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Research Article

Exponential Stability of Time-Switched Two-Subsystem Nonlinear Systems with Application to Intermittent Control

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This paper studies the exponential stability of a class of periodically time-switched nonlinear systems. Three cases of such systems which are composed, respectively, of a pair of unstable subsystems, of both stable and unstable subsystems, and of a pair of stable systems, are considered. For the first case, the proposed result shows that there exists periodically switching rule guaranteeing the exponential stability of the whole system with (sufficient) small switching period if there is a Hurwitz linear convex combination of two uncertain linear systems derived from two subsystems by certain linearization. For the second case, we present two general switching criteria by means of multiple and single Lyapunov function, respectively. We also investigate the stability issue of the third case, and the switching criteria of exponential stability are proposed. The present results for the second case are further applied to the periodically intermittent control. Several numerical examples are also given to show the effectiveness of theoretical results.

1. Introduction

In the recent years, motivated by the fact that many practical systems are inherently multimodal in the sense of that several dynamical systems are required to describe their behavior which may depend on various environmental factors [1, 2], and by the fact that the methods of intelligent control design are based on switching between different controllers [2, 3], the study of switched systems has been received an increasing attention in control theory and applications [2, 4–13]. It is also worth noting that the switching rule is naturally identified as two classes. One is known as state-switched system, and another is time-switched system. So far, most contributions to switched systems were made for the state-switched system. For example, the stability properties for state-dependent switched systems have been characterized in [4–7], and the switched control synthesis designs have been presented in [2, 8, 9]. For the time-switched systems, we refer the readers to [14–16]. One can see that the study of time-switched systems is almost limited to the linear subsystems, at most with the nonlinear perturbations. In [14], the switched system consists of only Hurwitz stable subsystems. The papers [15, 16] deal with the switched systems with both stable and unstable subsystems by means of average dwell time approach.

In the present paper, we study the exponential stability of time-periodically switched systems composed of a pair of nonlinear subsystems with neural-type Lipstchiz nonlinearities (i.e., the nonlinear vector-valued function $f(x)$ is of the form $f(x) = [f_1(x_1), f_2(x_2), \ldots, f_n(x_n)]^T$). Three cases of such systems will be dealt with.

Case 1. Periodically switched system with a pair of unstable nonlinear subsystems.

Case 2. Periodically switched system with both stable and unstable nonlinear subsystems.

Case 3. Periodically switched system with a pair of stable nonlinear subsystems.

For the first case, a linearization transformation is introduced to shift the exponential stability issue of the original system into the robustly exponential stability of the transformed system with a pair of linear time-varying subsystems, and we will use the average-system approach to analyze the robustly exponential stability of the transformed systems. For the second and third cases, the general theoretical frameworks based on the multiple Lyapunov functions will be established. We also suggest that if there exists a common Lyapunov function $V(x)$ such that (i) for the second case, $\dot{V}(x) \le -\lambda_1 V(x)$ for the first subsystem and $\dot{V}(x) \le \lambda_2 V(x)$ for the second subsystem, then the system as a whole will be globally exponentially stable for any switching period T and any switching rate α satisfying $1 > \alpha > \lambda_2/(\lambda_1 + \lambda_2)$; (ii) for the third case, $\dot{V}(x) \le -\lambda_i V(x)$, $i = 1, 2$, for the ith subsystem, then the system as a whole will be globally exponentially stable for any switching period T and any switching rate α satisfying $1 > \alpha > 0$. Note also that for the first case because both subsystems are unstable there is no such common Lyapunov function $V(x)$ that $\dot{V}(x)$ is negative definite for any subsystem. Based on the stability analysis of the second case, we address the periodically intermittent output feedback control problem. Several numerical examples will be presented to show the validity of the theoretical results.

The rest of the paper is organized as follows. In the next section, the problem to be dealt with is formulated and the necessary preliminaries are presented. Then, the theoretical results for three cases of time-switched systems are established in Sections 3–5, respectively. Section 6 deals with the stabilization problem by means of periodically intermittent control. In Section 7, several examples are given to verify the effectiveness of the theoretical results. Finally, conclusions are drawn in Section 8.

2. Problem Formulation and Preliminaries

Consider a class of periodically time-switched systems with m subsystems

$$\begin{aligned} \dot{x}(t) = A_i x(t) + B_i f_i(x(t)), \quad & kT + \tau_{i-1} \le t < kT + \tau_i, \\ & i = 1, 2, \ldots, m, \ k = 1, 2, \ldots, \end{aligned} \tag{2.1}$$

where $x \in R^n$ denotes the state vector, $A_i = (a_{kl}^{(i)}) \in R^{n\times n}$ and $B_i = (b_{kl}^{(i)}) \in R^{n\times n}$, $i = 1,2,\ldots,m$, are all constant matrices, $T > 0$ is called the switching period, $\Delta\tau_i = \tau_i - \tau_{i-1}$ is called the time duration of the ith subsystem with $0 = \tau_0 < \tau_1 < \cdots < \tau_{m-1} < \tau_m = T$. It is clear that the system (2.1) is a class of periodically time-switched systems with m nonlinear subsystems. Throughout this paper, we also assume that $f_i(x) = [f_{i1}(x_1), f_{i2}(x_2), \ldots, f_{in}(x_n)]^T$ with $f_i(0) = 0$ are continuous functions satisfying the following condition.

