

Research Article

On a Multiple Hilbert-Type Integral Operator and Applications

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By using the way of weight functions and the technic of real analysis, a multiple Hilbert-type integral operator with the homogeneous kernel of $-\lambda$ -degree ($\lambda \in \mathbf{R}$) and its norm are considered. As for applications, two equivalent inequalities with the best constant factors, the reverses, and some particular norms are obtained.

1. Introduction

If $p > 1, 1/p + 1/q = 1, f(\geq 0) \in L^p(0, \infty), g(\geq 0) \in L^q(0, \infty), \|f\|_p = \{\int_0^\infty f^p(x)dx\}^{1/p} > 0, \|g\|_q > 0$, then we have the following famous Hardy-Hilbert's integral inequality and its equivalent form (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \|f\|_p \|g\|_{q'} \quad (1.1)$$

$$\left\{ \int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx \right)^p dy \right\}^{1/p} < \frac{\pi}{\sin(\pi/p)} \|f\|_{p'} \quad (1.2)$$

where the constant factor $\pi / \sin(\pi/p)$ is the best possible. Define the Hardy-Hilbert's integral operator $T : L^p(0, \infty) \rightarrow L^p(0, \infty)$ as follows: for $f \in L^p(0, \infty), Tf(y) := \int_0^\infty (1/(x+y))f(x)dx$ ($y \in (0, \infty)$). Then in view of (1.2), it follows that $\|Tf\|_p < \pi / \sin(\pi/p) \|f\|_p$ and $\|T\| \leq (\pi / \sin(\pi/p))$. Since the constant factor in (1.2) is the best possible, we find that (cf. [2]) $\|T\| = \pi / \sin(\pi/p)$.

Inequalities (1.1) and (1.2) are important in analysis and its applications (cf. [3]). In 2002, reference [4] considered the property of Hardy-Hilbert's integral operator and gave an improvement of (1.1) (for $p = q = 2$). In 2004-2005, introducing another pair of conjugate exponents (r, s) ($r > 1, 1/r + 1/s = 1$) and an independent parameter $\lambda > 0$, [5, 6] gave two best extensions of (1.1) as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dx dy < \frac{\pi}{\lambda \sin(\pi/r)} \|f\|_{p,\phi} \|g\|_{q,\psi'} \quad (1.3)$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy < B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \|f\|_{p,\phi} \|g\|_{q,\psi'} \quad (1.4)$$

where $B(u, v)$ is the Beta function ($\phi(x) = x^{p(1-\lambda/r)-1}$, $\psi(x) = x^{q(1-\lambda/s)-1}$, $\|f\|_{p,\phi} := \{\int_0^\infty \phi(x) f^p(x) dx\}^{1/p} > 0$, $\|g\|_{q,\psi} > 0$). In 2009, [7, Theorem 9.1.1] gave the following multiple Hilbert-type integral inequality: suppose that $n \in \mathbf{N} \setminus \{1\}$, $p_i > 1$, $\sum_{i=1}^n (1/p_i) = 1$, $\lambda > 0$, then $k_\lambda(x_1, \dots, x_n) \geq 0$ is a measurable function of $-\lambda$ -degree in \mathbf{R}_+^n and for any (r_1, \dots, r_n) ($r_i > 1$) satisfies $\sum_{i=1}^n (1/r_i) = 1$ and

$$k_\lambda = \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1, \dots, u_{n-1}, 1) \prod_{j=1}^{n-1} u_j^{(\lambda/r_j)-1} du_1 \cdots du_{n-1} > 0. \quad (1.5)$$

If $\phi_i(x) = x^{p_i(1-\lambda/r_i)-1}$, $f_i(\geq 0) \in L_{\phi_i}^{p_i}(0, \infty)$, $\|f\|_{p_i,\phi_i} > 0$ ($i = 1, \dots, n$), then we have the following inequality:

$$\int_{\mathbf{R}_+^n} k_\lambda(x_1, \dots, x_n) \prod_{i=1}^n f_i(x_i) dx_1 \cdots dx_n < k_\lambda \prod_{i=1}^n \|f_i\|_{p_i,\phi_i'} \quad (1.6)$$

where the constant factor k_λ is the best possible. For $n = 2$, $k_\lambda(x, y) = 1/(x^\lambda + y^\lambda)$, and $1/(x+y)^\lambda$ in (1.6), we obtain (1.3) and (1.4). Inequality (1.6) is some extensions of the results in [6, 8–11]. In 2006, reference [12] also considered a multiple Hilbert-type integral operator with the homogeneous kernel of $-n+1$ -degree and its inequality with the norm, which is the best extension of (1.2).

In this paper, by using the way of weight functions and the technic of real analysis, a new multiple Hilbert-type integral operator with the norm is considered, which is an extension of the result in [12]. As for applications, an extended multiple Hilbert-type integral inequality and the equivalent form, the reverses, and some particular norms are obtained.

