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Research Article

An Interchangeable Theorem of q-Integral

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We give a sufficient condition for the interchangeability of the order of sum and q-integral by using inequality technique. As the application of the theorem, some interesting results on the hypergeometric series are obtained.

1. Introduction and Some Lemmas

q-series, which are also called basic hypergeometric series, plays a very important role in many fields, such as affine root systems, Lie algebras and groups, number theory, orthogonal polynomials, and physics. Inequality technique is one of the useful tools in the study of special functions. There are many papers about it (see [1–6]). First, we recall some definitions, notations, and known results which will be used in this paper. Throughout this paper, it is supposed that $0 < q < 1$. The q-shifted factorials are defined as

$$\begin{aligned} (a;q)_0 &= 1, \\ (a;q)_n &= \prod_{k=0}^{n-1}\left(1-aq^k\right), \quad n = 1,2,\ldots, \\ (a;q)_\infty &= \prod_{k=0}^{\infty}\left(1-aq^k\right). \end{aligned} \tag{1.1}$$

We also adopt the following compact notation for multiple q-shifted factorial:

$$\left(a_1,a_2,\ldots,a_m;q\right)_n = \left(a_1;q\right)_n\left(a_2;q\right)_n\cdots\left(a_m;q\right)_n, \tag{1.2}$$

where n is an integer or ∞.

The q-binomial theorem [2] tells us that

$$\sum_{k=0}^{\infty} \frac{(a;q)_k z^k}{(q;q)_k} = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1. \tag{1.3}$$

Replace a with $1/a$, and z with az and then set $a = 0$, we get

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2} z^k}{(q;q)_k} = (z;q)_\infty. \tag{1.4}$$

Heine [2] introduced the basic hypergeometric series ${}_2\phi_1$, which is defined by

$${}_2\phi_1(a_1, a_2; b_1; q, z) = \sum_{k=0}^{\infty} \frac{(a_1, a_2; q)_k}{(q, b_1; q)_k} z^k. \tag{1.5}$$

Thomae [7] defined the q-integral on interval $[0,1]$ by

$$\int_0^1 f(t) d_q t = (1-q) \sum_{n=0}^{\infty} f(q^n) q^n, \tag{1.6}$$

provided that the series converges.

Fubini's theorem. Suppose that f_{ij} is absolutely summary, that is

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |f_{ij}| < \infty, \tag{1.7}$$

then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} f_{ij}. \tag{1.8}$$

In order to prove the main result, we need to introduce two lemmas.

Lemma 1.1. *Let* b *be a given real number, satisfying* $b < 1$*. Then, for* $0 \le x \le 1$*, one has*

$$\frac{1}{1 - bx} \le e^{(|b|/(1-b))x}. \tag{1.9}$$

Proof. Let

$$f(x) = (1 - bx) e^{(b/(1-b))x}, \tag{1.10}$$

since

$$f'(x) = \frac{b^2(1-x)}{1-b} e^{(b/(1-b))x} \geq 0, \quad 0 \leq x \leq 1. \tag{1.11}$$

$f(x)$ is monotonous increasing function with respect to $0 \leq x \leq 1$. Hence,

$$\frac{1}{1-bx} \leq e^{(b/(1-b))x}, \tag{1.12}$$

(1.9) is proved. □

Lemma 1.2. *Let* a_i, b_i *be some real numbers, satisfying* $b_i < 1$ *with* $i = 1, 2, \ldots, r$. *Then, for all nonnegative integer* n, *one has*

$$\left| \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_r; q)_n} \right| \leq e^{(1/(1-q))\sum_{i=1}^{r}(|a_i|+|b_i|/(1-b_i))}. \tag{1.13}$$

Proof. When $n = 0$, it is obvious that (1.13) holds; when $n \geq 1$, for $0 \leq x \leq 1$ and $1 \leq i \leq r$, we have

$$|1 - a_i x| \leq 1 + |a_i| x \leq e^{|a_i| x}, \tag{1.14}$$

and by Lemma 1.1, we have

$$\left| \frac{1 - a_i q^k}{1 - b_i q^k} \right| \leq e^{(|a_i|+|b_i|/(1-b_i))q^k}, \quad (k = 0, 1, 2, \ldots). \tag{1.15}$$