(H1) There exist constant scalar numbers α_{ij} $(i = 1,2,\ldots,m;\ j = 1,2,\ldots,n)$ such that

$$\left|f_{ij}(y)\right| \le \alpha_{ij}\left|y\right|, \quad \text{for any } y \in R. \tag{2.2}$$

In order to linearize system (2.1), we define $m \times n$ functions s_{ij} $(i = 1,2,\ldots,m;\ j = 1,2,\ldots,n)$ as follows:

$$s_{ij}(t) := \begin{cases} \dfrac{f_{ij}(x_j(t))}{x_j(t)}, & x_j(t) \neq 0, \\ 0, & x_j(t) = 0. \end{cases} \tag{2.3}$$

Let $S_i = \operatorname{diag}(s_{i1}, s_{i2}, \ldots, s_{in})$, $i = 1,2,\ldots,m$. Then, system (2.1) can be rewritten as

$$\begin{aligned} \dot{x}(t) = (A_i + B_i S_i(t))x(t), \quad kT + \tau_{i-1} \le t < kT + \tau_i, \\ i = 1,2,\ldots,m,\ k = 1,2,\ldots. \end{aligned} \tag{2.4}$$

Notice that assumption (H1) implies that $|s_{ij}| \le \alpha_{ij}$ $(i = 1,2,\ldots,m;\ j = 1,2,\ldots,n)$. Therefore, we have, for any $t \ge t_0$ and $i = 1,2,\ldots,m$,

$$-L_i \le S_i \le L_i = \operatorname{diag}(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}). \tag{2.5}$$

Furthermore, let, for $i = 1,2,\ldots,m$, be

$$\begin{gathered} |B_i| = \left(\left|b_{kl}^{(i)}\right|\right)_{n\times n}, \qquad \underline{C}_i = \left(\underline{c}_{kl}^{(i)}\right)_{n\times n} \equiv A_i - |B_i|L_i, \qquad \overline{C}_i = \left(\overline{c}_{kl}^{(i)}\right)_{n\times n} \equiv A_i + |B_i|L_i, \\ M\left[\underline{C}_i, \overline{C}_i\right] = \left\{C = \left(c_{kl}^{(i)}\right)_{n\times n} : \underline{c}_{kl}^{(i)} \le c_{kl}^{(i)} \le \overline{c}_{kl}^{(i)},\ k,l = 1,2,\ldots,n\right\}. \end{gathered} \tag{2.6}$$

Then, for any $t \ge t_0$, we have

$$A_i + B_i S_i(t) \in M\left[\underline{C}_i, \overline{C}_i\right]. \tag{2.7}$$

In order to formulate the transformation, we still need the following quantities:

$$C_i = \frac{\underline{C}_i + \overline{C}_i}{2} = A_1, \quad H_i = \frac{\overline{C}_i - \underline{C}_i}{2} = |B_i| L_i, \quad i = 1, 2, \ldots, m. \tag{2.8}$$

Because the entries of matrix $H_i = (h_{kl}^{(i)}) \in R^{n\times n}$ $(i = 1, 2, \ldots, m)$ are nonnegative, we further define

$$E_i = \begin{bmatrix} \sqrt{h_{11}^{(i)}} & \cdots & \sqrt{h_{1n}^{(i)}} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \sqrt{h_{21}^{(i)}} & \cdots & \sqrt{h_{2n}^{(i)}} & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \sqrt{h_{n1}^{(i)}} & \cdots & \sqrt{h_{nn}^{(i)}} \end{bmatrix},$$
$$F_i = \begin{bmatrix} \sqrt{h_{11}^{(i)}} & \cdots & 0 \\ \cdots & \ddots & \cdots \\ 0 & \cdots & \sqrt{h_{1n}^{(i)}} \\ \sqrt{h_{21}^{(i)}} \cdots & \cdots & 0 \\ \cdots & \ddots & \cdots \\ 0 & \cdots & \sqrt{h_{2n}^{(i)}} \\ \cdots & \cdots & \cdots \\ \sqrt{h_{n1}^{(i)}} & \cdots & 0 \\ \cdots & \ddots & \cdots \\ 0 & \cdots & \sqrt{h_{nn}^{(i)}} \end{bmatrix}. \tag{2.9}$$

Simple computation yields, for $i = 1, 2, \ldots, m$,

$$E_i E_i^T = \operatorname{diag}\left(\sum_{j=1}^{n} h_{1j}^{(i)}, \sum_{j=1}^{n} h_{2j}^{(i)}, \ldots, \sum_{j=1}^{n} h_{nj}^{(i)}\right),$$
$$F_i^T F_i = \operatorname{diag}\left(\sum_{j=1}^{n} h_{j1}^{(i)}, \sum_{j=1}^{n} h_{j2}^{(i)}, \ldots, \sum_{j=1}^{n} h_{jn}^{(i)}\right). \tag{2.10}$$

We then have the following lemma which is of key importance for our results.

