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Research Article

Some New Results Related to Favard's Inequality

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Log-convexity of Favard's difference is proved, and Drescher's and Lyapunov's type inequalities for this difference are deduced. The weighted case is also considered. Related Cauchy type means are defined, and some basic properties are given.

1. Introduction and Preliminaries

Let f and p be two positive measurable real valued functions defined on $(a,b) \subseteq \mathbb{R}$ with $\int_a^b p(x)dx = 1$. From theory of convex means (cf. [1, 2]), the well-known Jensen's inequality gives that for $t < 0$ or $t > 1$,

$$\int_a^b p(x) f^t(x) dx \geq \left(\int_a^b p(x) f(x) dx \right)^t, \tag{1.1}$$

and reverse inequality holds for $0 < t < 1$. In [3], Simic considered the difference

$$D_s = D_s(a,b,f,p) = \int_a^b p(x) f^s(x) dx - \left(\int_a^b p(x) f(x) dx \right)^s. \tag{1.2}$$

He has given the following.

Theorem 1.1. *Let f and p be nonnegative and integrable functions on (a,b), with $\int_a^b p(x)dx = 1$, then for $0 < r < s < t$, $r,s,t \neq 1$, one has*

$$\left(\frac{D_s}{s(s-1)}\right)^{t-r} \leq \left(\frac{D_r}{r(r-1)}\right)^{t-s}\left(\frac{D_t}{t(t-1)}\right)^{s-r}. \tag{1.3}$$

Remark 1.2. For an extension of Theorem 1.1 see [3].

Let us write the well-known Favard's inequality.

Theorem 1.3. *Let f be a concave nonnegative function on $[a,b] \subset \mathbb{R}$. If $q > 1$, then*

$$\frac{2^q}{q+1}\left(\frac{1}{b-a}\int_a^b f(x)dx\right)^q \geq \frac{1}{b-a}\int_a^b f^q(x)dx. \tag{1.4}$$

If $0 < q < 1$, the reverse inequality holds in (1.4).

Note that (1.4) is a reversion of (1.1) in the case when $p(x) = 1/(b-a)$.

Let us note that Theorem 1.3 can be obtained from the following result and also obtained by Favard (cf. [4, page 212]).

Theorem 1.4. *Let f be a nonnegative continuous concave function on $[a,b]$, not identically zero, and let ϕ be a convex function on $[0, 2\tilde{f}]$, where*

$$\tilde{f} = \frac{1}{b-a}\int_a^b f(x)dx. \tag{1.5}$$

Then

$$\frac{1}{2\tilde{f}}\int_0^{2\tilde{f}} \phi(y)dy \geq \frac{1}{b-a}\int_a^b \phi(f(x))dx. \tag{1.6}$$

Karlin and Studden (cf. [5, page 412]) gave a more general inequality as follows.

Theorem 1.5. *Let f be a nonnegative continuous concave function on $[a,b]$, not identically zero; $\tilde{f}$ is defined in* (1.5)*, and let ϕ be a convex function on $[c, 2\tilde{f}-c]$, where c satisfies $0 < c \leq f_{\min}$ (where $f_{\min}$ is the minimum of f). Then*

$$\frac{1}{2\tilde{f}-2c}\int_c^{2\tilde{f}-c} \phi(y)dy \geq \frac{1}{b-a}\int_a^b \phi(f(x))dx. \tag{1.7}$$

For $\phi(y) = y^p$, $p > 1$, we can get the following from Theorem 1.5.

Theorem 1.6. *Let f be continuous concave function such that $0 < c \leq f_{\min}$; $\tilde{f}$ is defined in* (1.5). *If $p > 1$, then*

$$\frac{1}{(2\tilde{f} - 2c)(p+1)}\left((2\tilde{f} - c)^{p+1} - c^{p+1}\right) \geq \frac{1}{b-a}\int_a^b f^p(x)dx. \tag{1.8}$$

If $0 < p < 1$, the reverse inequality holds in (1.8).

In this paper, we give a related results to (1.3) for Favard's inequality (1.4) and (1.8). We need the following definitions and lemmas.

Definition 1.7. It is said that a positive function f is log-convex in the Jensen sense on some interval $I \subseteq \mathbb{R}$ if

$$f(s)f(t) \geq f^2\left(\frac{s+t}{2}\right) \tag{1.9}$$

holds for every $s, t \in I$.

We quote here another useful lemma from log-convexity theory (cf. [3]).

Lemma 1.8. *A positive function f is log-convex in the Jensen sense on an interval $I \subseteq \mathbb{R}$ if and only if the relation*

$$u^2 f(s) + 2uwf\left(\frac{s+t}{2}\right) + w^2 f(t) \geq 0 \tag{1.10}$$

holds for each real u, w and $s, t \in I$.

