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Research Article

Subordination for Higher-Order Derivatives of Multivalent Functions

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Differential subordination methods are used to obtain several interesting subordination results and best dominants for higher-order derivatives of p-valent functions. These results are next applied to yield various known results as special cases.

1. Motivation and preliminaries

For a fixed $p \in \mathbb{N} := \{1, 2, \ldots\}$, let $\mathcal{A}_p$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \tag{1.1}$$

which are p-valent in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A} := \mathcal{A}_1$. Upon differentiating both sides of (1.1) q-times with respect to z, the following differential operator is obtained:

$$f^{(q)}(z) = \lambda(p; q) z^{p-q} + \sum_{k=1}^{\infty} \lambda(k+p; q) a_{k+p} z^{k+p-q}, \tag{1.2}$$

where

$$\lambda(p; q) := \frac{p!}{(p-q)!} \quad (p \geq q;\ p \in \mathbb{N};\ q \in \mathbb{N} \cup \{0\}). \tag{1.3}$$

Several researchers have investigated higher-order derivatives of multivalent functions, see, for example, [1–10]. Recently, by the use of the well-known Jack's lemma [11, 12], Irmak and Cho [5] obtained interesting results for certain classes of functions defined by higher-order derivatives.

Let f and g be analytic in $\mathbb{U}$. Then f is *subordinate* to g, written as $f(z) \prec g(z)$ $(z \in \mathbb{U})$ if there is an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. In particular, if g is univalent in $\mathbb{U}$, then f subordinate to g is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$. A p-valent function $f \in \mathcal{A}_p$ is *starlike* if it satisfies the condition $(1/p)\Re(zf'(z)/f(z)) > 0$ $(z \in \mathbb{U})$. More generally, let $\phi(z)$ be an analytic function with positive real part in $\mathbb{U}$, $\phi(0) = 1$, $\phi'(0) > 0$, and $\phi(z)$ maps the unit disc $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The classes $S_p^*(\phi)$ and $C_p(\phi)$ consist, respectively, of p-valent functions f *starlike* with respect to ϕ and p-valent functions f *convex* with respect to ϕ in $\mathbb{U}$ given by

$$f \in S_p^*(\phi) \Longleftrightarrow \frac{1}{p}\frac{zf'(z)}{f(z)} \prec \phi(z), \qquad f \in C_p(\phi) \Longleftrightarrow \frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \phi(z). \tag{1.4}$$

These classes were introduced and investigated in [13], and the functions $h_{\phi,p}$ and $k_{\phi,p}$, defined, respectively, by

$$\begin{gathered} \frac{1}{p}\frac{zh'_{\phi,p}}{h_{\phi,p}} = \phi(z) \quad (z \in \mathbb{U}, h_{\phi,p} \in \mathcal{A}_p), \\ \frac{1}{p}\left(1 + \frac{zk''_{\phi,p}}{k'_{\phi,p}}\right) = \phi(z) \quad (z \in \mathbb{U}, k_{\phi,p} \in \mathcal{A}_p), \end{gathered} \tag{1.5}$$

are important examples of functions in $S_p^*(\phi)$ and $C_p^*(\phi)$. Ma and Minda [14] have introduced and investigated the classes $S^*(\phi) := S_1^*(\phi)$ and $C(\phi) := C_1(\phi)$. For $-1 \leq B < A \leq 1$, the class $S^*[A,B] = S^*((1 + Az)/(1 + Bz))$ is the class of Janowski starlike functions (cf. [15, 16]).

In this paper, corresponding to an appropriate subordinate function $Q(z)$ defined on the unit disk $\mathbb{U}$, sufficient conditions are obtained for a p-valent function f to satisfy the subordination

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} \prec Q(z), \qquad \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 \prec Q(z). \tag{1.6}$$

In the particular case when $q = 1$ and $p = 1$, and $Q(z)$ is a function with positive real part, the first subordination gives a sufficient condition for univalence of analytic functions, while the second subordination implication gives conditions for convexity of functions. If $q = 0$ and $p = 1$, the second subordination gives conditions for starlikeness of functions. Thus results obtained in this paper give important information on the geometric properties of functions satisfying differential subordination conditions involving higher-order derivatives.

The following lemmas are needed to prove our main results.