2. Some Lemmas

Lemma 2.1. *If $n \in \mathbf{N} \setminus \{1\}$, $p_i \in \mathbf{R} \setminus \{0, 1\}$, $\lambda_i \in \mathbf{R}$ ($i = 1, \dots, n$), $\sum_{i=1}^n 1/p_i = 1$, then*

$$A := \prod_{i=1}^n \left[x_i^{(\lambda_i-1)(1-p_i)} \prod_{j=1(j \neq i)}^n x_j^{\lambda_j-1} \right]^{1/p_i} = 1. \quad (2.1)$$

Proof. We find that

$$\begin{aligned}
 A &:= \prod_{i=1}^n \left[x_i^{(\lambda_i-1)(1-p_i)+1-\lambda_i} \prod_{j=1}^n x_j^{\lambda_j-1} \right]^{1/p_i} \\
 &= \prod_{i=1}^n \left[x_i^{(1-\lambda_i)p_i} \prod_{j=1}^n x_j^{\lambda_j-1} \right]^{1/p_i} = \prod_{i=1}^n x_i^{1-\lambda_i} \left(\prod_{j=1}^n x_j^{\lambda_j-1} \right)^{\sum_{i=1}^n (1/p_i)},
 \end{aligned} \tag{2.2}$$

and then (2.1) is valid. □

Definition 2.2. If $n \in \mathbf{N}$, $\mathbf{R}_+^n := \{(x_1, \dots, x_n) \mid x_i > 0 \ (i = 1, \dots, n)\}$, $\lambda \in \mathbf{R}$, and $k_\lambda(x_1, \dots, x_n)$ is a measurable function in \mathbf{R}_+^n such that for any $u > 0$ and $(x_1, \dots, x_n) \in \mathbf{R}_+^n$, $k_\lambda(ux_1, \dots, ux_n) = u^{-\lambda}k_\lambda(x_1, \dots, x_n)$, then call $k_\lambda(x_1, \dots, x_n)$ the homogeneous function of $-\lambda$ -degree in \mathbf{R}_+^n .

Lemma 2.3. *As for the assumption of Lemma 2.1, if $\sum_{i=1}^n \lambda_i = \lambda$, $k_\lambda(x_1, \dots, x_n) \geq 0$ is a homogeneous function of $-\lambda$ -degree in \mathbf{R}_+^n ,*

$$H(i) := \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1, \dots, u_{i-1}, 1, u_{i+1}, \dots, u_n) \prod_{j=1(j \neq i)}^n u_j^{\lambda_j-1} du_1 \cdots du_{i-1} du_{i+1} \cdots du_n \tag{2.3}$$

$(i = 1, \dots, n)$, and $H(n) = k_\lambda \in \mathbf{R}$, then each $H(i) = H(n) = k_\lambda (i = 1, \dots, n)$, and for any $i = 1, \dots, n$, it follows that

$$\omega_i(x_i) := x_i^{\lambda_i} \int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1, \dots, x_n) \prod_{j=1(j \neq i)}^n x_j^{\lambda_j-1} dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n = k_\lambda. \tag{2.4}$$

Proof. Setting $u_j = u_n v_j (j \neq i, n)$ in the integral $H(i)$, we find that

$$H(i) = \int_{\mathbf{R}_+^{n-1}} k_\lambda(v_1, \dots, v_{i-1}, u_n^{-1}, v_{i+1}, \dots, v_{n-1}, 1) \prod_{j=1(j \neq i)}^{n-1} v_j^{\lambda_j-1} u_n^{-1-\lambda_i} dv_1 \cdots dv_{i-1} dv_{i+1} \cdots dv_{n-1} du_n. \tag{2.5}$$

Setting $v_i = u_n^{-1}$ in the above integral, we obtain $H(i) = H(n)$. Setting $u_j = x_j/x_i (j \neq i)$ in (2.4), we find that $\omega_i(x_i) = H(i) = H(n) = k_\lambda$. □

Lemma 2.4. *As for the assumption of Lemma 2.3, setting*

$$k(\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-1}) := \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1, \dots, u_{n-1}, 1) \prod_{j=1}^{n-1} u_j^{\tilde{\lambda}_j-1} du_1 \cdots du_{n-1}, \tag{2.6}$$

then there exist $\delta_0 > 0$ and $I = \{(\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-1}) \mid \tilde{\lambda}_i = \lambda_i + \delta_i, |\delta_i| \leq \delta_0 \ (i = 1, \dots, n-1)\}$, such that for any $(\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-1}) \in I, k(\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-1}) \in \mathbf{R}$, if and only if $k(\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-1})$ is continuous at $(\lambda_1, \dots, \lambda_{n-1})$.

Proof. The sufficiency property is obvious. We prove the necessary property of the condition by mathematical induction. For $n = 2$, since

$$\begin{aligned} k(\lambda_1 + \delta_1) &= \int_0^1 k_\lambda(u_1, 1) u_1^{\lambda_1 + \delta_1 - 1} du_1 + \int_1^\infty k_\lambda(u_1, 1) u_1^{\lambda_1 + \delta_1 - 1} du_1, \\ k_\lambda(u_1, 1) u_1^{\lambda_1 + \delta_1 - 1} &\leq k_\lambda(u_1, 1) u_1^{\lambda_1 - \delta_0 - 1} du_1, \quad u_1 \in (0, 1], \\ k_\lambda(u_1, 1) u_1^{\lambda_1 + \delta_1 - 1} &\leq k_\lambda(u_1, 1) u_1^{\lambda_1 + \delta_0 - 1} du_1, \quad u_1 \in (1, \infty), \end{aligned} \quad (2.7)$$

and $k(\lambda_1 - \delta_0) + k(\lambda_1 + \delta_0) < \infty$, then by Lebesgue control convergence theorem (cf. [13]), it follows that $k(\lambda_1 + \delta_1) = k(\lambda_1) + o(1)(\delta_1 \rightarrow 0)$. Assuming that for $n(\geq 2), k(\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-1})$ is continuous at $(\lambda_1, \dots, \lambda_{n-1})$, then for $n + 1$, in view of the result for $n = 2$, we have that