Lemma 2.1 (See [17, 18]). *Let*

$$\Sigma^* = \left\{ \Sigma \in R^{n^2 \times n^2} \mid \Sigma = \operatorname{diag}(\varepsilon_{11}, \ldots, \varepsilon_{1n}, \ldots, \varepsilon_{n1}, \ldots, \varepsilon_{nn}),\ \left| \varepsilon_{ij} \right| \leq 1,\ \textit{for}\ i, j = 1, 2, \ldots, n \right\},$$
$$N\left[\underline{C}_i, \overline{C}_i\right] = \left\{ D_i = C_i + E_i \Sigma_i F_i \mid \Sigma_i \in \Sigma^* \right\}. \tag{2.11}$$

Then, $M[\underline{C}_i, \overline{C}_i] = N[\underline{C}_i, \overline{C}_i]$.

Proof. A proof of this lemma is presented in the appendix. □

Now, we consider the following uncertain linear system with $\Sigma_i \in \Sigma^*$, $i = 1, 2$,

$$\dot{x}(t) = (C_i + E_i \Sigma_i F_i) x(t), \qquad kT + \tau_{i-1} \leq t < kT + \tau_i. \tag{2.12}$$

It follows from Lemma 2.1 that the parameter uncertainties in (2.4) and (2.12) are identical, which implies that (2.12) is equivalent to system (2.4). It is also observed that the robust stability property of (2.4) implies the stability property of system (2.1). Therefore, in order to derive the sufficient conditions for stability of system (2.1), we consider the robust stability of system (2.12).

Remark 2.2. If the nonlinear function $f_i(x)$ satisfies the assumption

(H2) for $x_j \neq 0$ and $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$, $0 \leq f_{ij}(x_j)/x_j \leq \alpha_{ij}$.

Then, $0 \leq s_{ij} \leq \alpha_{ij}$ and $\underline{C}_i = A_i + (\min\{0, b_{kl}^{(i)} \alpha_{il}\})$, $\overline{C}_i = A_i + (\max\{0, b_{kl}^{(i)} \alpha_{il}\})$. This implies that $C_i = A_i + (1/2)(b_{kl}^{(i)} \alpha_{il})$, $H_i = (1/2)(|b_{kl}^{(i)}| \alpha_{il})$, where (d_{kl}) denotes a matrix with d_{kl} as the kth line and lth column entry.

For briefness, we mainly focus on system (2.1) with two subsystems in the sequel, that is, $m = 2$. In this case, we rewrite, respectively, systems (2.1) and (2.12) as

$$\begin{aligned} \dot{x}(t) &= A_1 x(t) + B_1 f(x(t)), \qquad kT \leq t < kT + \alpha T, \\ \dot{x}(t) &= A_2 x(t) + B_2 g(x(t)), \qquad kT + \alpha T \leq t < (k+1)T, \end{aligned} \tag{2.13}$$

$$\begin{aligned} \dot{x}(t) &= (C_1 + E_1 \Sigma_1 F_1) x(t), \qquad kT \leq t < kT + \alpha T, \\ \dot{x}(t) &= (C_2 + E_2 \Sigma_2 F_2) x(t), \qquad kT + \alpha T \leq t < (k+1)T, \end{aligned} \tag{2.14}$$

where $0 < \alpha < 1$ is called the switching rate. The results for (2.13) and/or (2.14) are easily extended to the general systems (2.1) and/or (2.12) with m subsystems.

The following two lemmas are useful in the sequel.

Lemma 2.3 (Sanchez and Perez [19]). *Given any real matrices $\Sigma_1, \Sigma_2, \Sigma_3$ of appropriate dimensions and a scalar $\varepsilon > 0$ such that $0 < \Sigma_3 = \Sigma_3^T$. Then, the following inequality holds:*

$$\Sigma_1^T \Sigma_2 + \Sigma_2^T \Sigma_1 \leq \varepsilon \Sigma_1^T \Sigma_3 \Sigma_1 + \varepsilon^{-1} \Sigma_2^T \Sigma_3^{-1} \Sigma_2, \tag{2.15}$$

where the superscript T means the transpose of a matrix.

Lemma 2.4 (Schur complement, Boyd et al. [20]). *The following LMI:*

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0, \tag{2.16}$$

where $Q(x) = Q^T(x)$, $R(x) = R^T(x)$, *and* $S(x)$ *depend affinely on* x, *is equivalent to*

$$R(x) > 0, \quad Q(x) - S(x)R^{-1}(x)S^T(x) > 0. \tag{2.17}$$

Throughout this paper, we denote by P^T the transpose of matrix P; $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ the minimal and maximal eigenvalues of a real symmetric matrix P, respectively; $P > 0$ $(\geq, <, \leq 0)$ the symmetrical and positive (semipositive, negative, seminegative) definite matrix P, and $\|P\|$ the Euclidian norm of the square matrix.