Lemma 1.1 (see [12, page 135, Corollary 3.4h.1]). *Let Q be univalent in $\mathbb{U}$, and φ be analytic in a domain D containing $Q(\mathbb{U})$. If $zQ'(z)\cdot\varphi[Q(z)]$ is starlike, and P is analytic in $\mathbb{U}$ with $P(0) = Q(0)$ and $P(\mathbb{U}) \subset D$, then*

$$zP'(z)\cdot\varphi\big[P(z)\big] \prec zQ'(z)\cdot\varphi\big[Q(z)\big] \Longrightarrow P \prec Q, \tag{1.7}$$

and Q is the best dominant.

Lemma 1.2 (see [12, page 135, Corollary 3.4h.2]). *Let Q be convex univalent in $\mathbb{U}$, and let θ be analytic in a domain D containing $Q(\mathbb{U})$. Assume that*

$$\Re\left[\theta'[Q(z)] + 1 + \frac{zQ''(z)}{Q'(z)}\right] > 0. \tag{1.8}$$

If P is analytic in $\mathbb{U}$ with $P(0) = Q(0)$ and $P(\mathbb{U}) \subset D$, then

$$zP'(z) + \theta\big[P(z)\big] \prec zQ'(z) + \theta\big[Q(z)\big] \Longrightarrow P \prec Q, \tag{1.9}$$

and Q is the best dominant.

2. Main results

The first four theorems below give sufficient conditions for a differential subordination of the form

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} \prec Q(z) \tag{2.1}$$

to hold.

Theorem 2.1. *Let $Q(z)$ be univalent and nonzero in $\mathbb{U}$, $Q(0) = 1$, and let $zQ'(z)/Q(z)$ be starlike in $\mathbb{U}$. If a function $f \in \mathcal{A}_p$ satisfies the subordination*

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \prec \frac{zQ'(z)}{Q(z)} + p - q, \tag{2.2}$$

then

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} \prec Q(z), \tag{2.3}$$

and Q is the best dominant.

Proof. Define the analytic function $P(z)$ by

$$P(z) := \frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}}. \tag{2.4}$$

Then a computation shows that

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} = \frac{zP'(z)}{P(z)} + p - q. \tag{2.5}$$

The subordination (2.2) yields

$$\frac{zP'(z)}{P(z)} + p - q \prec \frac{zQ'(z)}{Q(z)} + p - q, \tag{2.6}$$

or equivalently

$$\frac{zP'(z)}{P(z)} \prec \frac{zQ'(z)}{Q(z)}. \tag{2.7}$$

Define the function φ by $\varphi(w) := 1/w$. Then (2.7) can be written as $zP'(z)\cdot\varphi[P(z)] \prec zQ'(z)\cdot\varphi[Q(z)]$. Since $Q(z)\neq 0$, $\varphi(w)$ is analytic in a domain containing $Q(\mathbb{U})$. Also $zQ'(z)\cdot\varphi(Q(z)) = zQ'(z)/Q(z)$ is starlike. The result now follows from Lemma 1.1. □

Remark 2.2. For $f \in \mathcal{A}_p$, Irmak and Cho [5, page 2, Theorem 2.1] showed that

$$\Re\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} < p - q \Longrightarrow \left|f^{(q)}(z)\right| < \lambda(p;q)|z|^{p-q-1}. \tag{2.8}$$

However, it should be noted that the hypothesis of this implication cannot be satisfied by any function in $\mathcal{A}_p$ as the quantity

$$\left.\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\right|_{z=0} = p - q. \tag{2.9}$$

Theorem 2.1 is the correct formulation of their result in a more general setting.

Corollary 2.3. *Let* $-1 \le B < A \le 1$*. If* $f \in \mathcal{A}_p$ *satisfies*

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \prec \frac{z(A-B)}{(1+Az)(1+Bz)} + p - q, \tag{2.10}$$

then

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} \prec \frac{1+Az}{1+Bz}. \tag{2.11}$$

Proof. For $-1 \le B < A \le 1$, define the function Q by

$$Q(z) = \frac{1+Az}{1+Bz}. \tag{2.12}$$

Then a computation shows that

$$\begin{aligned} F(z) &:= \frac{zQ'(z)}{Q(z)} = \frac{(A-B)z}{(1+Az)(1+Bz)}, \\ h(z) &:= \frac{zF'(z)}{F(z)} = \frac{1-ABz^2}{(1+Az)(1+Bz)}. \end{aligned} \tag{2.13}$$