Throughout the paper, we will frequently use the following family of convex functions on $(0, \infty)$:

$$\varphi_s(x) = \begin{cases} \dfrac{x^s}{s(s-1)}, & s \neq 0, 1; \\ -\log x, & s = 0; \\ x \log x, & s = 1. \end{cases} \tag{1.11}$$

The following lemma is equivalent to the definition of convex function (see [4, page 2]).

Lemma 1.9. *If ϕ is convex on an interval $I \subseteq \mathbb{R}$, then*

$$\phi(s_1)(s_3 - s_2) + \phi(s_2)(s_1 - s_3) + \phi(s_3)(s_2 - s_1) \geq 0 \tag{1.12}$$

holds for every $s_1 < s_2 < s_3$, $s_1, s_2, s_3 \in I$.

Now, we will give our main results.

2. Favard's Inequality

In the following theorem, we construct another interesting family of functions satisfying the Lyapunov inequality. The proof is motivated by [3].

Theorem 2.1. *Let f be a positive continuous concave function on $[a,b]$; $\tilde{f}$ is defined in* (1.5), *and*

$$\Delta_s(f) := \begin{cases} \dfrac{1}{s(s-1)}\left[\dfrac{2^s}{s+1}\left(\dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx\right)^s - \dfrac{1}{b-a}\displaystyle\int_a^b f^s(x)dx\right], & s \neq 0,1; \\ 1-\log 2 - \log\tilde{f} + \dfrac{1}{b-a}\displaystyle\int_a^b \log f(x)dx, & s=0; \\ \log 2\tilde{f} + \tilde{f}\log\tilde{f} - \dfrac{1}{2}\tilde{f} - \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\log f(x)dx, & s=1. \end{cases} \tag{2.1}$$

Then $\Delta_s(f)$ is log-convex for $s \geq 0$, and the following inequality holds for $0 \leq r < s < t < +\infty$:

$$\Delta_s^{t-r}(f) \leq \Delta_r^{t-s}(f)\Delta_t^{s-r}(f). \tag{2.2}$$

Proof. Let us consider the function defined by

$$\phi(x) = u^2\varphi_s(x) + 2uw\varphi_r(x) + w^2\varphi_t(x), \tag{2.3}$$

where $r=(s+t)/2$, φ_s is defined by (1.11), and $u,w \in \mathbb{R}$. We have

$$\begin{aligned} \phi''(x) &= u^2x^{s-2} + 2uwx^{r-2} + w^2x^{t-2} \\ &= \left(ux^{s/2-1} + wx^{t/2-1}\right)^2 \geq 0, \quad x>0. \end{aligned} \tag{2.4}$$

Therefore, $\phi(x)$ is convex for $x>0$. Using Theorem 1.4,

$$\begin{aligned} &\frac{1}{2\tilde{f}}\int_0^{2\tilde{f}} \left(u^2\varphi_s(y) + 2uw\varphi_r(y) + w^2\varphi_t(y)\right)dy \\ &\quad\geq \frac{1}{b-a}\int_a^b \left(u^2\varphi_s(f(x)) + 2uw\varphi_r(f(x)) + w^2\varphi_t(f(x))\right)dx, \end{aligned} \tag{2.5}$$

or equivalently

$$u^2\left[\frac{1}{2\tilde{f}}\int_0^{2\tilde{f}}\varphi_s(y)dy - \frac{1}{b-a}\int_a^b \varphi_s(f(x)dx)\right]$$
$$+ 2uw\left[\frac{1}{2\tilde{f}}\int_0^{2\tilde{f}}\varphi_r(y)dy - \frac{1}{b-a}\int_a^b \varphi_r(f(x))dx\right] \tag{2.6}$$
$$+ w^2\left[\frac{1}{2\tilde{f}}\int_0^{2\tilde{f}}\varphi_t(y)dy - \frac{1}{b-a}\int_a^b \varphi_t(f(x))dx\right] \geq 0.$$

Since

$$\Delta_s(f) = \frac{1}{2\tilde{f}}\int_0^{2\tilde{f}}\varphi_s(y)dy - \frac{1}{b-a}\int_a^b \varphi_s(f(x))dx, \tag{2.7}$$

we have

$$u^2\Delta_s(f) + 2uw\Delta_r(f) + w^2\Delta_t(f) \geq 0. \tag{2.8}$$

By Lemma 1.8, we have

$$\Delta_s(f)\Delta_t(f) \geq \Delta_r^2(f) = \Delta_{(s+t)/2}^2(f), \tag{2.9}$$

that is, $\Delta_s(f)$ is log-convex in the Jensen sense for $s \geq 0$.