3. Stability Analysis for the First Case

In this section, we consider system (2.13) composed of a pair of unstable subsystems with neural-type nonlinearities. Since the robust stability property of system (2.14) implies the stability property of the original system (2.13), we will derive a sufficient condition of globally exponential stability. The theoretical result shows that similar switching criterion guaranteeing the exponential stability of the origin of time-switched LTI systems still holds for time-switched nonlinear systems with neural-type nonlinearities.

The main result in this section is as follows.

Theorem 3.1. *Suppose that there exist symmetric and positive definite matrix* P, *positive constants* q_1, q_2, *and* α $(0 < \alpha < 1)$ *such that*

$$\begin{aligned} \Omega_1 = {} & P(\alpha C_1 + (1-\alpha)C_2) + (\alpha C_1 + (1-\alpha)C_2)^T P + \alpha q_1^{-1} P E_1 E_1^T P + \alpha q_1 F_1^T F_1 \\ & + (1-\alpha) q_2^{-1} P E_2 E_2^T P + (1-\alpha) q_2 F_2^T F_2 < 0, \end{aligned} \tag{3.1}$$

where C_i, E_i, *and* F_i $(i = 1, 2)$ *are defined, respectively, in* (2.8). *Then, there exists (small) switching period* T *such that the origin of time-switched system* (2.14) *is globally robustly exponentially stable, and therefore system* (2.13) *is globally exponentially stable.*

Proof. We only need to show that inequality (3.1) implies the robustly exponential stability of system (2.14) with small switching period. Without loss of generality, we assume that the initial value is $x_0 = x(t_0)$ with the starting time $t_0 \in [0,\ \alpha T)$. Then, by piecewise integration, we have

$$\begin{aligned}
x(\alpha T) &= \exp((C_1 + E_1\Sigma_1 F_1)(\alpha T - t_0))x_0, \\
x(T) &= \exp((C_2 + E_2\Sigma_2 F_2)(1-\alpha)T)\exp((C_1 + E_1\Sigma_1 F_1)(\alpha T - t_0))x_0 \\
&= \exp((C_2 + E_2\Sigma_2 F_2)(1-\alpha)T)\exp((C_1 + E_1\Sigma_1 F_1)\alpha T)\exp(-(C_1 + E_1\Sigma_1 F_1)t_0)x_0 \\
&= \left[I + (C_2 + E_2\Sigma_2 F_2)(1-\alpha)T + \frac{1}{2!}(C_2 + E_2\Sigma_2 F_2)^2(1-\alpha)^2T^2 + \cdots\right] \\
&\quad \times \left[I + (C_1 + E_1\Sigma_1 F_1)\alpha T + \frac{1}{2!}(C_1 + E_1\Sigma_1 F_1)^2\alpha^2T^2 + \cdots\right] \\
&\quad \times \exp(-(C_1 + E_1\Sigma_1 F_1)t_0)x_0 \\
&= \{I + [(C_1 + E_1\Sigma_1 F_1)\alpha + (C_2 + E_2\Sigma_2 F_2)(1-\alpha)]T + O(T)\} \\
&\quad \times \exp(-(C_1 + E_1\Sigma_1 F_1)t_0)x_0.
\end{aligned} \tag{3.2}$$

Along this idea and omitting the terms $O(T)$ when T tends to zero, one observes that if we let T tend to zero, on any fixed time interval the solution of (2.14) will tend to the solution (with the same initial condition) of the following "averaged" system (see [21] for more details), for any $\Sigma_1,\ \Sigma_2 \in \Sigma^*$,

$$\dot{x}(t) = [(C_1 + E_1\Sigma_1 F_1)\alpha + (C_2 + E_2\Sigma_2 F_2)(1-\alpha)]x(t). \tag{3.3}$$

In particular, the stability properties of the switched system (2.14) will for sufficiently small T be determined by the robust stability properties of the averaged system (3.3). Note that system (3.3) is globally exponentially stable if and only if there exists $0 < \alpha < 1$ such that $A_e = (C_1 + E_1\Sigma_1 F_1)\alpha + (C_2 + E_2\Sigma_2 F_2)(1-\alpha)$ is Hurwitz for any $\Sigma_1,\ \Sigma_2 \in \Sigma^*$, which is equivalent to the fact that there exists symmetric and positive definite matrix P such that $A_e^T P + P A_e < 0$. A sufficient condition for this inequality is just $\Omega_1 < 0$, that is, inequality (3.1). This is because, for $x \neq 0$,

