption 4.1, we can conclude that

$$
\frac{|\theta/E|^{p(t_2)-1}}{N(2E+1)} \le \big|a(\lambda\delta N_f)\big| + \lambda\big|\delta F(N_f)(t)\big| < \frac{1}{N(2E+1)}\min_{t\in I}\left|\frac{\theta}{E}\right|^{p(t)-1}.
\tag{4.10}
$$

It is a contradiction. Summarizing this argument, for each $\lambda \in [0,1)$, the problem (4.2) with (1.2) has no solution on $\partial\Omega_\varepsilon$ when positive parameter δ is small enough.

(2°) According to Assumption 4.1, $u_0 \in \Omega_\varepsilon$, (where u_0 is defined in (3.23)), thus u_0 is the unique solution of $u = \Psi_f(u,0)$, then the Leray-Schauder degree

$$
d_{LS}\big[I - \Psi_f(\cdot,0),\Omega_\varepsilon,0\big] = d_{LS}\big[I - \Psi_f(\cdot,1),\Omega_\varepsilon,0\big] = 1 \neq 0.
\tag{4.11}
$$

This completes the proof. $\qquad\square$

Similar to the proof of Theorem 4.2, we have Theorem 4.3.

Theorem 4.3. *Assume $f(t,x,y) = \sigma(t)|x|^{q_1(t)-2}x + \mu(t)|y|^{q_2(t)-2}y$, where $q_1,q_2,\sigma,\mu \in C(I,\mathbb{R})$ satisfy $\max_{t\in I}p(t) < q_1(t),q_2(t)$, $\forall t \in I$. If $\delta = 1$ and $|e_0|$, $|e_1|$ are small enough, then the problem (1.1)-(1.2) possesses at least one solution.*

On the typical case, we have Corollary 4.4.

Corollary 4.4. *Assume that $f(t,x,y) = \sigma(t)|x|^{q_1(t)-2}x + \mu(t)|y|^{q_2(t)-2}y$, where $q_1,q_2,\sigma,\mu \in C(I,\mathbb{R})$ satisfy $\min_{t\in I}p(t) \le q_1(t)$, $q_2(t) \le \max_{t\in I}p(t)$. On the conditions of Theorem 4.2, the problem (1.1)-(1.2) possesses at least one solution.*

Acknowledgments

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