With $z = re^{i\theta}$, note that

$$\begin{aligned} \Re\left(h\left(re^{i\theta}\right)\right) &= \Re\frac{1-ABr^2e^{2i\theta}}{(1+Are^{i\theta})(1+Bre^{i\theta})} \\ &= \frac{(1-ABr^2)(1+ABr^2+(A+B)r\cos\theta)}{\left|(1+Are^{i\theta})(1+Bre^{i\theta})\right|^2}. \end{aligned} \tag{2.14}$$

Since $1+ABr^2+(A+B)r\cos\theta \ge (1-Ar)(1-Br) > 0$ for $(A+B) \ge 0$, and similarly, $1+ABr^2+(A+B)r\cos\theta \ge (1+Ar)(1+Br) > 0$ for $(A+B) \le 0$, it follows that $\Re h(z) > 0$, and hence $zQ'(z)/Q(z)$ is starlike. The desired result now follows from Theorem 2.1. □

Example 2.4. (1) For $0 < \beta < 1$, choose $A = \beta$ and $B = 0$ in Corollary 2.3. Since $w \prec \beta z/(1+\beta z)$ is equivalent to $|w| \le \beta|1-w|$, it follows that if $f \in \mathcal{A}_p$ satisfies

$$\left|\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + \frac{\beta^2}{1-\beta^2}\right| < \frac{\beta}{1-\beta^2}, \tag{2.15}$$

then

$$\left|\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} - 1\right| < \beta. \tag{2.16}$$

(2) With $A = 1$ and $B = 0$, it follows from Corollary 2.3 that whenever $f \in \mathcal{A}_p$ satisfies

$$\Re\left\{\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q\right\} < \frac{1}{2}, \tag{2.17}$$

then

$$\left|\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} - 1\right| < 1. \tag{2.18}$$

Taking $q = 0$ and $Q(z) = h_{\phi,p}/z^p$, Theorem 2.1 yields the following corollary.

Corollary 2.5 (see [13]). *If $f \in S_p^*(\phi)$, then*

$$\frac{f(z)}{z^p} \prec \frac{h_{\phi,p}}{z^p}. \tag{2.19}$$

Similarly, choosing $q = 1$ and $Q(z) = k'_{\phi,p}/pz^{p-1}$, Theorem 2.1 yields the following corollary.

Corollary 2.6 (see [13]). *If $f \in C_p^*(\phi)$, then*

$$\frac{f'(z)}{z^{p-1}} \prec \frac{k'_{\phi,p}}{z^{p-1}}. \tag{2.20}$$

Theorem 2.7. *Let $Q(z)$ be convex univalent in $\mathbb{U}$ and $Q(0) = 1$. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} \cdot \left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q\right) \prec zQ'(z), \tag{2.21}$$

then

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} \prec Q(z), \tag{2.22}$$

and Q is the best dominant.

Proof. Define the analytic function $P(z)$ by $P(z) := f^{(q)}(z)/\lambda(p;q)z^{p-q}$. Then it follows from (2.5) that

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} \cdot \left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q\right) = zP'(z). \tag{2.23}$$

By assumption, it follows that

$$zP'(z) \cdot \varphi[P(z)] \prec zQ'(z) \cdot \varphi[Q(z)], \tag{2.24}$$

where $\varphi(w) = 1$. Since $Q(z)$ is convex, and $zQ'(z) \cdot \varphi[Q(z)] = zQ'(z)$ is starlike, Lemma 1.1 gives the desired result. □

Example 2.8. When

$$Q(z) := 1 + \frac{z}{\lambda(p;q)}, \tag{2.25}$$

Theorem 2.7 is reduced to the following result in [5, page 4, Theorem 2.4]. For $f \in \mathcal{A}_p$,

$$\left| f^{(q)}(z) \cdot \left(\frac{z f^{(q+1)}(z)}{f^{(q)}(z)} - p + q \right) \right| \leq |z|^{p-q} \Longrightarrow \left| f^{(q)}(z) - \lambda(p;q) z^{p-q} \right| \leq |z|^{p-q}. \tag{2.26}$$

In the special case $q = 1$, this result gives a sufficient condition for the multivalent function $f(z)$ to be close-to-convex.

Theorem 2.9. *Let $Q(z)$ be convex univalent in $\mathbb{U}$ and $Q(0) = 1$. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{z f^{(q+1)}(z)}{\lambda(p;q) z^{p-q}} \prec z Q'(z) + (p-q) Q(z), \tag{2.27}$$

then

$$\frac{f^{(q)}(z)}{\lambda(p;q) z^{p-q}} \prec Q(z), \tag{2.28}$$

and Q is the best dominant.