$$\begin{aligned}
x^T\left[A_e^T P + P A_e\right]x &= x^T\Big\{[(C_1 + E_1\Sigma_1 F_1)\alpha + (C_2 + E_2\Sigma_2 F_2)(1-\alpha)]^T P \\
&\qquad + P\Big[(C_1 + E_1\Sigma_1 F_1)\alpha + (C_2 + E_2\Sigma_2 F_2)(1-\alpha)\Big]\Big\}x \\
&= x^T\Big\{[\alpha C_1 + (1-\alpha)C_2]^T P + P[\alpha C_1 + (1-\alpha)C_2]\Big\}x \\
&\quad + 2\alpha x^T P E_1\Sigma_1 F x + 2(1-\alpha)x^T P E_2\Sigma_2 F_2 x \\
&\leq x^T\Big\{[\alpha C_1 + (1-\alpha)C_2]^T P + P[\alpha C_1 + (1-\alpha)C_2]\Big\}x + \alpha q_1^{-1}x^T P E_1 E_1^T P x \\
&\quad + \alpha q_1 x^T F_1^T F_1 x + (1-\alpha)q_2^{-1}x^T P E_2 E_2^T P x + (1-\alpha)q_2 x^T F_2^T F_2 x
\end{aligned}$$

$$
\begin{aligned}
&= x^T \Big\{ [\alpha C_1 + (1-\alpha) C_2]^T P + P[\alpha C_1 + (1-\alpha) C_2] + \alpha q_1^{-1} P E_1 E_1^T P + \alpha q_1 F_1^T F_1 \\
&\qquad + (1-\alpha) q_2^{-1} P E_2 E_2^T P + (1-\alpha) q_2 F_2^T F_2 \Big\} x \\
&= x^T \Omega_1 x < 0.
\end{aligned}
\tag{3.4}
$$

This concludes the proof. □

Remark 3.2. This result is seen as the natural extension of that of time-switched linear systems [16].

4. Stability Analysis for the Second Case

In this section, we consider system (2.13) composed of both stable and unstable subsystems with neural-type nonlinearities. By means of, respectively, multiple and single Lyapunov function, we propose two general criteria, together with simple but effective sufficient conditions guaranteeing the globally exponential stability.

Reconsider the time-switched system (2.13). Here we assume that the first subsystem is globally exponentially stable at the origin, while the second one unstable. It is natural that the system as a whole will be globally exponentially stable for any given $0 < \alpha < 1$ when the switching period T approaches to infinite and $t_0 \in [kT, kT + \alpha T)$. Our interest here is to determine a region of the binary (T, α) composed of the switching period T and switching rate α such that the system as a whole is globally exponentially stable therein.

Theorem 4.1. *Suppose that there exist two scalar functions $V_i : R^n \to R^+$, $i = 1, 2$, a continuous and monotonously increasing function γ with $\gamma(0) = 0$ and constants $\lambda_1 > 0$, $\lambda_2 > 0$ and $\beta \ge 1$ such that the following conditions hold:*

(i) $\gamma(\|x\|) \le V_1(x)$;

(ii) *for any $k = 0, 1, 2, \ldots$, when $t \in [kT, kT + \alpha T)$, $\dot{V}_1(x) \le -\lambda_1 V_1(x)$, and when $t \in [kT + \alpha T, (k+1)T)$, $\dot{V}_2(x) \le \lambda_2 V_2(x)$;*

(iii) $V_i(x) \le \beta V_j(x)$, *for any $x \in R^n$ and $i, j \in \{1,\ 2\}$;*

(iv) $\varepsilon = \alpha\lambda_1 - (1-\alpha)\lambda_2 - (2/T)\ln\beta > 0$.

Then, the origin of the time-switched system (2.13) *is globally exponentially stable.*

Proof. When $t \in [kT, kT + \alpha T)$, it follows from condition (ii) that

$$
V_1(x) \le \exp\{-\lambda_1 (t - kT)\} V_1(kT).
\tag{4.1}
$$

Similarly, when $t \in [kT + \alpha T, (k+1)T)$, the differential inequality $\dot{V}_2(x) \le \lambda_2 V(x)$ implies

$$
V_2(x) \le \exp\{\lambda_2 (t - kT - \alpha T)\} V_2(kT + \alpha T).
\tag{4.2}
$$

From (4.1)-(4.2) and condition (iii), we have the following

(a) When $t \in [0, \alpha T)$, $V_1(x) \leq \exp\{-\lambda_1 t\} V_1(0)$.

(b) When $t \in [\alpha T, T)$,

$$\begin{aligned} V_2(x) &\leq \exp\{\lambda_2(t-\alpha T)\} V_2(\alpha T) \\ &\leq \beta \exp\{\lambda_2(t-\alpha T)\} V_1(\alpha T) \\ &\leq \beta \exp\{\lambda_2(t-\alpha T) - \lambda_1 \alpha T\} V_1(0) \\ &= \beta \exp\{-(\lambda_1+\lambda_2)\alpha T + \lambda_2 t\} V_1(0). \end{aligned} \tag{4.3}$$

(c) When $t \in [T, T+\alpha T)$,

$$\begin{aligned} V_1(x) &\leq \exp\{-\lambda_1(t-T)\} V_1(T) \\ &\leq \beta \exp\{-\lambda_1(t-T)\} V_2(T) \\ &\leq \beta^2 \exp\{-\lambda_1(t-T) - (\lambda_1+\lambda_2)\alpha T + \lambda_2 T\} V_1(0) \\ &= \beta^2 \exp\{(\lambda_1+\lambda_2)(1-\alpha)T - \lambda_1 t\} V_1(0). \end{aligned} \tag{4.4}$$

