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Research Article

Functional Equation $f(x) = pf(x-1) - qf(x-2)$ and Its Hyers-Ulam Stability

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We solve the functional equation, $f(x) = pf(x-1) - qf(x-2)$, and prove its Hyers-Ulam stability in the class of functions $f : \mathbb{R} \to X$, where X is a real (or complex) Banach space.

1. Introduction

In 1940, Ulam gave a wide-ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems [1]. Among those was the question concerning the stability of homomorphisms.

> Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given any $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In the following year, Hyers affirmatively answered in his paper [2] the question of Ulam for the case where G_1 and G_2 are Banach spaces.

Let $(G_1, \cdot)$ be a groupoid and let $(G_2, +)$ be a groupoid with the metric d. The equation of homomorphism

$$f(x \cdot y) = f(x) + f(y) \tag{1.1}$$

is stable in the Hyers-Ulam sense (or has the Hyers-Ulam stability) if for every $\delta > 0$ there exists an $\varepsilon > 0$ such that for every function $h : G_1 \to G_2$ satisfying

$$d[h(xy), h(x) + h(y)] \leq \varepsilon \tag{1.2}$$

for all $x, y \in G_1$ there exists a solution $g : G_1 \to G_2$ of the equation of homomorphism with

$$d[h(x), g(x)] \leq \delta \tag{1.3}$$

for any $x \in G_1$ (see [3, Definition 1]).

This terminology is also applied to the case of other functional equations. It should be remarked that we can find in the books [4–7] a lot of references concerning the stability of functional equations (see also [8–18]).

Throughout this paper, let p and q be fixed real numbers with $q \neq 0$ and $p^2 - 4q \neq 0$. By a and b we denote the distinct roots of the equation $x^2 - px + q = 0$. More precisely, we set

$$a = \frac{p + \sqrt{p^2 - 4q}}{2}, \qquad b = \frac{p - \sqrt{p^2 - 4q}}{2}. \tag{1.4}$$

Moreover, for any $n \in \mathbb{Z}$, we define

$$U_n = U_n(p, q) = \frac{a^n - b^n}{a - b}. \tag{1.5}$$

If p and q are integers, then $\{U_n(p, q)\}$ is called the Lucas sequence of the first kind. It is not difficult to see that

$$U_{n+2} = pU_{n+1} - qU_n \tag{1.6}$$

for any integer n. For any $x \in \mathbb{R}$, $[x]$ stands for the largest integer that does not exceed x.

In this paper, we will solve the functional equation

$$f(x) = pf(x - 1) - qf(x - 2) \tag{1.7}$$

and prove its Hyers-Ulam stability in the class of functions $f : \mathbb{R} \to X$, where X is a real (or complex) Banach space.

2. General Solution to (1.7)

In this section, let X be either a real vector space if $p^2 - 4q > 0$ or a complex vector space if $p^2 - 4q < 0$. In the following theorem, we investigate the general solution of the functional equation (1.7).

Theorem 2.1. *A function $f : \mathbb{R} \to X$ is a solution of the functional equation* (1.7) *if and only if there exists a function $h : [-1,1) \to X$ such that*

$$f(x) = U_{[x]+1}h(x-[x]) - qU_{[x]}h(x-[x]-1). \tag{2.1}$$

Proof. Since $a+b=p$ and $ab=q$, it follows from (1.7) that

$$\begin{aligned} f(x) - af(x-1) &= b\left[f(x-1) - af(x-2)\right], \\ f(x) - bf(x-1) &= a\left[f(x-1) - bf(x-2)\right]. \end{aligned} \tag{2.2}$$

By the mathematical induction, we can easily verify that

$$\begin{aligned} f(x) - af(x-1) &= b^n\left[f(x-n) - af(x-n-1)\right], \\ f(x) - bf(x-1) &= a^n\left[f(x-n) - bf(x-n-1)\right] \end{aligned} \tag{2.3}$$

for all $x \in \mathbb{R}$ and $n \in \{0,1,2,\ldots\}$. If we substitute $x+n$ $(n \geq 0)$ for x in (2.3) and divide the resulting equations by b^n, respectively, a^n, and if we substitute $-m$ for n in the resulting equations, then we obtain the equations in (2.3) with m in place of n, where $m \in \{0,-1,-2,\ldots\}$. Therefore, the equations in (2.3) are true for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