$$\begin{aligned} &\lim_{\delta_n \rightarrow 0} k(\lambda_1 + \delta_1, \dots, \lambda_n + \delta_n) \\ &= \lim_{\delta_n \rightarrow 0} \int_0^\infty \left(\int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1, \dots, u_n, 1) \prod_{j=1}^{n-1} u_j^{\lambda_j + \delta_j - 1} du_1 \cdots du_{n-1} \right) u_n^{\lambda_n + \delta_n - 1} du_n \\ &= \int_0^\infty \left(\int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1, \dots, u_n, 1) \prod_{j=1}^{n-1} u_j^{\lambda_j + \delta_j - 1} du_1 \cdots du_{n-1} \right) u_n^{\lambda_n - 1} du_n \\ &= \int_{\mathbf{R}_+^{n-1}} \left(\int_0^\infty k_\lambda(u_1, \dots, u_n, 1) u_n^{\lambda_n - 1} du_n \right) \prod_{j=1}^{n-1} u_j^{\lambda_j + \delta_j - 1} du_1 \cdots du_{n-1}, \end{aligned} \quad (2.8)$$

then by the assumption for n , it follows that

$$\lim_{\delta_i \rightarrow 0} k(\lambda_1 + \delta_1, \dots, \lambda_n + \delta_n) = k(\lambda_1, \dots, \lambda_n) + o(1) \quad (\delta_i \rightarrow 0, i = 1, \dots, n-1). \quad (2.9)$$

By mathematical induction, we prove that for $n \in \mathbf{N} \setminus \{1\}, k(\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-1})$ is continuous at $(\lambda_1, \dots, \lambda_{n-1})$. \square

Lemma 2.5. *As for the assumption of Lemma 2.4, if $0 < \varepsilon < \min_{1 \leq i \leq n} \{ |p_i| \} \delta_0$, then for $\varepsilon \rightarrow 0^+$,*

$$I_\varepsilon := \varepsilon \int_1^\infty \cdots \int_1^\infty k_\lambda(x_1, \dots, x_n) \prod_{j=1}^n x_j^{\lambda_j - \varepsilon/p_j - 1} dx_1 \cdots dx_n = k_\lambda + o(1). \quad (2.10)$$

Proof. Setting $u_j = x_j/x_n$ ($j = 1, \dots, n - 1$), we find that

$$I_\varepsilon = \varepsilon \int_1^\infty x_n^{-1-\varepsilon} \left[\int_{x_n^{-1}}^\infty \cdots \int_{x_n^{-1}}^\infty k_\lambda(u_1, \dots, u_{n-1}, 1) \prod_{j=1}^{n-1} u_j^{\lambda_j - \varepsilon/p_j - 1} du_1 \cdots du_{n-1} \right] dx_n. \tag{2.11}$$

Setting $D_j := \{(u_1, \dots, u_{n-1}) \mid u_j \in (0, x_n^{-1}), u_k \in (0, \infty) (k \neq j)\}$ and

$$A_j(x_n) := \int \cdots \int_{D_j} k_\lambda(u_1, \dots, u_{n-1}, 1) \prod_{j=1}^{n-1} u_j^{\lambda_j - \varepsilon/p_j - 1} du_1 \cdots du_{n-1}, \tag{2.12}$$

then by (2.11), it follows that

$$I_\varepsilon \geq \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1, \dots, u_{n-1}, 1) \prod_{j=1}^{n-1} u_j^{\lambda_j - \varepsilon/p_j - 1} du_1 \cdots du_{n-1} - \varepsilon \sum_{j=1}^{n-1} \int_1^\infty x_n^{-1} A_j(x_n) dx_n. \tag{2.13}$$

Without loses of generality, we estimate that $\int_1^\infty x_n^{-1} A_{n-1}(x_n) dx_n = O(1)$. In fact, setting $\alpha > 0$ such that $|\varepsilon/(p_{n-1}) + \alpha| < \delta_0$, since $-u_{n-1}^\alpha \ln u_{n-1} \rightarrow 0$ ($u_{n-1} \rightarrow 0^+$), there exists $M > 0$, such that $-u_{n-1}^\alpha \ln u_{n-1} \leq M(u_{n-1} \in (0, 1])$, and then by Fubini theorem, it follows that

$$\begin{aligned} 0 &\leq \int_1^\infty x_n^{-1} A_{n-1}(x_n) dx_n \\ &= \int_1^\infty x_n^{-1} \left[\int_{\mathbf{R}_+^{n-2}} \int_0^{x_n^{-1}} k_\lambda(u_1, \dots, u_{n-1}, 1) \prod_{j=1}^{n-1} u_j^{\lambda_j - \varepsilon/p_j - 1} du_{n-1} du_1 \cdots du_{n-2} \right] dx_n \\ &= \int_0^1 \int_{\mathbf{R}_+^{n-2}} k_\lambda(u_1, \dots, u_{n-1}, 1) \prod_{j=1}^{n-1} u_j^{\lambda_j - \varepsilon/p_j - 1} \left(\int_1^{u_{n-1}^{-1}} x_n^{-1} dx_n \right) du_1 \cdots du_{n-1} \\ &= \int_0^1 \int_{\mathbf{R}_+^{n-2}} k_\lambda(u_1, \dots, u_{n-1}, 1) \prod_{j=1}^{n-1} u_j^{\lambda_j - \varepsilon/p_j - 1} (-\ln u_{n-1}) du_1 \cdots du_{n-1} \\ &\leq M \int_0^1 \int_{\mathbf{R}_+^{n-2}} k_\lambda(u_1, \dots, u_{n-1}, 1) \prod_{j=1}^{n-2} u_j^{\lambda_j - \varepsilon/p_j - 1} u_{n-1}^{\lambda_{n-1} - (\varepsilon/p_{n-1} + \alpha) - 1} du_1 \cdots du_{n-1} \\ &\leq M \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1, \dots, u_{n-1}, 1) \prod_{j=1}^{n-2} u_j^{\lambda_j - \varepsilon/p_j - 1} u_{n-1}^{\lambda_{n-1} - (\varepsilon/p_{n-1} + \alpha) - 1} du_1 \cdots du_{n-1} \\ &= M \cdot k \left(\lambda_1 - \frac{\varepsilon}{p_1}, \dots, \lambda_{n-2} - \frac{\varepsilon}{p_{n-2}}, \lambda_{n-1} - \left(\frac{\varepsilon}{p_{n-1}} + \alpha \right) \right) < \infty. \end{aligned} \tag{2.14}$$