(d) When $t \in [T+\alpha T, 2T)$,

$$\begin{aligned} V_2(x) &\leq \exp\{\lambda_2(t-T-\alpha T)\} V_2(T+\alpha T) \\ &\leq \beta \exp\{\lambda_2(t-T-\alpha T)\} V_1(T+\alpha T) \\ &\leq \beta^3 \exp\{\lambda_2(t-T-\alpha T) + (\lambda_1+\lambda_2)(1-\alpha)T - \lambda_1(T+\alpha T)\} V_1(0) \\ &= \beta^3 \exp\{-2(\lambda_1+\lambda_2)\alpha T + \lambda_2 t\} V_1(0). \end{aligned} \tag{4.5}$$

By induction, we have the following.

(e) When $t \in [kT, kT+\alpha T)$ which implies $k \leq t/T$,

$$\begin{aligned} V_1(x) &\leq \beta^{2k} \exp\{k(\lambda_1+\lambda_2)(1-\alpha)T - \lambda_1 t\} V_1(0) \\ &\leq \exp\left\{ [(\lambda_1+\lambda_2)(1-\alpha)T + 2\ln\beta] \frac{t}{T} - \lambda_1 t \right\} V_1(0) \\ &\leq \exp\left\{ \left[(\lambda_1+\lambda_2)(1-\alpha) + \frac{2}{T}\ln\beta\right] t \right\} V_1(0) \\ &= \exp\left\{ -\left[\alpha\lambda_1 - (1-\alpha)\lambda_2 - \frac{2}{T}\ln\beta - \lambda_1\right] t \right\} V_1(0). \end{aligned} \tag{4.6}$$

(f) When $t \in [kT + \alpha T, (k+1)T)$ which implies $k + 1 \geq t/T \geq k$,

$$\begin{aligned}
V_2(x) &\leq \beta^{2k+1} \exp\{-(k+1)(\lambda_1 + \lambda_2)\alpha T + \lambda_2 t\} V_1(0) \\
&\leq \beta \exp\left\{ [-(\lambda_1 + \lambda_2)\alpha T + 2\ln\beta] \frac{t}{T} + \lambda_2 t \right\} V_1(0) \\
&\leq \beta \exp\left\{ -\left[(\lambda_1 + \lambda_2)\alpha - \frac{2}{T}\ln\beta - \lambda_2 \right] t \right\} V_1(0) \\
&= \beta \exp\left\{ -\left[\alpha\lambda_1 - (1-\alpha)\lambda_2 - \frac{2}{T}\ln\beta \right] t \right\} V_1(0).
\end{aligned} \tag{4.7}$$

Therefore, we can conclude the proof from (e)-(f) and conditions (i) and (iv). □

Based on Theorem 4.1, if we choose the quadratic Lyapunov function $V_i(x) = x^T P_i x2$ $(i = 1, 2)$ for the ith subsystem, the following result is immediate.

Corollary 4.2. *Suppose that there exist symmetric and positive definite matrices P_1 and P_2, the positive constants $\lambda_1, \lambda_2, \mu_1$, and μ_2 such that*

(i) $P_1(A_1 + 0.5\lambda_1 I) + (A_1 + 0.5\lambda_1 I)^T P_1 + \mu_1^{-1} P_1 B_1 B_1^T P_1 + \mu_1 L_1^2 \leq 0$,

(ii) $P_2(A_2 - 0.5\lambda_2 I) + (A_2 - 0.5\lambda_2 I)^T P_2 + \mu_2^{-1} P_2 B_2 B_2^T P_2 + \mu_2 L_2^2 \leq 0$,

(iii) $\varepsilon = \alpha\lambda_1 - (1-\alpha)\lambda_2 - (2/T)\ln\beta > 0$,

where $\beta = \sup_{1 \leq i \neq j \leq 2}(\lambda_{\max}(P_i)/\lambda_{\min}(P_j))$. Then, by the switching period T with switching rate α, the origin of time-switched system (2.13) *with both stable and unstable subsystems is globally exponentially stable. Moreover, the norm of state vector satisfies the following inequality:*

$$\|x(t)\| \leq \sqrt{\frac{\beta\lambda_{\max}(P_1)}{\lambda_{\min}(P_1)}} \|x_0\| \exp\left\{ -\frac{1}{2}\left[\alpha\lambda_1 - (1-\alpha)\lambda_2 - \frac{2}{T}\ln\beta \right] t \right\}. \tag{4.8}$$