Consequently,

$$\left| \frac{(a_i; q)_n}{(b_i; q)_n} \right| \leq e^{(|a_i|+|b_i|/(1-b_i))(1+q+\cdots+q^{n-1})} \leq e^{(1/(1-q))(|a_i|+|b_i|/(1-b_i))} \quad (i = 1, 2, \ldots, r). \tag{1.16}$$

Thus, (1.13) follows. We complete the proof. □

2. Main Result and Its Proof

We know that, whether the order of sum and q-integral is interchangeable is an important problem in the study of q-series. We obtain following result on the interchangeability.

Theorem 2.1. *Let* a_i, b_i *be some real numbers, satisfying* $b_i < 1$ *with* $i = 1, 2, \ldots, r$. *Suppose real function* $f_n(t)$ *is* q*-integrable absolutely with* $n = 0, 1, \ldots$ *and series* $\sum_{n=0}^{\infty}\int_0^1 |f_n(t)| d_q t$ *is convergent. Then*

$$\int_0^1 \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_r; q)_n} f_n(t) d_q t = \sum_{n=0}^{\infty} \int_0^1 \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_r; q)_n} f_n(t) d_q t. \tag{2.1}$$

Proof. Using (1.13) and (1.6), we have

$$\begin{aligned}
&(1-q)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left|\frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_r; q)_n} f_n(q^m)\right| q^m \\
&\quad \le e^{(1/(1-q))\sum_{i=1}^{r}(|a_i|+|b_i|/(1-b_i))} \sum_{n=0}^{\infty}(1-q)\sum_{m=0}^{\infty}|f_n(q^m)|q^m \\
&\quad = e^{(1/(1-q))\sum_{i=1}^{r}(|a_i|+|b_i|/(1-b_i))} \sum_{n=0}^{\infty}\int_0^1 |f_n(t)| d_q t.
\end{aligned} \tag{2.2}$$

Since, the series $\sum_{n=0}^{\infty}\int_0^1 |f_n(t)| d_q t$ is convergent, the series

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_r; q)_n} f_n(q^m) q^m \tag{2.3}$$

is absolutely convergent. Hence, by the Fubini's theorem, we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_r; q)_n} f_n(q^m) q^m = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_r; q)_n} f_n(q^m) q^m. \tag{2.4}$$

From (2.4) and (1.6), (2.1) holds. The proof is completed. □

3. Applications

As the application of Theorem 2.1, in this section, we obtain some results. First, we give following lemma.

Lemma 3.1. *Let* a *be a real number, satisfying* $a < 1$. *Then, for all nonnegative integer* n, *one has*

$$\int_0^1 \frac{(qt; q)_\infty}{(at; q)_\infty} t^n \, d_q t = \frac{1-q}{1-a} \frac{(q; q)_n}{(aq; q)_n}. \tag{3.1}$$

Proof. By (1.3) and (1.6), we have

$$\begin{aligned}\frac{(aq^{n+1};q)_\infty}{(q^{n+1};q)_\infty} &= \sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}q^{(n+1)k}\\ &= \frac{(a;q)_\infty}{(1-q)(q;q)_\infty}(1-q)\sum_{k=0}^{\infty}\frac{(q^{k+1};q)_\infty}{(aq^k;q)_\infty}q^{nk}q^k\\ &= \frac{(a;q)_\infty}{(1-q)(q;q)_\infty}\int_0^1\frac{(qt;q)_\infty}{(at;q)_\infty}t^n\,d_qt\\ &= \frac{(a;q)_{n+1}}{(1-q)(q;q)_n}\frac{(aq^{n+1};q)_\infty}{(q^{n+1};q)_\infty}\int_0^1\frac{(qt;q)_\infty}{(at;q)_\infty}t^n\,d_qt.\end{aligned} \tag{3.2}$$