Proof. Define the function $P(z)$ by $P(z) = f^{(q)}(z)/\lambda(p;q) z^{p-q}$. It follows from (2.5) that

$$z P'(z) + (p-q) P(z) \prec z Q'(z) + (p-q) Q(z), \tag{2.29}$$

that is,

$$z P'(z) + \theta[P(z)] \prec z Q'(z) + \theta[Q(z)], \tag{2.30}$$

where $\theta(w) = (p-q)w$. The conditions in Lemma 1.2 are clearly satisfied. Thus $f^{(q)}(z)/\lambda(p;q) z^{p-q} \prec Q(z)$, and Q is the best dominant. □

Taking $q = 0$, Theorem 2.9 yields the following corollary.

Corollary 2.10 (see [17, Corollary 2.11]). *Let $Q(z)$ be convex univalent in $\mathbb{U}$, and $Q(0) = 1$. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{f'(z)}{z^{p-1}} \prec z Q'(z) + p Q(z), \tag{2.31}$$

then

$$\frac{f(z)}{z^p} \prec Q(z). \tag{2.32}$$

With $p = 1$, Corollary 2.10 yields the following corollary.

Corollary 2.11 (see [17, Corollary 2.9]). *Let $Q(z)$ be convex univalent in $\mathbb{U}$, and $Q(0) = 1$. If $f \in \mathcal{A}$ satisfies*

$$f'(z) \prec zQ'(z) + Q(z), \tag{2.33}$$

then

$$\frac{f(z)}{z} \prec Q(z). \tag{2.34}$$

Theorem 2.12. *Let $Q(z)$ be univalent and nonzero in $\mathbb{U}$, $Q(0) = 1$, and $zQ'(z)/Q^2(z)$ be starlike. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{\lambda(p;q)z^{p-q}}{f^{(q)}(z)} \cdot \left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q\right) \prec \frac{zQ'(z)}{Q^2(z)}, \tag{2.35}$$

then

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} \prec Q(z), \tag{2.36}$$

and Q is the best dominant.

Proof. Define the function $P(z)$ by $P(z) = f^{(q)}(z)/\lambda(p;q)z^{p-q}$. It follows from (2.5) that

$$\frac{\lambda(p;q)z^{p-q}}{f^{(q)}(z)} \cdot \left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - (p-q)\right) = \frac{1}{P(z)} \cdot \frac{zP'(z)}{P(z)} = \frac{zP'(z)}{P^2(z)}. \tag{2.37}$$

By assumption,

$$\frac{zP'(z)}{P^2(z)} \prec \frac{zQ'(z)}{Q^2(z)}. \tag{2.38}$$

With $\varphi(w) := 1/w^2$, (2.38) can be written as $zP'(z) \cdot \varphi[P(z)] \prec zQ'(z) \cdot \varphi[Q(z)]$. The function $\varphi(w)$ is analytic in $\mathbb{C} - \{0\}$. Since $zQ'(z)\varphi[Q(z)]$ is starlike, it follows from Lemma 1.1 that $P(z) \prec Q(z)$, and $Q(z)$ is the best dominant. □

The next four theorems give sufficient conditions for the following differential subordination

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 \prec Q(z) \tag{2.39}$$

to hold.

Theorem 2.13. *Let $Q(z)$ be univalent and nonzero in $\mathbb{U}$, $Q(0) = 1$, $Q(z) \neq q - p + 1$, and $zQ'(z)/[Q(z)(Q(z) + p - q - 1)]$ be starlike in $\mathbb{U}$. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{1 + (zf^{(q+2)}(z)/f^{(q+1)}(z)) - p + q + 1}{(zf^{(q+1)}(z)/f^{(q)}(z)) - p + q + 1} \prec 1 + \frac{zQ'(z)}{Q(z)(Q(z) + p - q - 1)}, \tag{2.40}$$

then

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 \prec Q(z), \tag{2.41}$$

and Q is the best dominant.