We multiply the first and the second equations of (2.3) by b and a, respectively. If we subtract the first resulting equation from the second one, then we obtain

$$f(x) = U_{n+1}f(x-n) - qU_nf(x-n-1) \tag{2.4}$$

for any $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

If we put $n = [x]$ in (2.4), then

$$f(x) = U_{[x]+1}f(x-[x]) - qU_{[x]}f(x-[x]-1) \tag{2.5}$$

for all $x \in \mathbb{R}$.

Since $0 \leq x-[x] < 1$ and $-1 \leq x-[x]-1 < 0$, if we define a function $h : [-1,1) \to X$ by $h := f|_{[-1,1)}$, then we see that f is a function of the form (2.1).

Now, we assume that f is a function of the form (2.1), where $h : [-1,1) \to X$ is an arbitrary function. Then, it follows from (2.1) that

$$\begin{aligned} f(x) &= U_{[x]+1}h(x-[x]) - qU_{[x]}h(x-[x]-1), \\ f(x-1) &= U_{[x]}h(x-[x]) - qU_{[x]-1}h(x-[x]-1), \\ f(x-2) &= U_{[x]-1}h(x-[x]) - qU_{[x]-2}h(x-[x]-1) \end{aligned} \tag{2.6}$$

for any $x \in \mathbb{R}$. Thus, by (1.6), we obtain

$$\begin{aligned} f(x) - pf(x-1) + qf(x-2) &= \left(U_{[x]+1} - pU_{[x]} + qU_{[x]-1}\right)h(x-[x]) \\ &\quad - q\left(U_{[x]} - pU_{[x]-1} + qU_{[x]-2}\right)h(x-[x]-1) \\ &= 0, \end{aligned} \tag{2.7}$$

which completes the proof. □

Remark 2.2. It should be remarked that the functional equation (1.7) is a particular case of the linear equation $\sum_{i=0}^{n} p_i f(g^i(x)) = 0$ with $g(x) = x-1$ and $n = 2$. Moreover, a substantial part of proof of Theorem 2.1 can be derived from theorems presented in the books [19, 20]. However, the theorems in [19, 20] deal with solutions of the linear equation under some regularity conditions, for example, the continuity, convexity, differentiability, analyticity and so on, while Theorem 2.1 deals with the general solution of (1.7) without regularity conditions.

3. Hyers-Ulam Stability of (1.7)

In this section, we denote by a and b the distinct roots of the equation $x^2-px+q = 0$ satisfying $|a| > 1$ and $0 < |b| < 1$. Moreover, let $(X, \|\cdot\|)$ be either a real Banach space if $p^2 - 4q > 0$ or a complex Banach space if $p^2 - 4q < 0$.

We can prove the Hyers-Ulam stability of the functional equation (1.7) as we see in the following theorem.

Theorem 3.1. *If a function $f : \mathbb{R} \to X$ satisfies the inequality*

$$\left\| f(x) - pf(x-1) + qf(x-2) \right\| \leq \varepsilon \tag{3.1}$$

for all $x \in \mathbb{R}$ and for some $\varepsilon \geq 0$, then there exists a unique solution function $F : \mathbb{R} \to X$ of the functional equation (1.7) *such that*

$$\left\| f(x) - F(x) \right\| \leq \frac{|a| - |b|}{|a-b|} \frac{\varepsilon}{(|a|-1)(1-|b|)} \tag{3.2}$$

for all $x \in \mathbb{R}$.