From (3.2), (3.1) holds. ☐

Theorem 3.2. *Let a, b be two real numbers, satisfying $a < 1,\ |b| < 1$. Then*

$$\sum_{n=0}^{\infty}\frac{(q^{n+1}, abq^{n+1};q)_\infty}{(aq^n, bq^n;q)_\infty}q^n = \frac{1}{(1-a)(1-b)}. \tag{3.3}$$

Proof. By (1.6), we have

$$\sum_{n=0}^{\infty}\frac{(q^{n+1}, abq^{n+1};q)_\infty}{(aq^n, bq^n;q)_\infty}q^n = \frac{1}{1-q}\int_0^1\frac{(qt, abqt;q)_\infty}{(at, bt;q)_\infty}d_qt. \tag{3.4}$$

By (1.3), we have

$$\int_0^1\frac{(qt, abqt;q)_\infty}{(at, bt;q)_\infty}d_qt = \int_0^1\frac{(qt;q)_\infty}{(at;q)_\infty}\sum_{m=0}^{\infty}\frac{(aq;q)_m}{(q;q)_m}(bt)^m\,d_qt. \tag{3.5}$$

Using Theorem 2.1, we have

$$\int_0^1\frac{(qt;q)_\infty}{(at;q)_\infty}\sum_{m=0}^{\infty}\frac{(aq;q)_m}{(q;q)_m}(bt)^m\,d_qt = \sum_{m=0}^{\infty}\frac{(aq;q)_m}{(q;q)_m}b^m\int_0^1\frac{(qt;q)_\infty}{(at;q)_\infty}t^m\,d_qt. \tag{3.6}$$

By Lemma 3.1, we have

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{(aq;q)_m}{(q;q)_m}b^m\int_0^1\frac{(qt;q)_\infty}{(at;q)_\infty}t^m\,d_qt &= \sum_{m=0}^{\infty}\frac{(aq;q)_m}{(q;q)_m}b^m\frac{(1-q)(q;q)_m}{(1-a)(aq;q)_m}\\ &= \sum_{m=0}^{\infty}b^m\frac{1-q}{1-a}\\ &= \frac{1-q}{(1-a)(1-b)}.\end{aligned}\tag{3.7}$$

Combining (3.4)–(3.7), (3.3) holds. □

In (3.5), replacing abq by c, we obtain the following result.

Corollary 3.3. *Let a, b, c be some real numbers, satisfying $a<1$, $|b|<1$. Then*

$$\int_0^1\frac{(qt,ct;q)_\infty}{(at,bt;q)_\infty}d_qt = \frac{1-q}{1-a}\sum_{m=0}^{\infty}\frac{(c/b;q)_m}{(aq;q)_m}b^m.\tag{3.8}$$

Corollary 3.4. *Let c be a real number. Then*

$$\sum_{n=0}^{\infty}(c/q;q)_nq^n = \sum_{n=0}^{\infty}\frac{(-1)^nq^{n(n-1)/2}}{(q;q)_{n+1}}c^n.\tag{3.9}$$

Proof. Taking $a=0,\ b=q$ in (3.8), we have

$$\int_0^1(ct;q)_\infty d_qt = (1-q)\sum_{n=0}^{\infty}(c/q;q)_nq^n.\tag{3.10}$$

On the other hand, by (1.4) and Theorem 2.1, we have

$$\begin{aligned}\int_0^1(ct;q)_\infty d_qt &= \int_0^1\sum_{n=0}^{\infty}\frac{(-1)^nq^{n(n-1)/2}(ct)^n}{(q;q)_n}d_qt\\ &= \sum_{n=0}^{\infty}\frac{(-1)^nq^{n(n-1)/2}c^n}{(q;q)_n}\int_0^1t^n\,d_qt\\ &= (1-q)\sum_{n=0}^{\infty}\frac{(-1)^nq^{n(n-1)/2}}{(q;q)_{n+1}}c^n,\end{aligned}\tag{3.11}$$

which by combining with (3.10), implies (3.9). □

Take $c=1$, (3.9) implies the following result.