Proof. Let the function $P(z)$ be defined by

$$P(z) = \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1. \tag{2.42}$$

Upon differentiating logarithmically both sides of (2.42), it follows that

$$\frac{zP'(z)}{P(z) + p - q - 1} = 1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)}. \tag{2.43}$$

Thus

$$1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - p + q + 1 = \frac{zP'(z)}{P(z) + p - q - 1} + P(z). \tag{2.44}$$

The equations (2.42) and (2.44) yield

$$\frac{1 + (zf^{(q+2)}(z)/f^{(q+1)}(z)) - p + q + 1}{(zf^{(q+1)}(z)/f^{(q)}(z)) - p + q - 1} = \frac{zP'(z)}{P(z)(P(z) + p - q - 1)} + 1. \tag{2.45}$$

If $f \in \mathcal{A}_p$ satisfies the subordination (2.40), (2.45) gives

$$\frac{zP'(z)}{P(z)(P(z) + p - q - 1)} \prec \frac{zQ'(z)}{Q(z)(Q(z) + p - q - 1)}, \tag{2.46}$$

that is,

$$zP'(z) \cdot \varphi[P(z)] \prec zQ'(z) \cdot \varphi[Q(z)] \tag{2.47}$$

with $\varphi(w) := 1/w(w + p - q - 1)$. The desired result is now established by an application of Lemma 1.1. □

Theorem 2.13 contains a result in [18, page 122, Corollary 4] as a special case. In particular, we note that Theorem 2.13 with $p = 1$, $q = 0$, and $Q(z) = (1 + Az)/(1 + Bz)$ for $-1 \leq B < A \leq 1$ yields the following corollary.

Corollary 2.14 (see [18, page 123, Corollary 6]). *Let $-1 \leq B < A \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$\frac{1 + (zf''(z)/f'(z))}{zf'(z)/f(z)} \prec 1 + \frac{(A - B)z}{(1 + Az)^2}, \tag{2.48}$$

then $f \in S^*[A, B]$.

For $A = 0,\ B = b$ and $A = 1,\ B = -1$, Corollary 2.14 gives the results of Obradovič and Tuneski [19].

Theorem 2.15. *Let $Q(z)$ be univalent and nonzero in $\mathbb{U}$, $Q(0) = 1$, $Q(z) \neq q - p + 1$, and let $zQ'(z)/[Q(z) + p - q - 1]$ be starlike in $\mathbb{U}$. If $f \in \mathcal{A}_p$ satisfies*

$$1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \prec \frac{zQ'(z)}{Q(z) + p - q - 1}, \tag{2.49}$$

then

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 \prec Q(z), \tag{2.50}$$

and Q is the best dominant.

Proof. Let the function $P(z)$ be defined by (2.42). It follows from (2.43) and the hypothesis that

$$\frac{zP'(z)}{P(z) + p - q - 1} \prec \frac{zQ'(z)}{Q(z) + p - q - 1}. \tag{2.51}$$

Define the function φ by $\varphi(w) := 1/(w + p - q - 1)$. Then (2.51) can be written as

$$zP'(z) \cdot \varphi[P(z)] \prec zQ'(z) \cdot \varphi[Q(z)]. \tag{2.52}$$

Since $\varphi(w)$ is analytic in a domain containing $Q(\mathbb{U})$, and $zQ'(z) \cdot \varphi[Q(z)]$ is starlike, the result follows from Lemma 1.1. □

Theorem 2.16. *Let $Q(z)$ be a convex function in $\mathbb{U}$, and $Q(0) = 1$. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\left[2 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\right] \prec zQ'(z) + Q(z) + p - q - 1, \tag{2.53}$$

then

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 \prec Q(z), \tag{2.54}$$

and Q is the best dominant.

Proof. Let the function $P(z)$ be defined by (2.42). Using (2.43), it follows that

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\left(1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\right) = zP'(z), \tag{2.55}$$

and, therefore,

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\left(2 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\right) = zP'(z) + P(z) + p - q - 1. \tag{2.56}$$

By assumption,

$$zP'(z) + P(z) + p - q - 1 \prec zQ'(z) + Q(z) + p - q - 1, \tag{2.57}$$

or

$$zP'(z) + \theta[P(z)] \prec zQ'(z) + \theta[Q(z)], \tag{2.58}$$

where the function $\theta(w) = w + p - q + 1$. The proof is completed by applying Lemma 1.2. □

Theorem 2.17. *Let $Q(z)$ be a convex function in $\mathbb{U}$, with $Q(0) = 1$. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\left[1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\right] \prec zQ'(z), \tag{2.59}$$

then

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 \prec Q(z), \tag{2.60}$$

and Q is the best dominant.

Proof. Let the function $P(z)$ be defined by (2.42). It follows from (2.43) that $zP'(z) \cdot \varphi[P(z)] \prec zQ'(z) \cdot \varphi[Q(z)]$, where $\varphi(w) = 1$. The result follows easily from Lemma 1.1. □

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