Note that $\Delta_s(f)$ is continuous for $s \geq 0$ since

$$\lim_{s\to 0}\Delta_s(f) = \Delta_0(f) \quad \text{and} \quad \lim_{s\to 1}\Delta_s(f) = \Delta_1(f). \tag{2.10}$$

This implies $\Delta_s(f)$ is continuous; therefore, it is log-convex.

Since $\Delta_s(f)$ is log-convex, that is, $s \mapsto \log\Delta_s(f)$ is convex, by Lemma 1.9 for $0 \leq r < s < t < +\infty$ and taking $\phi(s) = \log\Delta_s(f)$, we get

$$\log\Delta_s^{t-r}(f) \leq \log\Delta_r^{t-s}(f) + \log\Delta_t^{s-r}(f), \tag{2.11}$$

which is equivalent to (2.2). □

Theorem 2.2. *Let f, $\Delta_s(f)$ be defined as in Theorem 2.1, and let t, s, u, v be nonnegative real numbers such that $s \leq u$, $t \leq v$, $s \neq t$, and $u \neq v$. Then*

$$\left(\frac{\Delta_t(f)}{\Delta_s(f)}\right)^{1/(t-s)} \leq \left(\frac{\Delta_v(f)}{\Delta_u(f)}\right)^{1/(v-u)}. \tag{2.12}$$

Proof. An equivalent form of (1.12) is

$$\frac{\varphi(x_2)-\varphi(x_1)}{x_2-x_1}\le\frac{\varphi(y_2)-\varphi(y_1)}{y_2-y_1}, \tag{2.13}$$

where $x_1\le y_1$, $x_2\le y_2$, $x_1\ne x_2$, and $y_1\ne y_2$. Since by Theorem 2.1, $\Delta_s(f)$ is log-convex, we can set in (2.13): $\varphi(x)=\log\Delta_x(f)$, $x_1=s$, $x_2=t$, $y_1=u$, and $y_2=v$. We get

$$\frac{\log\Delta_t(f)-\log\Delta_s(f)}{t-s}\le\frac{\log\Delta_v(f)-\log\Delta_u(f)}{v-u}, \tag{2.14}$$

from which (2.12) trivially follows. □

The following extensions of Theorems 2.1 and 2.2 can be deduced in the same way from Theorem 1.5.

Theorem 2.3. *Let f be a continuous concave function on $[a,b]$ such that $0<c\le f_{\min}$; $\tilde{f}$ is defined in* (1.5)*, and*

$$\tilde{\Delta}_s(f):=\begin{cases}\dfrac{1}{s(s-1)}\left[\dfrac{(2\tilde{f}-c)^{s+1}}{(2\tilde{f}-2c)(s+1)}-\dfrac{c^{s+1}}{(2\tilde{f}-2c)(s+1)}-\dfrac{1}{b-a}\displaystyle\int_a^b f^s(x)dx\right], & s\ne 0,1;\\ \dfrac{1}{2\tilde{f}-2c}\left[2\tilde{f}+c\log c-2c-(2\tilde{f}-c)\log(2\tilde{f}-c)\right]+\dfrac{1}{b-a}\displaystyle\int_a^b\log f(x)dx, & s=0;\\ \dfrac{1}{2(2\tilde{f}-2c)}\left[(2\tilde{f}-c)^2\log(2\tilde{f}-c)-2\tilde{f}^2+2c\tilde{f}-c^2\log c+2c\right] & \\ \quad-\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\log f(x)dx, & s=1.\end{cases} \tag{2.15}$$

Then $\tilde{\Delta}_s(f)$ is log-convex for $s\ge 0$, and the following inequality holds for $0\le r<s<t<+\infty$:

$$\tilde{\Delta}_s^{t-r}(f)\le\tilde{\Delta}_r^{t-s}(f)\tilde{\Delta}_t^{s-r}(f). \tag{2.16}$$

Theorem 2.4. *Let f, $\tilde{\Delta}_s(f)$ be defined as in Theorem 2.3, and let t,s,u,v be nonnegative real numbers such that $s\le u, t\le v, s\ne t$, and $u\ne v$, one has*

$$\left(\frac{\tilde{\Delta}_t(f)}{\tilde{\Delta}_s(f)}\right)^{1/(t-s)}\le\left(\frac{\tilde{\Delta}_v(f)}{\tilde{\Delta}_u(f)}\right)^{1/(v-u)}. \tag{2.17}$$

3. Weighted Favard's Inequality

The weighted version of Favard's inequality was obtained by Maligranda et al. in [6].