Hence by (2.13), we have that

$$I_\varepsilon \geq \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1, \dots, u_{n-1}, 1) \prod_{j=1}^{n-1} u_j^{\lambda_j - \varepsilon/p_j - 1} du_1 \cdots du_{n-1} - o_1(1). \quad (2.15)$$

By Lemma 2.4, we find that

$$\begin{aligned} I_\varepsilon &\leq \varepsilon \int_1^\infty x_n^{-1-\varepsilon} \left[\int_0^\infty \cdots \int_0^\infty k_\lambda(u_1, \dots, u_{n-1}, 1) \right. \\ &\quad \left. \times \prod_{j=1}^{n-1} u_j^{\lambda_j - \varepsilon/p_j - 1} du_1 \cdots du_{n-1} \right] dx_n \\ &= \int_0^\infty \cdots \int_0^\infty k_\lambda(u_1, \dots, u_{n-1}, 1) \prod_{j=1}^{n-1} u_j^{\lambda_j - \varepsilon/p_j - 1} du_1 \cdots du_{n-1} \\ &= k\left(\lambda_1 - \frac{\varepsilon}{p_1}, \dots, \lambda_{n-1} - \frac{\varepsilon}{p_{n-1}}\right) = k_\lambda + o_2(1), \end{aligned} \quad (2.16)$$

Then by combination with (2.15), we have (2.10). \square

Lemma 2.6. *Suppose that $n \in \mathbf{N} \setminus \{1\}$, $p_1 \in \mathbf{R}_+ \setminus \{1\}$, $\sum_{i=1}^n (1/p_i) = 1$, $1/q_n = 1 - 1/p_n$, $(\lambda_1, \dots, \lambda_n) \in \mathbf{R}^n$, $\sum_{i=1}^n \lambda_i = \lambda$, then $k_\lambda(x_1, \dots, x_n) \geq 0$ is a measurable function of $-\lambda$ -degree in \mathbf{R}_+^n such that*

$$k_\lambda = \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1, \dots, u_{n-1}, 1) \prod_{j=1}^{n-1} u_j^{\lambda_j - 1} du_1 \cdots du_{n-1} \in \mathbf{R}. \quad (2.17)$$

If $f_i \geq 0$ are measurable functions in \mathbf{R}_+ ($i = 1, \dots, n-1$), then (1) for $p_i > 1$ ($i = 1, \dots, n$),

$$\begin{aligned} J &:= \left\{ \int_0^\infty x_n^{q_n \lambda_n - 1} \left[\int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1, \dots, x_n) \prod_{i=1}^{n-1} f_i(x_i) dx_1 \cdots dx_{n-1} \right]^{q_n} dx_n \right\}^{1/q_n} \\ &\leq k_\lambda \prod_{i=1}^{n-1} \left\{ \int_0^\infty x^{p_i(1-\lambda_i) - 1} f^{p_i}(x) dx \right\}^{1/p_i}, \end{aligned} \quad (2.18)$$

(2) for $0 < p_1 < 1$, $p_i < 0$ ($i = 2, \dots, n$), the reverse of (2.18) is obtained.

Proof. (1) For $p_i > 1$ ($i = 1, \dots, n$), by Hölder's inequality (cf. [14]) and (2.4), it follows that

$$\begin{aligned}
& \left[\int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1, \dots, x_n) \prod_{i=1}^{n-1} f_i(x_i) dx_1 \cdots dx_{n-1} \right]^{q_n} \\
&= \left\{ \int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1, \dots, x_n) \prod_{i=1}^{n-1} \left[x_i^{(\lambda_i-1)(1-p_i)} \prod_{j=1(j \neq i)}^n x_j^{\lambda_j-1} \right]^{1/p_i} f_i(x_i) \right. \\
&\quad \left. \times \left[x_n^{(\lambda_n-1)(1-p_n)} \prod_{j=1}^{n-1} x_j^{\lambda_j-1} \right]^{1/p_n} dx_1 \cdots dx_{n-1} \right\}^{q_n} \\
&\leq \int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1, \dots, x_n) \prod_{i=1}^{n-1} \left[x_i^{(\lambda_i-1)(1-p_i)} \prod_{j=1(j \neq i)}^n x_j^{\lambda_j-1} \right]^{q_n/p_i} \\
&\quad \times f_i^{q_n}(x_i) dx_1 \cdots dx_{n-1} \\
&\quad \times \left\{ \int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1, \dots, x_n) x_n^{(\lambda_n-1)(1-p_n)} \prod_{j=1}^{n-1} x_j^{\lambda_j-1} dx_1 \cdots dx_{n-1} \right\}^{q_n-1} \\
&= (k_\lambda)^{q_n-1} x_n^{1-q_n \lambda_n} \int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1, \dots, x_n) \\
&\quad \times \prod_{i=1}^{n-1} \left[x_i^{(\lambda_i-1)(1-p_i)} \prod_{j=1(j \neq i)}^n x_j^{\lambda_j-1} \right]^{q_n/p_i} f_i^{q_n}(x_i) dx_1 \cdots dx_{n-1}, \\
J &\leq (k_\lambda)^{1/p_n} \left\{ \int_0^\infty \int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1, \dots, x_n) \right. \\
&\quad \left. \times \prod_{i=1}^{n-1} \left[x_i^{(\lambda_i-1)(1-p_i)} \prod_{j=1(j \neq i)}^n x_j^{\lambda_j-1} \right]^{q_n/p_i} f_i^{q_n}(x_i) dx_1 \cdots dx_{n-1} dx_n \right\}^{1/q_n} \\
&= (k_\lambda)^{1/p_n} \left\{ \int_{\mathbf{R}_+^{n-1}} \left(\int_0^\infty k_\lambda(x_1, \dots, x_n) x_n^{\lambda_n-1} dx_n \right) \right. \\
&\quad \left. \times \prod_{i=1}^{n-1} \left[x_i^{(\lambda_i-1)(1-p_i)} \prod_{j=1(j \neq i)}^{n-1} x_j^{\lambda_j-1} \right]^{q_n/p_i} f_i^{q_n}(x_i) dx_1 \cdots dx_{n-1} \right\}^{1/q_n}.
\end{aligned} \tag{2.20}$$