Proof. Analogously to the first equation of (2.2), it follows from (3.1) that

$$\left\| f(x) - af(x-1) - b\left[f(x-1) - af(x-2)\right] \right\| \leq \varepsilon \tag{3.3}$$

for each $x \in \mathbb{R}$. If we replace x by $x-k$ in the last inequality, then we have

$$\left\| f(x-k) - af(x-k-1) - b\left[f(x-k-1) - af(x-k-2)\right] \right\| \leq \varepsilon \tag{3.4}$$

and further

$$\left\| b^k \left[f(x-k) - af(x-k-1) \right] - b^{k+1} \left[f(x-k-1) - af(x-k-2) \right] \right\| \leq |b|^k \varepsilon \tag{3.5}$$

for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. By (3.5), we obviously have

$$\begin{aligned} &\| f(x) - af(x-1) - b^n [f(x-n) - af(x-n-1)] \| \\ &\quad \leq \sum_{k=0}^{n-1} \Big\| b^k [f(x-k) - af(x-k-1)] \\ &\qquad\qquad - b^{k+1} [f(x-k-1) - af(x-k-2)] \Big\| \\ &\quad \leq \sum_{k=0}^{n-1} |b|^k \varepsilon \end{aligned} \tag{3.6}$$

for $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

For any $x \in \mathbb{R}$, (3.5) implies that the sequence $\{b^n[f(x-n)-af(x-n-1)]\}$ is a Cauchy sequence (note that $0 < |b| < 1$.) Therefore, we can define a function $F_1 : \mathbb{R} \to X$ by

$$F_1(x) = \lim_{n \to \infty} b^n \left[f(x-n) - af(x-n-1) \right], \tag{3.7}$$

since X is complete. In view of the previous definition of F_1, we obtain

$$\begin{aligned} &pF_1(x-1) - qF_1(x-2) \\ &\quad = pb^{-1} \lim_{n \to \infty} b^{n+1} \left[f(x-(n+1)) - af(x-(n+1)-1) \right] \\ &\qquad - qb^{-2} \lim_{n \to \infty} b^{n+2} \left[f(x-(n+2)) - af(x-(n+2)-1) \right] \\ &\quad = pb^{-1} F_1(x) - qb^{-2} F_1(x) \\ &\quad = F_1(x) \end{aligned} \tag{3.8}$$

for all $x \in \mathbb{R}$, since $b^2 = pb - q$. If n goes to infinity, then (3.6) yields that

$$\| f(x) - af(x-1) - F_1(x) \| \leq \frac{\varepsilon}{1-|b|} \tag{3.9}$$

for every $x \in \mathbb{R}$.

On the other hand, it also follows from (3.1) that

$$\| f(x) - bf(x-1) - a\left[f(x-1) - bf(x-2) \right] \| \leq \varepsilon \tag{3.10}$$

(see the second equation in (2.2)). Analogously to (3.5), replacing x by $x+k$ in the previous inequality and then dividing by $|a|^k$ both sides of the resulting inequality, then we have

$$\left\| a^{-k}\left[f(x+k)-bf(x+k-1)\right]-a^{-k+1}\left[f(x+k-1)-bf(x+k-2)\right]\right\| \leq |a|^{-k}\varepsilon \tag{3.11}$$

for all $x\in\mathbb{R}$ and $k\in\mathbb{Z}$. By using (3.11), we further obtain

$$\begin{aligned}
&\left\| a^{-n}\left[f(x+n)-bf(x+n-1)\right]-\left[f(x)-bf(x-1)\right]\right\| \\
&\quad\leq \sum_{k=1}^{n}\left\| a^{-k}\left[f(x+k)-bf(x+k-1)\right]\right. \\
&\qquad\qquad \left. -\,a^{-k+1}\left[f(x+k-1)-bf(x+k-2)\right]\right\| \\
&\quad\leq \sum_{k=1}^{n}|a|^{-k}\varepsilon
\end{aligned} \tag{3.12}$$

for $x\in\mathbb{R}$ and $n\in\mathbb{N}$.

