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Research Article

On the Symmetric Properties for the Generalized Twisted Bernoulli Polynomials

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We study the symmetry for the generalized twisted Bernoulli polynomials and numbers. We give some interesting identities of the power sums and the generalized twisted Bernoulli polynomials using the symmetric properties for the p-adic invariant integral.

1. Introduction

Let p be a fixed prime number. Throughout this paper, the symbols $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of rational integers, the ring of p-adic integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined as

$$I(f) = \int_{\mathbb{Z}_p} f(x)dx = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) \tag{1.1}$$

(see [1]). From (1.1), we note that

$$I(f_1) = I(f) + f'(0), \tag{1.2}$$

where $f'(0) = df(x)/dx|_{x=0}$ and $f_1(x) = f(x+1)$. For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$. Then we can derive the following equation from (1.2):

$$I(f_n) = I(f) + \sum_{i=0}^{n-1} f'(i) \tag{1.3}$$

(see [1–7]).

Let d be a fixed positive integer. For $n \in \mathbb{N}$, let

$$\begin{gathered} X = X_d = \varprojlim_N \mathbb{Z}/dp^N\mathbb{Z}, \quad X_1 = \mathbb{Z}_p, \\ X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\,\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p = \left\{ x \in X \mid x \equiv a \left(\bmod dp^N \right) \right\}, \end{gathered} \tag{1.4}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$. It is easy to see that

$$\int_X f(x)dx = \int_{\mathbb{Z}_p} f(x)dx, \quad \text{for } f \in UD(\mathbb{Z}_p). \tag{1.5}$$

The ordinary Bernoulli polynomials $B_n(x)$ are defined as

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1.6}$$

and the Bernoulli numbers B_n are defined as $B_n = B_n(0)$ (see [1–19]).

For $n \in \mathbb{N}$, let T_p be the p-adic locally constant space defined by

$$T_p = \bigcup_{n \ge 1} \mathbb{C}_{p^n} = \lim_{n \to \infty} \mathbb{C}_{p^n}, \tag{1.7}$$

where $\mathbb{C}_{p^n} = \{\omega \mid \omega^{p^n} = 1\}$ is the cyclic group of order p^n. It is well known that the twisted Bernoulli polynomials are defined as

$$\frac{t}{\xi e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\xi}(x) \frac{t^n}{n!}, \quad \xi \in T_p, \tag{1.8}$$

and the twisted Bernoulli numbers $B_{n,\xi}$ are defined as $B_{n,\xi} = B_{n,\xi}(0)$ (see [15–18]).

Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$. Then the generalized twisted Bernoulli polynomials $B_{n,\chi,\xi}(x)$ attached to χ are defined as follows:

$$\sum_{a=0}^{d-1} \frac{\chi(a)\xi^a e^{at} t}{\xi^d e^{dt} - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi,\xi}(x) \frac{t^n}{n!}, \quad \xi \in T_p. \tag{1.9}$$

The generalized twisted Bernoulli numbers attached to χ, $B_{n,\chi,\xi}$, are defined as $B_{n,\chi,\xi} = B_{n,\chi,\xi}(0)$ (see [16]).

Recently, many authors have studied the symmetric properties of the p-adic invariant integrals on $\mathbb{Z}_p$, which gave some interesting identities for the Bernoulli and the Euler polynomials (cf. [3, 6, 7, 13, 14, 20–27]). The authors of this paper have established various identities by the symmetric properties of the p-adic invariant integrals and investigated interesting relationships between the power sums and the Bernoulli polynomials (see [2, 3, 6, 7, 13]).

The twisted Bernoulli polynomials and numbers and the twisted Euler polynomials and numbers are very important in several fields of mathematics and physics(cf. [15–18]). The second author has been interested in the twisted Euler numbers and polynomials and the twisted Bernoulli polynomials and studied the symmetry of power sum and twisted Bernoulli polynomials (see [11–13]).