Theorem 3.1. (1) *Let f be a positive increasing concave function on $[a,b]$. Assume that ϕ is a convex function on $[0,\infty)$, where*

$$\widetilde{f}_i = \frac{(b-a)\int_a^b f(t)w(t)dt}{2\int_a^b (t-a)w(t)dt}. \tag{3.1}$$

Then

$$\frac{1}{b-a}\int_a^b \phi(f(t))w(t)dt \le \int_0^1 \phi(2r\widetilde{f}_i)w[a(1-r)+br]dr. \tag{3.2}$$

If f is an increasing convex function on $[a,b]$ and $f(a)=0$, then the reverse inequality in (3.2) *holds.*

(2) *Let f be a positive decreasing concave function on $[a,b]$. Assume that ϕ is a convex function on $[0,\infty)$, where*

$$\widetilde{f}_d = \frac{(b-a)\int_a^b f(t)w(t)dt}{2\int_a^b (b-t)w(t)dt}. \tag{3.3}$$

Then

$$\frac{1}{b-a}\int_a^b \phi(f(t))w(t)dt \le \int_0^1 \phi(2r\widetilde{f}_d)w[ar+b(1-r)]dr. \tag{3.4}$$

If f is a decreasing convex function on $[a,b]$ and $f(b)=0$, then the reverse inequality in (3.4) *holds.*

Theorem 3.2. (1) *Let f be a positive increasing concave function on $[a,b]$; $\widetilde{f}_i$ is defined in* (3.1)*, and*

$$\Pi_s(f) := \int_0^1 \varphi_s(2r\widetilde{f}_i)w[a(1-r)+br]dr - \frac{1}{b-a}\int_a^b \varphi_s(f(t))w(t)dt. \tag{3.5}$$

Then $\Pi_s(f)$ is log-convex on $[0,\infty)$, and the following inequality holds for $0 \le r < s < t < +\infty$:

$$\Pi_s^{t-r}(f) \le \Pi_r^{t-s}(f)\Pi_t^{s-r}(f). \tag{3.6}$$

(2) *Let f be an increasing convex function on $[a,b]$, $f(a)=0$, $\widetilde{\Pi}_s(f) := -\Pi_s(f)$. Then $\widetilde{\Pi}_s(f)$ is log-convex on $[0,\infty)$, and the following inequality holds for $0 \le r < s < t < +\infty$:*

$$\widetilde{\Pi}_s^{t-r}(f) \le \widetilde{\Pi}_r^{t-s}(f)\widetilde{\Pi}_t^{s-r}(f). \tag{3.7}$$

Proof. As in the proof of Theorem 2.1, we use Theorem 3.1(1) instead of Theorem 1.4. □

Theorem 3.3. (1) *Let f and $\Pi_s(f)$ be defined as in Theorem 3.2(1), and let $t, s, u, v \geq 0$ be such that $s \leq u$, $t \leq v$, $s \neq t$, and $u \neq v$. Then*

$$\left(\frac{\Pi_t(f)}{\Pi_s(f)}\right)^{1/(t-s)} \leq \left(\frac{\Pi_v(f)}{\Pi_u(f)}\right)^{1/(v-u)}. \tag{3.8}$$

(2) *Let f and $\widetilde{\Pi}_s(f)$ be defined as in Theorem 3.2(2), and let $t, s, u, v \geq 0$ be such that $s \leq u$, $t \leq v$, $s \neq t$, and $u \neq v$. Then,*

$$\left(\frac{\widetilde{\Pi}_t(f)}{\widetilde{\Pi}_s(f)}\right)^{1/(t-s)} \leq \left(\frac{\widetilde{\Pi}_v(f)}{\widetilde{\Pi}_u(f)}\right)^{1/(v-u)}. \tag{3.9}$$

Proof. Similar to the proof of Theorem 2.2. □

Theorem 3.4. (1) *Let f be a positive decreasing concave function on $[a,b]$; $\widetilde{f}_d$ is defined as in* (3.3), *and*

$$\Gamma_s(f) := \int_0^1 \varphi_s(2r\widetilde{f}_d)w[ar + b(1-r)]dr - \frac{1}{b-a}\int_a^b \varphi_s(f(t))w(t)dt. \tag{3.10}$$

Then $\Gamma_s(f)$ is log-convex on $[0,\infty)$, and the following inequality holds for $0 \leq r < s < t < +\infty$:

$$\Gamma_s^{t-r}(f) \leq \Gamma_r^{t-s}(f)\Gamma_t^{s-r}(f). \tag{3.11}$$

(2) *Let f be a decreasing convex function on $[a,b]$, $f(b) = 0$, $\widetilde{\Gamma}_s(f) := -\Gamma_s(f)$. Then $\widetilde{\Gamma}_s$ is log-convex on $[0,\infty)$, and the following inequality holds for $0 \leq r < s < t < +\infty$:*

$$\widetilde{\Gamma}_s^{t-r}(f) \leq \widetilde{\Gamma}_r^{t-s}(f)\widetilde{\Gamma}_t^{s-r}(f). \tag{3.12}$$