For $n \geq 3$, by Hölder's inequality again, it follows that

$$\begin{aligned}
 J &\leq (k_\lambda)^{1/p_n} \left\{ \prod_{i=1}^{n-1} \left[\int_{\mathbf{R}_+^{n-1}} \left(\int_0^\infty k_\lambda(x_1, \dots, x_n) x_n^{\lambda_n-1} dx_n \right) \right. \right. \\
 &\quad \left. \left. \times x_i^{(\lambda_i-1)(1-p_i)} \prod_{j=1(j \neq i)}^{n-1} x_j^{\lambda_j-1} f_i^{p_i}(x_i) dx_1 \cdots dx_{n-1} \right]^{q_n/p_i} \right\}^{1/q_n} \\
 &= (k_\lambda)^{1/p_n} \prod_{i=1}^{n-1} \left\{ \int_0^\infty \left[\int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1, \dots, x_n) \right. \right. \\
 &\quad \left. \left. \times x_i^{\lambda_i} \prod_{j=1(j \neq i)}^n x_j^{\lambda_j-1} dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n \right] x_i^{p_i(1-\lambda_i)-1} f_i^{p_i}(x_i) dx_i \right\}^{1/p_i} \\
 &= (k_\lambda)^{1/p_n} \prod_{i=1}^{n-1} \left\{ \int_0^\infty \omega_i(x_i) x_i^{p_i(1-\lambda_i)-1} f_i^{p_i}(x_i) dx_i \right\}^{1/p_i}.
 \end{aligned} \tag{2.21}$$

Then by (2.4), we have (2.18) (note that for $n = 2$, we do not use Hölder's inequality again).
 (2) For $0 < p_1 < 1, p_i < 0$ ($i = 2, \dots, n$), by the reverse Hölder's inequality and the same way, we obtain the reverses of (2.18). \square

3. Main Results and Applications

As for the assumption of Lemma 2.6, setting $\phi_i(x) := x^{p_i(1-\lambda_i)-1}$ ($x \in (0, \infty); i = 1, \dots, n$), then we find that $\phi_n^{1/(1-p_n)}(x) = x^{q_n \lambda_n - 1}$. If $p_i > 1$ ($i = 1, \dots, n$), then define the following real function spaces:

$$\begin{aligned}
 L_{\phi_i}^{p_i}(0, \infty) &:= \left\{ f; \|f\|_{p_i, \phi_i} = \left\{ \int_0^\infty \phi_i(x) |f(x)|^{p_i} dx \right\}^{1/p_i} < \infty \right\} \quad (i = 1, \dots, n), \\
 \prod_{i=1}^{n-1} L_{\phi_i}^{p_i}(0, \infty) &:= \left\{ (f_1, \dots, f_{n-1}); f_i \in L_{\phi_i}^{p_i}(0, \infty), i = 1, \dots, n-1 \right\},
 \end{aligned} \tag{3.1}$$

and a multiple Hilbert-type integral operator $T : \prod_{i=1}^{n-1} L_{\phi_i}^{p_i}(0, \infty) \rightarrow L_{\phi_n}^{q_n}$ as follows: for $f = (f_1, \dots, f_{n-1}) \in \prod_{i=1}^{n-1} L_{\phi_i}^{p_i}(0, \infty)$,

$$(Tf)(x_n) := \int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1, \dots, x_n) \prod_{i=1}^{n-1} f_i(x_i) dx_1 \cdots dx_{n-1}, \quad x_n \in (0, \infty). \tag{3.2}$$

Then by (2.18), it follows that $Tf \in L_{\phi_n}^{q_n, 1/(1-p_n)}$, T is bounded, $\|Tf\|_{q_n, \phi_n^{1/(1-p_n)}} \leq k_\lambda \prod_{i=1}^{n-1} \|f_i\|_{p_i, \phi_i}$, and $\|T\| \leq k_\lambda$, where

$$\|T\| := \sup_{f \in \prod_{i=1}^{n-1} L_{\phi_i}^{p_i}(0, \infty) (f_i \neq \theta, i=1, \dots, n-1)} \frac{\|Tf\|_{q_n, \phi_n^{1/(1-p_n)}}}{\prod_{i=1}^{n-1} \|f_i\|_{p_i, \phi_i}}. \tag{3.3}$$

Define the formal inner product of $T(f_1, \dots, f_{n-1})$ and f_n as

$$(T(f_1, \dots, f_{n-1}), f_n) := \int_{\mathbf{R}_+^n} k_\lambda(x_1, \dots, x_n) \prod_{i=1}^n f_i(x_i) dx_1 \cdots dx_n. \tag{3.4}$$