The purpose of this paper is to study the symmetry for the generalized twisted Bernoulli polynomials and numbers attached to χ. In Section 2, we give interesting identities for the power sums and the generalized twisted Bernoulli polynomials using the symmetric properties for the p-adic invariant integral.

2. Symmetry for the Generalized Twisted Bernoulli Polynomials

Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$. For $\xi \in T_p$, we have

$$\int_X \chi(x)\xi^x e^{xt}dx = \frac{t\sum_{i=0}^{d-1}\chi(i)\xi^i e^{it}}{\xi^d e^{dt}-1} = \sum_{n=0}^{\infty} B_{n,\chi,\xi}\frac{t^n}{n!}, \tag{2.1}$$

where $B_{n,\chi,\xi}$ are the nth generalized twisted Bernoulli numbers attached to χ. We also see that the generalized twisted Bernoulli polynomials attached to χ are given by

$$\int_X \chi(y)\xi^y e^{(x+y)t}dy = \frac{t\sum_{i=0}^{d-1}\chi(i)\xi^i e^{it}}{\xi^d e^{dt}-1}e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi,\xi}(x)\frac{t^n}{n!}. \tag{2.2}$$

By (2.1) and (2.2), we see that

$$\int_X \chi(x)\xi^x x^n dx = B_{n,\chi,\xi}, \qquad \int_X \chi(y)\xi^y (x+y)^n dy = B_{n,\chi,\xi}(x). \tag{2.3}$$

From (2.3), we derive that

$$B_{n,\chi,\xi}(x) = \sum_{l=0}^{n}\binom{n}{l}B_{l,\chi,\xi}x^{n-l}. \tag{2.4}$$

By (1.5) and (2.3), we see that

$$\int_X \chi(x)\xi^x e^{xt}dx = \frac{1}{d}\sum_{a=0}^{d-1}\chi(a)e^{at}\xi^a\int_{\mathbb{Z}_p}\xi^{dx}e^{dxt}dx = \frac{1}{d}\sum_{a=0}^{d-1}\chi(a)\xi^a\frac{dt}{\xi^d e^{dt}-1}e^{(a/d)dt}. \tag{2.5}$$

From (2.2) and (2.5), we obtain that

$$\int_X \chi(x)\xi^x e^{xt}dx = \sum_{n=0}^{\infty}\left\{d^{n-1}\sum_{a=0}^{d-1}\chi(a)\xi^a B_{n,\xi^d}\left(\frac{a}{d}\right)\right\}\frac{t^n}{n!}. \tag{2.6}$$

Thus we have the following theorem from (2.1) and (2.6).

Theorem 2.1. *For $\xi \in T_p$, one has*

$$B_{n,\chi,\xi} = d^{n-1}\sum_{a=0}^{d-1}\chi(a)\xi^a B_{n,\xi^d}\left(\frac{a}{d}\right). \tag{2.7}$$

By (1.3) and (1.5), we have that for $n \in \mathbb{N}$,

$$\int_X f(x+n)dx = \int_X f(x)dx + \sum_{i=0}^{n-1}f'(i), \tag{2.8}$$

where $f'(i) = df(x)/dx|_{x=i}$. Taking $f(x) = \chi(x)\xi^x e^{xt}$ in (2.8), it follows that

$$\begin{aligned}&\frac{1}{t}\left(\int_X \chi(x)\xi^{nd+x}e^{(nd+x)t}dx - \int_X \chi(x)\ \xi^x e^{xt}dx\right)\\ &\quad= \frac{nd\int_X \chi(x)\ \xi^x e^{xt}dx}{\int_X \xi^{ndx}e^{ndxt}dx} = \frac{\xi^{nd}e^{ndt}-1}{\xi^d e^{dt}-1}\left(\sum_{i=0}^{d-1}\chi(i)\xi^i e^{it}\right).\end{aligned} \tag{2.9}$$