Proof. As in the proof of Theorem 2.1, we use Theorem 3.1(2) instead of Theorem 1.4. □

Theorem 3.5. (1) *Let f and $\Gamma_s(f)$ be defined as in Theorem 3.4(1), and let $t, s, u, v \geq 0$ be such that $s \leq u$, $t \leq v$, $s \neq t$, and $u \neq v$. Then*

$$\left(\frac{\Gamma_t(f)}{\Gamma_s(f)}\right)^{1/(t-s)} \leq \left(\frac{\Gamma_v(f)}{\Gamma_u(f)}\right)^{1/(v-u)}. \tag{3.13}$$

(2) *Let f and $\widetilde{\Gamma}_s(f)$ be defined as in Theorem 3.4(2), and let $t, s, u, v \geq 0$ be such that $s \leq u$, $t \leq v$, $s \neq t$, and $u \neq v$. Then*

$$\left(\frac{\widetilde{\Gamma}_t(f)}{\widetilde{\Gamma}_s(f)}\right)^{1/(t-s)} \leq \left(\frac{\widetilde{\Gamma}_v(f)}{\widetilde{\Gamma}_u(f)}\right)^{1/(v-u)}. \tag{3.14}$$

Proof. Similar to the proof of Theorem 2.2. □

Remark 3.6. Let $w \equiv 1$. If f is a positive concave function on $[a,b]$, then the decreasing rearrangement f^* is concave on $[a,b]$. By applying Theorem 3.4 to f^*, we obtain that $\Gamma_s(f^*)$ is log-convex. Equimeasurability of f with f^* gives $\Gamma_s(f) = \Gamma_s(f^*)$ and we see that Theorem 3.4 is equivalent to Theorem 2.1.

Remark 3.7. Let $w(t) = t^\alpha$ with $\alpha > -1$. Then Theorem 3.2 gives that if f is a positive increasing concave function on $[0,1]$, then Π_s^α is log-convex, and

$$
\begin{aligned}
\Pi_s^\alpha &= \frac{1}{s(s-1)}\left[\frac{(\alpha+2)^s}{(\alpha+s+1)}\left(\int_0^1 f(t)t^\alpha dt\right)^s - \int_0^1 f^s(t)t^\alpha dt\right], \quad s\neq 0,1,\\
\Pi_0^\alpha &= \int_0^1 \log f(t)t^\alpha dt - \frac{\log\left((\alpha+2)\int_0^1 f(t)t^\alpha dt\right)}{\alpha+1} + \frac{1}{(\alpha+1)^2},\\
\Pi_1^\alpha &= \log\left((\alpha+2)\int_0^1 f(t)t^\alpha dt\right)\int_0^1 f(t)t^\alpha dt - \frac{\int_0^1 f(t)t^\alpha dt}{\alpha+2} - \int_0^1 f(t)\log f(t)t^\alpha dt,
\end{aligned}
\tag{3.15}
$$

with zero for the function $f(t) = t$.

If f is a positive decreasing concave function on $[0,1]$, then Theorem 3.4 gives that Γ_s^α is log-convex, and

$$
\begin{aligned}
\Gamma_s^\alpha &= \frac{1}{s(s-1)}\left[(\alpha+1)^s(\alpha+2)^s B(s+1,\alpha+1)\left(\int_0^1 f(t)t^\alpha dt\right)^s - \int_0^1 f^s(t)t^\alpha dt\right], \quad s\neq 0,1,\\
\Gamma_0^\alpha &= \int_0^1 \log f(t)t^\alpha dt + \frac{1}{\alpha+1}H(\alpha+1) - \frac{1}{\alpha+1}\log\left[(\alpha+1)(\alpha+2)\int_0^1 f(t)t^\alpha dt\right],\\
\Gamma_1^\alpha &= \left(1 - H(\alpha+2) + \log\left[(\alpha+1)(\alpha+2)\right]\right)\int_0^1 f(t)t^\alpha dt\\
&\quad + \int_0^1 f(t)t^\alpha dt \log\int_0^1 f(t)t^\alpha dt - \int_0^1 f(t)\log f(t)t^\alpha dt,
\end{aligned}
\tag{3.16}
$$

with zero for the function $f(t) = 1-t$, where $B(\cdot,\cdot)$ is the beta function, and $H(\alpha)$ is the harmonic number defined for $\alpha > -1$ with $H(\alpha) = \psi(\alpha+1) + \gamma$, where ψ is the digamma function and $\gamma = 0.577215\ldots$ the Euler constant.

4. Cauchy Means

Let us note that (2.12), (2.17), (3.8), (3.9), (3.13), and (3.14) have the form of some known inequalities between means (e.g., Stolarsky means, Gini means, etc.). Here we will prove that expressions on both sides of (3.8) are also means. The proofs in the remaining cases are analogous.