Corollary 3.5. *The following equation holds:*

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q;q)_{n+1}} = q. \tag{3.12}$$

Take $c = 1/q$, (3.9) implies the following result.

Corollary 3.6. *The following equation holds:*

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-3)/2}}{(q;q)_{n+1}} = q^2. \tag{3.13}$$

Remark 3.7. Taking $c = q^{-k}$, where k is positive integer, (3.9) readily yields many equations.

Corollary 3.8. *Let a be a real number, satisfying $|a| < 1$. Then*

$$\sum_{n=0}^{\infty} \frac{q^n}{(a;q)_{n+1}} = \sum_{n=0}^{\infty} \frac{a^n}{(q;q)_{n+1}}. \tag{3.14}$$

Proof. Taking $b = q,\ c = 0$ in (3.8), we have

$$\int_0^1 \frac{1}{(at;q)_\infty} d_q t = \frac{1-q}{1-a} \sum_{n=0}^{\infty} \frac{q^n}{(aq;q)_n}. \tag{3.15}$$

On the other hand, by Theorem 2.1 and set $a = 0$ then replace z with at in (1.3), we have

$$\int_0^1 \frac{1}{(at;q)_\infty} d_q t = \int_0^1 \sum_{n=0}^{\infty} \frac{(at)^n}{(q;q)_n} d_q t = \sum_{n=0}^{\infty} \frac{a^n}{(q;q)_n} \frac{1-q}{1-q^{n+1}}. \tag{3.16}$$

Combining (3.15) and (3.16), (3.14) follows. □

Theorem 3.9. *Let $a,\ b$ be two real numbers, satisfying $|ab| < 1$. Then*

$${}_2\phi_1(q/a, q/b; q^2; q, ab) = \frac{1-q}{1-ab}\, {}_2\phi_1(a, b; abq; q, q). \tag{3.17}$$

Proof. We recall the Heines transformation formula [7]

$${}_2\phi_1(a, b; c; q, z) = \frac{(abz/c;q)_\infty}{(z;q)_\infty}\, {}_2\phi_1(c/a, c/b; c; q, abz/c). \tag{3.18}$$

In (3.18), replacing $c,\ z$ by $q,\ qt$, respectively, (3.18) yields

$$\frac{(qt;q)_\infty}{(abt;q)_\infty}\, {}_2\phi_1(a, b; q; q, qt) = {}_2\phi_1(q/a, q/b; q; q, abt). \tag{3.19}$$

Taking the q-integral on both sides of (3.19) with respect to variable t, we have

$$\int_0^1 \frac{(qt;q)_\infty}{(abt;q)_\infty}\,{}_2\phi_1(a,b;q;q,qt)d_qt = \int_0^1 {}_2\phi_1(q/a,q/b;q;q,abt)d_qt. \tag{3.20}$$

Applying (1.5) to (3.20) yields

$$\int_0^1 \frac{(qt;q)_\infty}{(abt;q)_\infty}\sum_{n=0}^{\infty}\frac{(a,b;q)_n}{(q,q;q)_n}(qt)^n d_qt = \int_0^1 \sum_{n=0}^{\infty}\frac{(q/a,q/b;q)_n}{(q,q;q)_n}(abt)^n d_qt. \tag{3.21}$$

Applying Theorem 2.1 and Lemma 3.1 to (3.21), we have

$$\sum_{n=0}^{\infty}\frac{(a,b;q)_n}{(q,q;q)_n}q^n\frac{1-q}{1-ab}\frac{(q;q)_n}{(abq;q)_n} = \sum_{n=0}^{\infty}\frac{(q/a,q/b;q)_n}{(q,q;q)_n}(ab)^n\frac{1-q}{1-q^{n+1}}, \tag{3.22}$$

hence,

$$\frac{1-q}{1-ab}\sum_{n=0}^{\infty}\frac{(a,b;q)_n}{(q,abq;q)_n}q^n = \sum_{n=0}^{\infty}\frac{(q/a,q/b;q)_n}{(q,q^2;q)_n}(ab)^n. \tag{3.23}$$

From (3.23) and (1.5), (3.17) follows. □

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