Theorem 3.1. *Suppose that $n \in \mathbf{N} \setminus \{1\}$, $p_1 \in \mathbf{R}_+ \setminus \{1\}$, $\sum_{i=1}^n (1/p_i) = 1$, $1/q_n = 1 - 1/p_n$, then $k_\lambda(x_1, \dots, x_n) \geq 0$ is a measurable function of $-\lambda$ -degree in \mathbf{R}_+^n , and for any $(\lambda_1, \dots, \lambda_n) \in \mathbf{R}^n$, it satisfies $\sum_{i=1}^n \lambda_i = \lambda$ and*

$$k_\lambda = \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1, \dots, u_{n-1}, 1) \prod_{j=1}^{n-1} u_j^{\lambda_j-1} du_1 \cdots du_{n-1} > 0. \tag{3.5}$$

If $f_i(\geq 0) \in L_{\phi_i}^{p_i}(0, \infty)$, $\|f\|_{p_i, \phi_i} > 0 (i = 1, \dots, n)$, then (1) for $p_i > 1 (i = 1, \dots, n)$, $\|T\| = k_\lambda$ and the following equivalent inequalities are obtained:

$$\|T(f_1, \dots, f_{n-1})\|_{q_n, \phi_n^{1/(1-p_n)}} < k_\lambda \prod_{i=1}^{n-1} \|f_i\|_{p_i, \phi_i}, \tag{3.6}$$

$$(T(f_1, \dots, f_{n-1}), f_n) < k_\lambda \prod_{i=1}^n \|f_i\|_{p_i, \phi_i}, \tag{3.7}$$

where the constant factor k_λ is the best possible; (2) for $0 < p_1 < 1, p_i < 0 (i = 2, \dots, n)$, using the formal symbols of the case in $p_i > 1 (i = 1, \dots, n)$, the equivalent reverses of (3.6) and (3.7) with the best constant factor are given.

Proof. (1) For $p_i > 1 (i = 1, \dots, n)$, if (2.18) takes the form of equality, then for $n \geq 3$ in (2.21), there exist constants C_i and $C_k (i \neq k)$ such that they are not all zero and

$$\begin{aligned} & C_i x_i^{(\lambda_i-1)(1-p_i)} \prod_{j=1(j \neq i)}^{n-1} x_j^{\lambda_j-1} f_i^{p_i}(x_i) \\ &= C_k x_k^{(\lambda_k-1)(1-p_k)} \prod_{j=1(j \neq k)}^{n-1} x_j^{\lambda_j-1} f_k^{p_k}(x_k) \text{ a.e. in } \mathbf{R}_+^n, \end{aligned} \tag{3.8}$$

viz. $C_i x_i^{p_i(1-\lambda_i)} f_i^{p_i}(x_i) = C_k x_k^{p_k(1-\lambda_k)} f_k^{p_k}(x_k) = C$ a.e. in \mathbf{R}_+^n . Assuming that $C_i > 0$, then $x_i^{p_i(1-\lambda_i)-1} f_i^{p_i}(x_i) = C/(C_i x_i)$, which contradicts $\|f\|_{p_i, \phi_i} > 0$. (Note that for $n = 2$, we only

consider (2.19) for $f_k^{pk}(x_k) = 1$ in the above). Hence we have (3.6). By Hölder's inequality, it follows that

$$\begin{aligned} (Tf, f_n) &= \int_0^\infty \left(x_n^{\lambda_n - 1/q_n} \int_{\mathbf{R}^{n-1}} k_\lambda(x_1, \dots, x_n) \prod_{i=1}^{n-1} f_i(x_i) dx_1 \cdots dx_{n-1} \right) \\ &\quad \times \left(x_n^{1/q_n - \lambda_n} f_n(x_n) \right) dx_n \leq \|T(f_1, \dots, f_{n-1})\|_{q_n, \phi_n^{1/(1-p_n)}} \|f_n\|_{p_n, \phi_n}, \end{aligned} \quad (3.9)$$

and then by (3.6), we have (3.7). Assuming that (3.7) is valid, setting

$$f_n(x_n) := x_n^{q_n \lambda_n - 1} \left[\int_{\mathbf{R}^{n-1}} k_\lambda(x_1, \dots, x_n) \prod_{i=1}^{n-1} f_i(x_i) dx_1 \cdots dx_{n-1} \right]^{q_n - 1}, \quad (3.10)$$

then $J = \left\{ \int_0^\infty x_n^{p_n(1-\lambda_n)-1} f_n^{p_n}(x_n) dx_n \right\}^{1/q_n}$. By (2.18), it follows that $J < \infty$. If $J = 0$, then (3.6) is naturally valid. Assuming that $0 < J < \infty$, by (3.7), it follows that

$$\begin{aligned} \int_0^\infty x_n^{p_n(1-\lambda_n)-1} f_n^{p_n}(x_n) dx_n &= J^{q_n} = (Tf, f_n) < k_\lambda \prod_{i=1}^n \|f_i\|_{p_i, \phi_i}, \\ \left\{ \int_0^\infty x_n^{p_n(1-\lambda_n)-1} f_n^{p_n}(x_n) dx_n \right\}^{1/q_n} &= J < k_\lambda \prod_{i=1}^{n-1} \|f_i\|_{p_i, \phi_i}, \end{aligned} \quad (3.11)$$

and then (3.6) is valid, which is equivalent to (3.7).