Lemma 4.1. *Let $h \in C^2(I)$, I interval in $\mathbb{R}$, be such that h'' is bounded, that is, $m \le h'' \le M$. Then the functions ϕ_1, ϕ_2 defined by*

$$\phi_1(t) = \frac{M}{2}t^2 - h(t), \qquad \phi_2(t) = h(t) - \frac{m}{2}t^2, \tag{4.1}$$

are convex functions.

Theorem 4.2. *Let w be a nonnegative integrable function on (a,b) with $\int_a^b w(x)dx = 1$. Let f be a positive increasing concave function on $[a,b]$, $h \in C^2([0,2\widetilde{f}_i])$. Then there exists $\xi \in [0,2\widetilde{f}_i]$, such that*

$$\begin{aligned}\int_0^1 h(2r\widetilde{f}_i)w[a(1-r)+br]dr &- \frac{1}{b-a}\int_a^b h(f(t))w(t)dt \\ &= \frac{h''(\xi)}{2}\left[\int_0^1 (2r\widetilde{f}_i)^2 w[a(1-r)+br]dr - \frac{1}{b-a}\int_a^b f^2(t)w(t)dt\right].\end{aligned} \tag{4.2}$$

Proof. Set $m = \min_{x\in[0,2\widetilde{f}_i]} h''(x)$, $M = \max_{x\in[0,2\widetilde{f}_i]} h''(x)$. Applying (3.2) for ϕ_1 and ϕ_2 defined in Lemma 4.1, we have

$$\begin{aligned}\int_0^1 \phi_1(2r\widetilde{f}_i)w[a(1-r)+br]dr &\ge \frac{1}{b-a}\int_a^b \phi_1(f(t))w(t)dt, \\ \int_0^1 \phi_2(2r\widetilde{f}_i)w[a(1-r)+br]dr &\ge \frac{1}{b-a}\int_a^b \phi_2(f(t))w(t)dt,\end{aligned} \tag{4.3}$$

that is,

$$\begin{aligned}\frac{M}{2}&\left[\int_0^1 (2r\widetilde{f}_i)^2 w[a(1-r)+br]dr - \frac{1}{b-a}\int_a^b f^2(t)w(t)dt\right] \\ &\ge \int_0^1 h(2r\widetilde{f}_i)w[a(1-r)+br]dr - \frac{1}{b-a}\int_a^b h(f(t))w(t)dt,\end{aligned} \tag{4.4}$$

$$\begin{aligned}\int_0^1 h(2r\widetilde{f}_i)w[a(1-r)+br]dr &- \frac{1}{b-a}\int_a^b h(f(t))w(t)dt \\ &\ge \frac{m}{2}\left[\int_0^1 \left(2r\widetilde{f}_i\right)^2 w[a(1-r)+br]dr - \frac{1}{b-a}\int_a^b f^2(t)w(t)dt\right].\end{aligned} \tag{4.5}$$

By combining (4.4) and (4.5), (4.2) follows from continuity of h''. □

Theorem 4.3. *Let f be a positive increasing concave nonlinear function on $[a,b]$, and let w be a nonnegative integrable function on (a,b) with $\int_a^b w(x)dx = 1$. If $h_1, h_2 \in C^2([0,2\tilde{f}_i])$, then there exists $\xi \in [0,2\tilde{f}_i]$ such that*

$$\frac{h_1''(\xi)}{h_2''(\xi)} = \frac{\int_0^1 h_1(2r\tilde{f}_i)w[a(1-r)+br]dr - (1/(b-a))\int_a^b h_1(f(t))w(t)dt}{\int_0^1 h_2(2r\tilde{f}_i)w[a(1-r)+br]dr - (1/(b-a))\int_a^b h_2(f(t))w(t)dt}, \tag{4.6}$$

provided that $h_2''(x) \neq 0$ for every $x \in [0,2\tilde{f}_i]$.

Proof. Define the functional $\Phi : C^2([0,2\tilde{f}_i]) \to \mathbb{R}$ with

$$\Phi(h) = \int_0^1 h(2r\tilde{f}_i)w[a(1-r)+br]dr - \frac{1}{b-a}\int_a^b h(f(t))w(t)dt, \tag{4.7}$$

and set $h_0 = \Phi(h_2)h_1 - \Phi(h_1)h_2$. Obviously, $\Phi(h_0) = 0$. Using Theorem 4.2 , there exists $\xi \in [0,2\tilde{f}_i]$ such that

$$\Phi(h_0) = \frac{h_0''(\xi)}{2}\left[\int_0^1 (2r\tilde{f}_i)^2 w[a(1-r)+br]dr - \frac{1}{b-a}\int_a^b f^2(t)w(t)dt\right]. \tag{4.8}$$