On account of (3.11), we see that the sequence $\{a^{-n}[f(x+n)-bf(x+n-1)]\}$ is a Cauchy sequence for any fixed $x\in\mathbb{R}$ (note that $|a|>1$.) Hence, we can define a function $F_2:\mathbb{R}\to X$ by

$$F_2(x)=\lim_{n\to\infty}a^{-n}\left[f(x+n)-bf(x+n-1)\right]. \tag{3.13}$$

Using the previous definition of F_2, we get

$$\begin{aligned}
&pF_2(x-1)-qF_2(x-2) \\
&\quad= pa^{-1}\lim_{n\to\infty}a^{-(n-1)}\left[f(x+n-1)-bf(x+(n-1)-1)\right] \\
&\qquad -\,qa^{-2}\lim_{n\to\infty}a^{-(n-2)}\left[f(x+n-2)-bf(x+(n-2)-1)\right] \\
&\quad= pa^{-1}F_2(x)-qa^{-2}F_2(x) \\
&\quad= F_2(x)
\end{aligned} \tag{3.14}$$

for any $x\in\mathbb{R}$. If we let n go to infinity, then it follows from (3.12) that

$$\left\|F_2(x)-f(x)+bf(x-1)\right\|\leq\frac{\varepsilon}{|a|-1} \tag{3.15}$$

for $x\in\mathbb{R}$.

By (3.9) and (3.15), we have

$$\begin{aligned}
&\left\| f(x) - \left[\frac{b}{b-a}F_1(x) - \frac{a}{b-a}F_2(x)\right] \right\| \\
&\quad = \frac{1}{|b-a|}\left\|(b-a)f(x) - [bF_1(x) - aF_2(x)]\right\| \\
&\quad \le \frac{1}{|a-b|}\left\|bf(x) - abf(x-1) - bF_1(x)\right\| \\
&\qquad + \frac{1}{|a-b|}\left\|aF_2(x) - af(x) + abf(x-1)\right\| \\
&\quad \le \frac{|a|-|b|}{|a-b|}\frac{\varepsilon}{(|a|-1)\ (1-|b|)}
\end{aligned} \tag{3.16}$$

for all $x \in \mathbb{R}$. We now define a function $F : \mathbb{R} \to X$ by

$$F(x) = \frac{b}{b-a}F_1(x) - \frac{a}{b-a}F_2(x) \tag{3.17}$$

for all $x \in \mathbb{R}$. Then, it follows from (3.8) and (3.14) that

$$\begin{aligned}
&pF(x-1) - qF(x-2) \\
&\quad = \frac{pb}{b-a}F_1(x-1) - \frac{pa}{b-a}F_2(x-1) - \frac{qb}{b-a}F_1(x-2) + \frac{qa}{b-a}F_2(x-2) \\
&\quad = \frac{b}{b-a}\left[pF_1(x-1) - qF_1(x-2)\right] - \frac{a}{b-a}\left[pF_2(x-1) - qF_2(x-2)\right] \\
&\quad = \frac{b}{b-a}F_1(x) - \frac{a}{b-a}F_2(x) \\
&\quad = F(x)
\end{aligned} \tag{3.18}$$

for each $x \in \mathbb{R}$; that is, F is a solution of (1.7). Moreover, by (3.16), we obtain the inequality (3.2).

Now, it only remains to prove the uniqueness of F. Assume that $F, G : \mathbb{R} \to X$ are solutions of (1.7) and that there exist positive constants C_1 and C_2 with

$$\|f(x) - F(x)\| \le C_1, \qquad \|f(x) - G(x)\| \le C_2 \tag{3.19}$$

for all $x \in \mathbb{R}$. According to Theorem 2.1, there exist functions $h, g : [-1, 1) \to X$ such that

$$\begin{aligned}
F(x) &= U_{[x]+1}h(x-[x]) - qU_{[x]}h(x-[x]-1), \\
G(x) &= U_{[x]+1}g(x-[x]) - qU_{[x]}g(x-[x]-1)
\end{aligned} \tag{3.20}$$

for any $x \in \mathbb{R}$, since F and G are solutions of (1.7).