For $\varepsilon > 0$ small enough, setting $\tilde{f}_i(x)$ as follows: $\tilde{f}_i(x) = 0, x \in (0, 1)$; $\tilde{f}_i(x) = x^{\lambda_i - \varepsilon/p_i - 1}, x \in [1, \infty)$ ($i = 1, \dots, n$), if there exists $k \leq k_\lambda$, such that (3.7) is still valid as we replace k_λ by k , then in particular, by Lemma 2.5, we have that

$$k_\lambda + o(1) = I_\varepsilon = \varepsilon \left(T(\tilde{f}_1, \dots, \tilde{f}_{n-1}), \tilde{f}_n \right) < \varepsilon k \prod_{i=1}^n \|\tilde{f}_i\|_{p_i, \phi_i} = k, \quad (3.12)$$

and $k_\lambda \leq k$ ($\varepsilon \rightarrow 0^+$). Hence $k = k_\lambda$ is the best value of (3.7). We conform that the constant factor k_λ in (3.6) is the best possible; otherwise, we can get a contradiction by (3.9) that the constant factor in (3.7) is not the best possible. Therefore $\|T\| = k_\lambda$.

(2) For $0 < p_1 < 1, p_i < 0$ ($i = 2, \dots, n$), by using the reverse Hölder's inequality and the same way, we have the equivalent reverses of (3.6) and (3.7) with the same best constant factor. \square

Example 3.2. For $\lambda > 0, \lambda_i = (\lambda/r_i)$ ($i = 1, \dots, n$), $\sum_{i=1}^n (1/r_i) = 1, k_\lambda(x_1, \dots, x_n) = 1/(\sum_{i=1}^n x_i)^\lambda$, we obtain $k_\lambda = (1/\Gamma(\lambda))\prod_{i=1}^n \Gamma(\lambda/r_i)$ (cf. [7, (9.1.19)]). By Theorem 3.1, it follows that $\|T\| = k_\lambda = (1/\Gamma(\lambda))\prod_{i=1}^n \Gamma(\lambda/r_i)$, and then by (3.7), we find that

$$\int_{\mathbb{R}_+^n} \frac{1}{(\sum_{i=1}^n x_i)^\lambda} \prod_{i=1}^n F_i(x_i) dx_1 \cdots dx_n < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{r_i}\right) \left\{ \int_0^\infty x_i^{p_i(1-\lambda/r_i)-1} F_i^{p_i}(x_i) dx_i \right\}^{1/p_i} . \tag{3.13}$$

Setting $F_i(x_i) = x_i^{\beta/n} f_i(x_i)$ and $\lambda_i = \lambda/r_i - \beta/n$ ($i = 1, \dots, n$) in (3.13), we obtain $\sum_{i=1}^n \lambda_i = \lambda - \beta, \min_{1 \leq i \leq n} \{\lambda_i\} > -\beta/n$ and

$$\int_{\mathbb{R}_+^n} \frac{(\sqrt[n]{\prod_{i=1}^n x_i})^\beta}{(\sum_{i=1}^n x_i)^\lambda} \prod_{i=1}^n f_i(x_i) dx_1 \cdots dx_n < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\lambda_i + \frac{\beta}{n}\right) \left\{ \int_0^\infty x_i^{p_i(1-\lambda_i)-1} f_i^{p_i}(x_i) dx_i \right\}^{1/p_i} . \tag{3.14}$$

It is obvious that (3.13) and (3.14) are equivalent in which the constant factors are all the best possible. Hence for $k_{\lambda-\beta}(x_1, \dots, x_n) = (\sqrt[n]{\prod_{i=1}^n x_i})^\beta / (\sum_{i=1}^n x_i)^\lambda$ ($\lambda > 0, \min_{1 \leq i \leq n} \{\lambda_i\} > -\beta/n$), we can show that $\|T\| = k_{\lambda-\beta} = (1/\Gamma(\lambda))\prod_{i=1}^n \Gamma(\lambda_i + \beta/n)$.

Example 3.3. For $\lambda > 0, \lambda_i = \lambda/r_i$ ($i = 1, \dots, n$), $\sum_{i=1}^n (1/r_i) = 1, k_\lambda(x_1, \dots, x_n) = 1/(\max_{1 \leq i \leq n} \{x_i\})^\lambda$, we obtain $k_\lambda = (1/\lambda^{n-1})\prod_{i=1}^n r_i$ (cf. [7, (9.1.24)]). By Theorem 3.1, it follows that $\|T\| = k_\lambda = (1/\lambda^{n-1})\prod_{i=1}^n r_i$, and then by (3.7), we find that

$$\int_{\mathbb{R}_+^n} \frac{1}{(\max_{1 \leq i \leq n} \{x_i\})^\lambda} \prod_{i=1}^n F_i(x_i) dx_1 \cdots dx_n < \frac{1}{\lambda^{n-1}} \prod_{i=1}^n r_i \left\{ \int_0^\infty x_i^{p_i(1-\lambda/r_i)-1} F_i^{p_i}(x_i) dx_i \right\}^{1/p_i} . \tag{3.15}$$

Setting $F_i(x_i) = x_i^{\beta/n} f_i(x_i)$ and $\lambda_i = \lambda/r_i - \beta/n$ ($i = 1, \dots, n$) in (3.15), we obtain $\sum_{i=1}^n \lambda_i = \lambda - \beta, \min_{1 \leq i \leq n} \{\lambda_i\} > -\beta/n$ and

$$\int_{\mathbb{R}_+^n} \frac{(\sqrt[n]{\prod_{i=1}^n x_i})^\beta}{(\max_{1 \leq i \leq n} \{x_i\})^\lambda} \prod_{i=1}^n f_i(x_i) dx_1 \cdots dx_n < \lambda \prod_{i=1}^n \frac{1}{\lambda_i + \beta/n} \left\{ \int_0^\infty x_i^{p_i(1-\lambda_i)-1} f_i^{p_i}(x_i) dx_i \right\}^{1/p_i} . \tag{3.16}$$

Hence for $k_{\lambda-\beta}(x_1, \dots, x_n) = \left(\sqrt[n]{\prod_{i=1}^n x_i}\right)^\beta / (\max_{1 \leq i \leq n} \{x_i\})^\lambda$ ($\lambda > 0, \min_{1 \leq i \leq n} \{\lambda_i\} > -\beta/n$), we can show that $\|T\| = k_{\lambda-\beta} = \lambda \prod_{i=1}^n (1/(\lambda_i + \beta/n))$.