Proof. Consider the Lyapunov function $V_i(x) = x^T P_i x$ $(i = 1,\ 2)$. When $t \in [kT, kT + \alpha T)$, from condition (i) and Lemma 2.3 the derivative of V_1 along the trajectories of the first subsystem is calculated and estimated as follows:

$$\begin{aligned}
\dot{V}_1(x) &= 2x^T P_1 [A_1 x(t) + B_1 f(x(t))] \\
&= x^T(t) \left[P_1 A_1 + A_1^T P_1 \right] x(t) + 2x^T(t) P_1 B_1 f(x(t)) \\
&\leq x^T(t) \left[P_1 A_1 + A_1^T P_1 \right] x(t) + \mu_1^{-1} x^T(t) P_1 B_1 B_1^T P_1 x(t) + \mu_1 f^T(x(t)) f(x(t)) \\
&\leq x^T(t) \left[P_1 A_1 + A_1^T P_1 + \mu_1^{-1} P_1 B_1 B_1^T P_1 \right] x(t) + \mu_1 x^T(t) L_1^2 x(t) \\
&= x^T(t) \left[P_1 A_1 + A_1^T P_1 + \mu_1^{-1} P_1 B_1 B_1^T P_1 + \mu_1 L_1^2 \right] x(t) \\
&= -\lambda_1 V_1(x) + x^T(t) \left[P_1 A_1 + A_1^T P_1 + \mu_1^{-1} P_1 B_1 B_1^T P_1 + \mu_1 L_1^2 + \lambda_1 P_1 \right] x(t) \\
&\leq -\lambda_1 V_1(x).
\end{aligned} \tag{4.9}$$

Similarly, based on condition (ii) and Lemma 2.3, for $t \in [kT + \alpha T, (k+1)T)$, we have

$$\begin{aligned}
\dot{V}_2(x) &= 2x^T P_2 [A_2 x(t) + B_2 g(x(t))] \\
&= x^T(t) \left[P_2 A_2 + A_2^T P_2\right] x(t) + 2x^T(t) P_2 B_2 g(x(t)) \\
&\leq x^T(t) \left[P_2 A_2 + A_2^T P_2\right] x(t) + \mu_2^{-1} x^T(t) P_2 B_2 B_2^T P_2 x(t) + \mu_2 g^T(x(t)) g(x(t)) \\
&\leq x^T(t) \left[P_2 A_2 + A_2^T P_2 + \mu_2^{-1} P_2 B_2 B_2^T P_2\right] x(t) + \mu_2 x^T(t) L_2^2 x(t) \\
&= x^T(t) \left[P_2 A_2 + A_2^T P_2 + \mu_2^{-1} P_2 B_2 B_2^T P_2 + \mu_2 L_2^2\right] x(t) \\
&= \lambda_2 V_2(x) + x^T(t) \left[P_2 A_2 + A_2^T P_2 + \mu_2^{-1} P_2 B_2 B_2^T P_2 + \mu_2 L_2^2 - \lambda_2 P_2\right] x(t) \\
&\leq \lambda_2 V_2(x).
\end{aligned} \tag{4.10}$$

Therefore, conditions (i)-(ii) in Theorem 4.1 hold. Obviously, the definition of β implies $V_i(x) \leq \beta V_j(x)$ which leads to condition (iii) in Theorem 4.1. Thus, we complete the proof. □

Remark 4.3. Condition (iv) in Theorem 4.1 and condition (iii) in Corollary 4.2 hold for large enough switching period T if $\alpha\lambda_1 - (1-\alpha)\lambda_2 > 0$. This is completely consistent with the extreme case that only the stable system is activated, that is, $T = \infty$.

Remark 4.4. Condition (iii) in this theorem can help to derive an estimated region Ω of period T and switching rate α, where each binary $(\alpha, T) \in \Omega$ can guarantee the exponential stability of system (2.13). For computational consideration, we suggest the following steps.

(a) Find the maximum λ_1 and the corresponding P_1 from condition (i) which is equivalent to the following linear matrix inequality with respect to P_1 and μ_1 (this follows from Lemma 2.4):

$$\begin{bmatrix} P_1 A_1 + A_1^T P + \lambda_1 P + \mu_1 L_1^2 & -PB_1 \\ -B_1^T P & -\mu_1 I \end{bmatrix} \leq 0. \tag{4.11}$$

(b) Therefore, this step is changed into solving the optimization problem

$$\begin{aligned}
&\max \lambda_1 \\
&\text{s.t.,} \quad \begin{bmatrix} P_1 A_1 + A_1^T P + \lambda_1 P + \mu_1 L_1^2 & -PB_1 \\ -B_1^T P & -\mu_1 I \end{bmatrix} \leq 0.
\end{aligned} \tag{4.12}$$

(c) Find the minimum λ_2 and the corresponding P_2 from condition (ii) by solving the optimization problem

$$\min \lambda_2$$
$$\text{s.t.,} \quad \begin{bmatrix} P_2A_2 + A_2^TP_2 + \mu_2L_2^2 - \lambda_2P_2 & -P_2B_2 \\ -B_2^TP_2 & -\mu_2I \end{bmatrix} \leq 0. \tag{4.13}$$

(d) Estimate the region of (α, T)

$$\Omega = \left\{(\alpha, T) \ : \ \alpha\lambda_1 - (1-\alpha)\lambda_2 > 0, \ T > \frac{2\ln\beta}{\alpha\lambda_1 - (1-\alpha)\lambda_2}\right\}. \tag{4.14}$$

Then, for any $(\alpha, T) \in \Omega$, system (2.13) is globally exponentially stable.