Fix a t with $0 \le t < 1$. It then follows from (3.19) and (3.20) that

$$\begin{aligned}
&\left\| U_{n+1}[h(t)-g(t)] + U_n[qg(t-1)-qh(t-1)] \right\| \\
&\quad = \left\| [U_{n+1}h(t) - qU_n h(t-1)] - [U_{n+1}g(t) - qU_n g(t-1)] \right\| \\
&\quad = \|F(n+t) - G(n+t)\| \\
&\quad \le \|F(n+t) - f(n+t)\| + \|f(n+t) - G(n+t)\| \\
&\quad \le C_1 + C_2
\end{aligned} \tag{3.21}$$

for each $n \in \mathbb{Z}$, that is,

$$\left\| \frac{a^{n+1}-b^{n+1}}{a-b}[h(t)-g(t)] + \frac{a^n-b^n}{a-b}[qg(t-1)-qh(t-1)] \right\| \le C_1 + C_2 \tag{3.22}$$

for every $n \in \mathbb{Z}$. Dividing both sides by $|a|^n$ yields that

$$\left\| \frac{a-(b/a)^n b}{a-b}[h(t)-g(t)] + \frac{1-(b/a)^n}{a-b}[qg(t-1)-qh(t-1)] \right\| \le \frac{C_1+C_2}{|a|^n}, \tag{3.23}$$

and by letting $n \to \infty$, we obtain

$$a[h(t)-g(t)] + q[g(t-1)-h(t-1)] = 0. \tag{3.24}$$

Analogously, if we divide both sides of (3.22) by $|b|^n$ and let $n \to -\infty$, then we get

$$b[h(t)-g(t)] + q[g(t-1)-h(t-1)] = 0. \tag{3.25}$$

By (3.24) and (3.25), we have

$$\begin{pmatrix} a & q \\ b & q \end{pmatrix} \begin{pmatrix} h(t)-g(t) \\ g(t-1)-h(t-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.26}$$

Because $aq - bq \ne 0$ (where both a and b are nonzero and so $q = ab \ne 0$), it should hold that

$$h(t) - g(t) = g(t-1) - h(t-1) = 0 \tag{3.27}$$

for any $t \in [0,1)$, that is, $h(t) = g(t)$ for all $t \in [-1,1)$. Therefore, we conclude that $F(x) = G(x)$ for any $x \in \mathbb{R}$. (The presented proof of uniqueness of F is somewhat long and involved. Indeed, the referee has remarked that the uniqueness can be obtained directly from [21, Proposition 1].) □

Remark 3.2. The functional equation (1.7) is a particular case of the linear equations of higher orders and the Hyers-Ulam stability of the linear equations has been proved in [21, Theorem 2]. Indeed, Brzdęk et al. have proved an interesting theorem, from which the following corollary follows (see also [22, 23]):

Corollary 3.3. *Let a function* $f : \mathbb{R} \to X$ *satisfy the inequality* (3.1) *for all* $x \in \mathbb{R}$ *and for some* $\varepsilon \geq 0$ *and let* a, b *be the distinct roots of the equation* $x^2 - px + q = 0$*. If* $|a| > 1$*,* $0 < |b| < 1$ *and* $|b| \neq 1/2$*, then there exists a solution function* $F : \mathbb{R} \to X$ *of* (1.7) *such that*

$$\|f(x) - F(x)\| \leq \frac{4\varepsilon}{|2|a| - 1| \; |2|b| - 1|} \tag{3.28}$$

for all $x \in \mathbb{R}$*.*

If either $0 < |b| < 1/2$ and $|a| > 3/2 - |b|$ or $1/2 < |b| < 3/4$ and $|a| > (5 - 6|b|)/(6 - 8|b|)$, then

$$\frac{4\varepsilon}{|2|a| - 1| \; |2|b| - 1|} > \frac{\varepsilon}{(|a| - 1)(1 - |b|)} \geq \frac{|a| - |b|}{|a - b|} \frac{\varepsilon}{(|a| - 1)(1 - |b|)}. \tag{3.29}$$

Hence, the estimation (3.2) of Theorem 3.1 is better in these cases than the estimation (3.28).

Remark 3.4. As we know, $\{U_n(1, -1)\}_{n=1,2,\ldots}$ is the Fibonacci sequence. So if we set $p = 1$ and $q = -1$ in Theorems 2.1 and 3.1, then we obtain the same results as in [24, Theorems 2.1, 3.1, and 3.3].

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