We give a proof that the expression in square brackets in (4.8) is nonzero (actually strictly positive by inequality (3.2)) for nonlinear function f. Suppose that the expression in square brackets in (4.8) is equal to zero, which is by simple rearrangements equivalent to equality

$$\int_a^b (t-a)^2 w(t)dt = \int_a^b g^2(t)w(t)dt, \quad \text{where } g(t) = \frac{\int_a^b (t-a)w(t)dt}{\int_a^b f(t)w(t)dt} f(t). \tag{4.9}$$

Since g is positive concave function, it is easy to see that $g(t)/(t-a)$ is decreasing function on $(a,b]$ (see [6]), thus

$$1 = \frac{1}{\int_a^b (t-a)w(t)dt}\int_a^b g(t)w(t)dt \le \frac{1}{\int_a^x (t-a)w(t)dt}\int_a^x g(t)w(t)dt, \quad x \in (a,b], \tag{4.10}$$

so $\int_a^x (t-a)w(t)dt \le \int_a^x g(t)w(t)dt$ for every $x \in [a,b]$. Set

$$F(x) = \int_a^x [(t-a) - g(t)]w(t)dt. \tag{4.11}$$

Obviously, $F(x) \le 0$, $F(a) = F(b) = 0$. By (4.9), obvious estimations and integration by parts, we have

$$0 = \int_a^b \left[(t-a)^2 - g^2(t)\right] w(t)dt \ge \int_a^b 2g(t)\left[(t-a) - g(t)\right] w(t)dt$$
$$= \int_a^b 2g(t)dF(t) = -\int_a^b F(t)d\left[2g(t)\right] \ge 0. \tag{4.12}$$

This implies $\int_a^b [(t-a)^2 - g^2(t)]w(t)dt = \int_a^b 2g(t)[(t-a) - g(t)]w(t)dt$, which is equivalent to $\int_a^b [(t-a) - g(t)]^2 w(t)dt = 0$. This gives that g is a linear function, which obviously implies that f is a linear function.

Since the function f is nonlinear, the expression in square brackets in (4.8) is strictly positive which implies that $h_0''(\xi) = 0$, and this gives (4.6). Notice that Theorem 4.2 for $h = h_2$ implies that the denominator of the right-hand side of (4.6) is nonzero. □

Corollary 4.4. *Let w be a nonnegative integrable function with $\int_a^b w(x)dx = 1$. If f is a positive increasing concave nonlinear function on $[a,b]$, then for $0 < s \ne t \ne 1 \ne s$ there exists $\xi \in (0, 2\tilde{f}_i]$ such that*

$$\xi^{t-s} = \frac{s(s-1)}{t(t-1)} \frac{\int_0^1 (2r\tilde{f}_i)^t w\left[a(1-r)+br\right]dr - (1/(b-a))\int_a^b f^t(r)w(r)dr}{\int_0^1 (2r\tilde{f}_i)^s w\left[a(1-r)+br\right]dr - (1/(b-a))\int_a^b f^s(r)w(r)dr}. \tag{4.13}$$

Proof. Set $h_1(x) = x^t$ and $h_2(x) = x^s$, $t \ne s \ne 0, 1$ in (4.6), then we get (4.13). □

Remark 4.5. Since the function $\xi \mapsto \xi^{t-s}$ is invertible, then from (4.13) we have

$$0 < \left(\frac{s(s-1)}{t(t-1)} \frac{\int_0^1 (2r\tilde{f}_i)^t w[a(1-r)+br]dr - (1/(b-a))\int_a^b f^t(r)w(r)dr}{\int_0^1 (2r\tilde{f}_i)^s w[a(1-r)+br]dr - (1/(b-a))\int_a^b f^s(r)w(r)dr}\right)^{1/(t-s)} \le 2\tilde{f}_i\,. \tag{4.14}$$

In fact, similar result can also be given for (4.6). Namely, suppose that h_1''/h_2'' has inverse function. Then from (4.6), we have

$$\xi = \left(\frac{h_1''}{h_2''}\right)^{-1} \left(\frac{\int_0^1 h_1(2r\tilde{f}_i)w\left[a(1-r)+br\right]dr - (1/(b-a))\int_a^b h_1(f(t))w(t)dt}{\int_0^1 h_2(2r\tilde{f}_i)w\left[a(1-r)+br\right]dr - (1/(b-a))\int_a^b h_2(f(t))w(t)dt}\right). \tag{4.15}$$

So, we have that the expression on the right-hand side of (4.15) is also a mean.