Thus, we have

$$\frac{1}{t}\left(\int_X \chi(x)\xi^{nd+x}e^{(nd+x)t}dx - \int_X \chi(x)\xi^x e^{xt}dx\right) = \sum_{k=0}^{\infty}\left(\sum_{l=0}^{nd-1}\chi(l)\ \xi^l l^k\right)\frac{t^k}{k!}. \tag{2.10}$$

For $k \in \mathbb{Z}_+$, let us define the p-adic functional $K(\chi,\xi,k:n)$ as follows:

$$K(\chi,\xi,k:n) = \sum_{l=0}^{n}\chi(l)\xi^l l^k. \tag{2.11}$$

By (2.10) and (2.11), we see that for $k, n, d \in \mathbb{N}$,

$$\int_X \chi(x)\xi^{nd+x}(nd+x)^k dx - \int_X \chi(x)\xi^x x^k dx = kK(\chi,\xi,k-1:nd-1). \tag{2.12}$$

From (2.3) and (2.12), we have the following result.

Theorem 2.2. *For $\xi \in T_p$ and $k, n, d \in \mathbb{N}$, one has*

$$\xi^{nd} B_{k,\chi,\xi}(nd) - B_{k,\chi,\xi} = kK(\chi, \xi, k-1 : nd-1). \tag{2.13}$$

Let $w_1, w_2, d \in \mathbb{N}$. Then we have that

$$\begin{aligned}
&\frac{d\int_X \int_X \chi(x_1)\chi(x_2)\ \xi^{w_1x_1+w_2x_2} e^{(w_1x_1+w_2x_2)t} dx_1 dx_2}{\int_X \xi^{dw_1w_2x} e^{dw_1w_2xt} dx} \\
&\quad = \frac{t\left(\xi^{dw_1w_2} e^{dw_1w_2t} - 1\right)}{\left(\xi^{w_1d} e^{dw_1t} - 1\right)\left(\xi^{w_2d} e^{dw_2t} - 1\right)} \left(\sum_{a=0}^{d-1} \chi(a) \xi^{w_1a} e^{w_1at}\right)\left(\sum_{b=0}^{d-1} \chi(b) \xi^{w_2b} e^{w_2bt}\right).
\end{aligned} \tag{2.14}$$

By (2.9), (2.10), and (2.11), we see that

$$\frac{w_1 d\int_X \chi(x) \xi^x e^{xt} dx}{\int_X \xi^{dw_1x} e^{dw_1xt} dx} = \sum_{k=0}^{\infty} K(\chi, \xi, k : dw_1 - 1)\frac{t^k}{k!}. \tag{2.15}$$

Now let us define the p-adic functional $Y_{\chi,\xi}(w_1, w_2)$ as follows:

$$Y_{\chi,\xi}(w_1, w_2) = \frac{d\int_X \int_X \chi(x_1)\chi(x_2) \xi^{w_1x_1+w_2x_2} e^{(w_1x_1+w_2x_2+w_1w_2x)t} dx_1 dx_2}{\int_X \xi^{dw_1w_2x_3} e^{dw_1w_2x_3t} dx_3}. \tag{2.16}$$

Then it follows from (2.14) that

$$Y_{\chi,\xi}(w_1, w_2) = \frac{t\left(\xi^{dw_1w_2} e^{dw_1w_2t} - 1\right) e^{w_1w_2xt}}{\left(\xi^{w_1d} e^{dw_1t} - 1\right)\left(\xi^{w_2d} e^{dw_2t} - 1\right)} \left(\sum_{a=0}^{d-1} \chi(a) \xi^{w_1a} e^{w_1at}\right)\left(\sum_{b=0}^{d-1} \chi(b) \xi^{w_2b} e^{w_2bt}\right). \tag{2.17}$$