Example 3.4. For $\lambda > 0, \lambda_i = -\lambda/r_i, \sum_{i=1}^n (1/r_i) = 1$ ($i = 1, \dots, n$), $k_{-\lambda}(x_1, \dots, x_n) = (\min_{1 \leq i \leq n} \{x_i\})^\lambda$, by mathematical induction, we can show that

$$k_{-\lambda} = \int_{\mathbb{R}_+^{n-1}} (\min\{u_1, \dots, u_{n-1}, 1\})^\lambda \prod_{j=1}^{n-1} u_j^{-\lambda/r_j-1} du_1 \cdots du_{n-1} = \frac{\prod_{i=1}^n r_i}{\lambda^{n-1}}. \quad (3.17)$$

In fact, for $n = 2$, we obtain

$$k_{-\lambda} = \int_0^1 u_1^{\lambda/r_2-1} du_1 + \int_1^\infty u_1^{-\lambda/r_1-1} du_1 = \frac{1}{\lambda} r_1 r_2. \quad (3.18)$$

Assuming that for $n(\geq 2)$ (3.17) is valid, then for $n + 1$, it follows that

$$\begin{aligned} k_{-\lambda} &= \int_{\mathbb{R}_+^{n-1}} \prod_{j=2}^n u_j^{-\lambda/r_j-1} \left[\int_0^\infty (\min\{u_1, \dots, u_n, 1\})^\lambda u_1^{-\lambda/r_1-1} du_1 \right] du_2 \cdots du_n \\ &= \int_{\mathbb{R}_+^{n-1}} \prod_{j=2}^n u_j^{-\lambda/r_j-1} \left[\int_0^{\min\{u_2, \dots, u_n, 1\}} u_1^\lambda u_1^{-\lambda/r_1-1} du_1 \right. \\ &\quad \left. + \int_{\min\{u_2, \dots, u_n, 1\}}^\infty (\min\{u_2, \dots, u_n, 1\})^\lambda u_1^{-\lambda/r_1-1} du_1 \right] du_2 \cdots du_n \quad (3.19) \\ &= \frac{r_1^2}{\lambda(r_1-1)} \int_{\mathbb{R}_+^{n-1}} (\min\{u_2, \dots, u_n, 1\})^{\lambda(1-1/r_1)} \prod_{j=2}^n u_j^{-\lambda(1-1/r_1)/(1-1/r_1)r_j-1} du_2 \cdots du_n \\ &= \frac{r_1^2}{\lambda(r_1-1)} \frac{1}{[\lambda(1-1/r_1)]^{n-1}} \prod_{i=2}^{n+1} \left(1 - \frac{1}{r_1}\right) r_i = \frac{1}{\lambda^n} \prod_{i=1}^{n+1} r_i. \end{aligned}$$

Then by mathematical induction, (3.17) is valid for $n \in \mathbf{N} \setminus \{1\}$.

By Theorem 3.1, it follows that $\|T\| = k_{-\lambda} = (1/\lambda^{n-1}) \prod_{i=1}^n r_i$, and by (3.7), we find that

$$\begin{aligned} &\int_{\mathbb{R}_+^n} \left(\min_{1 \leq i \leq n} \{x_i\}\right)^\lambda \prod_{i=1}^n F_i(x_i) dx_1 \cdots dx_n \\ &< \frac{1}{\lambda^{n-1}} \prod_{i=1}^n r_i \left\{ \int_0^\infty x_i^{p_i(1+\lambda/r_i)-1} F_i^{p_i}(x_i) dx_i \right\}^{1/p_i}. \end{aligned} \quad (3.20)$$

Setting $F_i(x_i) = x_i^{-\beta/n} f_i(x_i)$ and $\lambda_i = -\lambda/r_i + \beta/n$ ($i = 1, \dots, n$) in (3.20), we obtain $\sum_{i=1}^n \lambda_i = \beta - \lambda$, $\max_{1 \leq i \leq n} \{\lambda_i\} < \beta/n$ and

$$\int_{\mathbf{R}_+^n} \frac{(\min_{1 \leq i \leq n} \{x_i\})^\lambda}{\left(\sqrt[n]{\prod_{i=1}^n x_i}\right)^\beta} \prod_{i=1}^n f_i(x_i) dx_1 \cdots dx_n$$

$$< \lambda \prod_{i=1}^n \frac{1}{\beta/n - \lambda_i} \left\{ \int_0^\infty x_i^{p_i(1-\lambda_i)-1} f_i^{p_i}(x_i) dx_i \right\}^{1/p_i}.$$
(3.21)

Hence for $k_{\beta-\lambda}(x_1, \dots, x_n) = (\min_{1 \leq i \leq n} \{x_i\})^\lambda / \left(\sqrt[n]{\prod_{i=1}^n x_i}\right)^\beta$ ($\lambda > 0$, $\max_{1 \leq i \leq n} \{\lambda_i\} < \beta/n$), we can show that $\|T\| = k_{\beta-\lambda} = \lambda \prod_{i=1}^n (1/(\beta/n - \lambda_i))$.

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