By the inequality (4.14), we can consider

$$M_{t,s}(f;w) = \left(\frac{s(s-1)}{t(t-1)} \frac{\int_0^1 (2r\tilde{f}_i)^t w\left[a(1-r)+br\right]dr - (1/(b-a))\int_a^b f^t(r)w(r)dr}{\int_0^1 (2r\tilde{f}_i)^s w\left[a(1-r)+br\right]dr - (1/(b-a))\int_a^b f^s(r)w(r)dr}\right)^{1/(t-s)} \tag{4.16}$$

for $0 < t \neq s \neq 1 \neq t$ as means in broader sense. Moreover, we can extend these means in other cases.

Denote, $\mu(r) = w[a(1-r) + br]$ and $\nu(r) = w(r)$. So by limit, we have

$$\log M_{t,t}(f;w) = \frac{\int_0^1 (2r\tilde{f}_i)^t \log(2r\tilde{f}_i)\mu(r)\,dr - (1/(b-a))\int_a^b f^t(r)\log f(r)\nu(r)dr}{\int_0^1 (2r\tilde{f}_i)^t \mu(r)\,dr - (1/(b-a))\int_a^b f^t(r)\nu(r)\,dr} - \frac{2t-1}{t(t-1)}, \quad t \neq 0,1,$$

$$\log M_{0,0}(f;w) = \frac{(1/(b-a))\int_a^b \log^2 f(r)\nu(r)\,dr - \int_0^1 \log^2(2r\tilde{f}_i)\mu(r)\,dr}{(2/(b-a))\int_a^b \log f(r)\nu(r)\,dr - 2\int_0^1 \log(2r\tilde{f}_i)\mu(r)\,dr} - \frac{2\int_0^1 \log(2r\tilde{f}_i)\mu(r)dr - (2/(b-a))\int_a^b \log f(r)\nu(r)\,dr}{(2/(b-a))\int_a^b \log f(r)\nu(r)\,dr - 2\int_0^1 \log(2r\tilde{f}_i)\mu(r)\,dr},$$

$$\log M_{1,1}(f;w) = \frac{2\tilde{f}_i\int_0^1 r\log(2r\tilde{f}_i)\left(\log(2r\tilde{f}_i) - 2\right)\mu(r)\,dr - (1/(b-a))\int_a^b f(r)\log f(r)\left(\log f(r) - 2\right)\nu(r)\,dr}{2\int_0^1 2r\tilde{f}_i\log(2r\tilde{f}_i)\mu(r)\,dr - (2/(b-a))\int_a^b f(r)\log f(r)\nu(r)\,dr}. \tag{4.17}$$

In our next result, we prove that this new mean is monotonic.

Theorem 4.6. *Let $t \leq u$, $r \leq s$, then the following inequality is valid:*

$$M_{t,r}(f;w) \leq M_{u,s}(f;w). \tag{4.18}$$

Proof. Since Π_s is log-convex, therefore by (3.8) we get (4.18). □

Remark 4.7. If $w \equiv 1$, then the above means become

$$M_{t,s}(f;1) = \left(\frac{(1/2\tilde{f})\int_0^{2\tilde{f}}\varphi_t(y)dy - (1/(b-a))\int_a^b \varphi_t(f(x))dx}{(1/2\tilde{f})\int_0^{2\tilde{f}}\varphi_s(y)dy - (1/(b-a))\int_a^b \varphi_s(f(x))dx}\right)^{1/(t-s)}, \quad 0 < t \neq s,$$

$$\log M_{t,t}(f;1) = \frac{(2^t/(t+1))\tilde{f}^t\log\tilde{f} + \tilde{f}^t(2^t\log 2/(t+1)) - \tilde{f}^t(2^t/(t+1)^2) - (1/(b-a))\int_a^b f^t(x)\log f(x)dx}{t(t-1)\Delta_t(f)} - \frac{2t-1}{t(t-1)}, \quad t \neq 0,1,$$

$$\log M_{0,0}(f;1) = \frac{(1/(b-a))\int_a^b \log^2 f(x)dx + (2/(b-a))\int_a^b \log f(x)dx - \log^2\tilde{f} - 2\log 2\log\tilde{f} - \log^2 2}{2 + (2/(b-a))\int_a^b \log f(x)dx - 2\log(2\tilde{f})},$$

$$\log M_{1,1}(f;1) = \frac{\tilde{f}\log\tilde{f}\log(4\tilde{f}/e^3) + \tilde{f}(\log^2 2 - \log 8 + 3/2) - (1/(b-a))\int_a^b f(x)\log f(x)(\log f(x) - 2)dx}{2\tilde{f}\log(2\tilde{f}) - \tilde{f} - (2/(b-a))\int_a^b f(x)\log f(x)dx}. \tag{4.19}$$

In this way (4.18) for $w \equiv 1$ gives an extension of (2.12) (see Remark 3.6).

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