By (2.15) and (2.16), we obtain that

$$\begin{aligned}
Y_{\chi,\xi}(w_1, w_2) &= \left(\frac{1}{w_1}\int_X \chi(x_1) \xi^{w_1x_1} e^{w_1(x_1+w_2x)t} dx_1\right)\left(\frac{dw_1\int_X \chi(x_2) \xi^{w_2x_2} e^{w_2x_2t} dx_2}{\int_X \xi^{dw_1w_2x} e^{dw_1w_2xt} dx}\right) \\
&= \sum_{l=0}^{\infty}\left(\sum_{i=0}^{l}\binom{l}{i} B_{i,\chi,\xi^{w_1}}(w_2x) K\left(\chi, \xi^{w_2}, l-i : dw_1 - 1\right) w_1^{i-1} w_2^{l-i}\right)\frac{t^l}{l!}.
\end{aligned} \tag{2.18}$$

On the other hand, the symmetric property of $Y_{\chi,\xi}(w_1, w_2)$ shows that

$$Y_{\chi,\xi}(w_1, w_2) = \left(\frac{1}{w_2}\int_X \chi(x_2)\xi^{w_2x_2}e^{w_2(x_2+w_1x)t}dx_2\right)\left(\frac{dw_2\int_X \chi(x_1)\xi^{w_1x_1}e^{w_1x_1t}dx_1}{\int_X \xi^{dw_1w_2x}e^{dw_1w_2xt}dx}\right)$$
$$= \sum_{l=0}^{\infty}\left(\sum_{i=0}^{l}\binom{l}{i}B_{i,\chi,\xi^{w_2}}(w_1x)K\left(\chi,\xi^{w_1}, l-i : dw_2-1\right)w_2^{i-1}w_1^{l-i}\right)\frac{t^l}{l!}. \tag{2.19}$$

Comparing the coefficients on the both sides of (2.18) and (2.19), we have the following theorem.

Theorem 2.3. *Let $\xi \in T_p$ and $d, w_1, w_2 \in \mathbb{N}$. Then one has*

$$\sum_{i=0}^{l}\binom{l}{i}B_{i,\chi,\xi^{w_1}}(w_2x)K\left(\chi,\xi^{w_2}, l-i : dw_1-1\right)w_1^{i-1}w_2^{l-i}$$
$$= \sum_{i=0}^{l}\binom{l}{i}B_{i,\chi,\xi^{w_2}}(w_1x)K\left(\chi,\xi^{w_1}, l-i : dw_2-1\right)w_2^{i-1}w_1^{l-i}. \tag{2.20}$$

We also derive some identities for the generalized twisted Bernoulli numbers. Taking $x = 0$ in Theorem 2.3, we have the following corollary.

Corollary 2.4. *Let $\xi \in T_p$ and $d, w_1, w_2 \in \mathbb{N}$. Then one has*

$$\sum_{i=0}^{l}\binom{l}{i}B_{i,\chi,\xi^{w_1}}K\left(\chi,\xi^{w_2}, l-i : dw_1-1\right)w_1^{i-1}w_2^{l-i}=\sum_{i=0}^{l}\binom{l}{i}B_{i,\chi,\xi^{w_2}}K\left(\chi,\xi^{w_1}, l-i : dw_2-1\right)w_2^{i-1}w_1^{l-i}. \tag{2.21}$$

Now we will derive another identities for the generalized twisted Bernoulli polynomials using the symmetric property of $Y_{\chi,\xi}(w_1, w_2)$. From (1.2), (2.15) and (2.17), we see that

$$Y_{\chi,\xi}(w_1, w_2) = \left(\frac{e^{w_1w_2xt}}{w_1}\int_X \chi(x_1)\xi^{w_1x_1}e^{w_1x_1t}dx_1\right)\left(\frac{dw_1\int_X \chi(x_2)\xi^{w_2x_2}e^{w_2x_2t}dx_2}{\int_X \xi^{dw_1w_2x}e^{dw_1w_2xt}dx}\right)$$
$$= \frac{1}{w_1}\sum_{i=0}^{dw_1-1}\chi(i)\xi^{w_2i}\int_X \chi(x_1)\xi^{w_1x_1}e^{w_1(x_1+w_2x+(w_2/w_1)i)t}dx_1$$
$$= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{dw_1-1}\chi(i)\xi^{w_2i}B_{k,\chi,\xi^{w_1}}\left(w_2x+\frac{w_2}{w_1}i\right)w_1^{k-1}\right)\frac{t^k}{k!}. \tag{2.22}$$

From the symmetric property of $Y_{\chi,\xi}(w_1, w_2)$, we also see that

$$Y_{\chi,\xi}(w_1, w_2) = \left(\frac{e^{w_1 w_2 x t}}{w_2}\int_X \chi(x_2)\xi^{w_2 x_2} e^{w_2 x_2 t} dx_2\right)\left(\frac{d w_2 \int_X \chi(x_1)\xi^{w_1 x_1} e^{w_1 x_1 t} dx_1}{\int_X \xi^{d w_1 w_2 x} e^{d w_1 w_2 x t} dx}\right)$$
$$= \frac{1}{w_2}\sum_{i=0}^{dw_2-1}\chi(i)\xi^{w_1 i}\int_X \chi(x_2)\xi^{w_2 x_2} e^{w_2(x_2+w_1 x+(w_1/w_2)i)t} dx_2 \tag{2.23}$$
$$= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{dw_2-1}\chi(i)\xi^{w_1 i} B_{k,\chi,\xi^{w_2}}\left(w_1 x + \frac{w_1}{w_2} i\right) w_2^{k-1}\right)\frac{t^k}{k!}.$$

Comparing the coefficients on the both sides of (2.22) and (2.23), we obtain the following theorem.

Theorem 2.5. *Let $\xi \in T_p$ and $d, w_1, w_2 \in \mathbb{N}$. Then one has*

$$\sum_{i=0}^{dw_1-1}\chi(i)\xi^{w_2 i} B_{k,\chi,\xi^{w_1}}\left(w_2 x + \frac{w_2}{w_1} i\right) w_1^{k-1} = \sum_{i=0}^{dw_2-1}\chi(i)\xi^{w_1 i} B_{k,\chi,\xi^{w_2}}\left(w_1 x + \frac{w_1}{w_2} i\right) w_2^{k-1}. \tag{2.24}$$

If we take $x = 0$ in Theorem 2.5, we also derive the interesting identity for the generalized twisted Bernoulli numbers as follows: for $d, w_1, w_2 \in \mathbb{N}$,

$$\sum_{i=0}^{dw_1-1}\chi(i)\xi^{w_2 i} B_{k,\chi,\xi^{w_1}}\left(\frac{w_2}{w_1} i\right) w_1^{k-1} = \sum_{i=0}^{dw_2-1}\chi(i)\xi^{w_1 i} B_{k,\chi,\xi^{w_2}}\left(\frac{w_1}{w_2} i\right) w_2^{k-1}. \tag{2.25}$$

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References

[1] T. Kim, “q-Volkenborn integration,” *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.
[2] T. Kim, “On the symmetries of the q-Bernoulli polynomials,” *Abstract and Applied Analysis*, vol. 2008, Article ID 914367, 7 pages, 2008.
[3] T. Kim, “Symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials,” *Journal of Difference Equations and Applications*, vol. 14, no. 12, pp. 1267–1277, 2008.
[4] T. Kim, “On the multiple q-Genocchi and Euler numbers,” *Russian Journal of Mathematical Physics*, vol. 15, no. 4, pp. 481–486, 2008.
[5] T. Kim, “Note on q-Genocchi numbers and polynomials,” *Advanced Studies in Contemporary Mathematics*, vol. 17, no. 1, pp. 9–15, 2008.
[6] T. Kim, “Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on $\mathbb{Z}_p$,” *Russian Journal of Mathematical Physics*, vol. 16, no. 1, pp. 93–96, 2009.
[7] T. Kim, S.-H. Rim, and B. Lee, “Some identities of symmetry for the generalized Bernoulli numbers and polynomials,” *Abstract and Applied Analysis*, vol. 2009, Article ID 848943, 8 pages, 2009.
[8] L. Carlitz, “q-Bernoulli numbers and polynomials,” *Duke Mathematical Journal*, vol. 15, pp. 987–1000, 1948.

[9] M. Cenkci, Y. Simsek, and V. Kurt, "Further remarks on multiple p-adic q-L-function of two variables," *Advanced Studies in Contemporary Mathematics*, vol. 14, no. 1, pp. 49–68, 2007.
[10] A. S. Hegazi and M. Mansour, "A note on q-Bernoulli numbers and polynomials," *Journal of Nonlinear Mathematical Physics*, vol. 13, no. 1, pp. 9–18, 2006.
[11] Y.-H. Kim, "On the p-adic interpolation functions of the generalized twisted (h, q)-Euler numbers," *International Journal of Mathematical Analysis*, vol. 3, no. 18, pp. 897–904, 2008.
[12] Y.-H. Kim, W. Kim, and L.-C. Jang, "On the q-extension of Apostol-Euler numbers and polynomials," *Abstract and Applied Analysis*, vol. 2008, Article ID 296159, 10 pages, 2008.
[13] Y.-H. Kim and K.-W. Hwang, "Symmetry of power sum and twisted Bernoulli polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 2, pp. 127–133, 2009.
[14] B. A. Kupershmidt, "Reflection symmetries of q-Bernoulli polynomials," *Journal of Nonlinear Mathematical Physics*, vol. 12, pp. 412–422, 2005.
[15] Y. Simsek, "Theorems on twisted L-function and twisted Bernoulli numbers," *Advanced Studies in Contemporary Mathematics*, vol. 11, no. 2, pp. 205–218, 2005.
[16] Y. Simsek, "On p-adic twisted q-L-functions related to generalized twisted Bernoulli numbers," *Russian Journal of Mathematical Physics*, vol. 13, no. 3, pp. 340–348, 2006.
[17] Y. Simsek, "Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation functions," *Advanced Studies in Contemporary Mathematics*, vol. 16, no. 2, pp. 251–278, 2008.
[18] Y. Simsek, V. Kurt, and D. Kim, "New approach to the complete sum of products of the twisted (h, q)-Bernoulli numbers and polynomials," *Journal of Nonlinear Mathematical Physics*, vol. 14, no. 1, pp. 44–56, 2007.
[19] H. M. Srivastava, T. Kim, and Y. Simsek, "q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series," *Russian Journal of Mathematical Physics*, vol. 12, no. 2, pp. 241–268, 2005.
[20] K. T. Atanassov and M. V. Vassilev-Missana, "On one of Murthy-Ashbacher's conjectures related to Euler's totient function," *Proceedings of the Jangjeon Mathematical Society*, vol. 9, no. 1, pp. 47–49, 2006.
[21] T. Kim, "A note on some formulae for the q-Euler numbers and polynomials," *Proceedings of the Jangjeon Mathematical Society*, vol. 9, no. 2, pp. 227–232, 2006.
[22] T. Kim, "Note on Dedekind type DC sums," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 2, pp. 249–260, 2009.
[23] T. Kim, "A note on the generalized q-Euler numbers," *Proceedings of the Jangjeon Mathematical Society*, vol. 12, no. 1, pp. 45–50, 2009.
[24] T. Kim, "New approach to q-Euler, Genocchi numbers and their interpolation functions," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 2, pp. 105–112, 2009.
[25] T. Kim, "Symmetry identities for the twisted generalized Euler polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 19, no. 2, pp. 151–155, 2009.
[26] Y.-H. Kim, W. Kim, and C. S. Ryoo, "On the twisted q-Euler zeta function associated with twisted q-Euler numbers," *Proceedings of the Jangjeon Mathematical Society*, vol. 12, no. 1, pp. 93–100, 2009.
[27] H. Ozden, I. N. Cangul, and Y. Simsek, "Remarks on q-Bernoulli numbers associated with Daehee numbers," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 1, pp. 41–48, 2009.