Similarly, following the idea of the proof of Theorem 4.1, we have the simpler result when a common Lyapunov function is chosen.

Theorem 4.5. *Suppose that there exist a Lyapunov function $V : R^n \to R^+$, a continuous and monotonously increasing function γ with $\gamma(0) = 0$, and constants $\lambda_1 > 0$, $\lambda_2 > 0$, and $\beta \geq 1$ such that the following conditions hold:*

(i) $\gamma(\|x\|) \leq V(x)$;

(ii) *for any $k = 0, 1, 2, \ldots$, when $t \in [kT, kT + \alpha T)$, $\dot{V}(x) \leq -\lambda_1 V(x)$, and when $t \in [kT + \alpha T, (k+1)T)$, $\dot{V}(x) \leq \lambda_2 V(x)$;*

(iii) $\varepsilon = \alpha\lambda_1 - (1-\alpha)\lambda_2 > 0$.

Then, the origin of the time-switched system (2.13) *is globally exponentially stable for any switching period $T > 0$.*

Proof. The proof is similar with that of Theorem 4.1, and omitted here. □

From this theorem, a standard Lyapunov function $V(x) = x^TPx$ will yield the following corollary.

Corollary 4.6. *Suppose that there exist symmetric and a positive definite matrix P, four positive constants $\lambda_1, \lambda_2, \mu_1,$ and μ_2 such that*

(i) $P(A_1 + 0.5\lambda_1 I) + (A_1 + 0.5\lambda_1 I)^T P + \mu_1^{-1} P B_1 B_1^T P + \mu_1 L_1^2 \leq 0$,

(ii) $P(A_2 - 0.5\lambda_2 I) + (A_2 - 0.5\lambda_2 I)^T P + \mu_2^{-1} P B_2 B_2^T P + \mu_2 L_2^2 \leq 0$,

(iii) $\varepsilon = \alpha\lambda_1 - (1-\alpha)\lambda_2 > 0$.

Then, for any switching period $T > 0$, the origin of time-switched system (2.13) *is globally exponentially stable. Moreover, the norm of state vector satisfies the following inequality:*

$$\|x(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \|x_0\| \exp\left\{-\frac{1}{2}[\alpha\lambda_1 - (1-\alpha)\lambda_2]t\right\}. \tag{4.15}$$

We now study this problem by using the linearization system (2.14) and Theorem 4.5. A simpler sufficient condition for exponential stability of system (2.13) with both stable and unstable subsystems is established.

Corollary 4.7. *Suppose that there exist symmetric and positive definite matrix P, positive constants $\lambda_1, \lambda_2, \mu_1,$ and μ_2 such that*

(i) $PC_1 + C_1^T P + \mu_1^{-1} PE_1E_1^T P + \mu_1 F_1^T F_1 + \lambda_1 P \leq 0$,

(ii) $PC_2 + C_2^T P + \mu_2^{-1} PE_2E_2^T P + \mu_2 F_2^T F_2 - \lambda_2 P \leq 0$,

then, for arbitrary $T > 0$, if $1 > \alpha > \lambda_2/(\lambda_1 + \lambda_2)$, the origin of time-switched system (2.13) *with both stable and unstable subsystems is globally exponentially stable.*

Proof. Consider the common Lyapunov function $V(x) = x^T Px$ for both subsystems. The rest of the proof is similar to that of Theorem 4.1, and hence omitted. □

Remark 4.8. In Corollaries 4.6 and 4.7, condition (i) is to guarantee the exponential stability of the first subsystem, while the instability of the second subsystem follows condition (ii).

Remark 4.9. For computational consideration, we suggest the following algorithm.

(a) Find the maximum λ_1 and the corresponding P from condition (i) in Corollary 4.6 (or Corollary 4.7) by using the convex optimization algorithm.

(b) Find the minimum λ_2 from condition (ii) in Corollary 4.6 (or Corollary 4.7) by using the convex optimization algorithm. Note that the matrix P is solved in (a).

(c) Calculate the low bound of α, $\lambda_2/(\lambda_1+\lambda_2)$. Then, for any $T > 0$, if $1 > \alpha > \lambda_2/(\lambda_1+\lambda_2)$, system (2.13) is globally exponentially stable.

5. Stability Analysis for the Third Case

In this section, we consider the time-switched system (2.13) with a pair of stable subsystems. The contribution is twofold. Firstly, for the linear case, we characterize the stability properties in four aspects. Secondly, we extend the results for the linear case to the nonlinear system with neural-type nonlinearities.

5.1. Linear Case

Consider the following time-switched system with a pair of stable linear subsystems:

$$\begin{gathered}\dot{x}(t) = A_1 x(t), \qquad kT \leq t < kT + \alpha T, \\ \dot{x}(t) = A_2 x(t), \qquad kT + \alpha T \leq t < (k+1)T, \\ x(0) = x_0,\end{gathered} \tag{5.1}$$

where $x \in R^n$ denotes the state vector, $A_i = (a_{kl}^{(i)}) \in R^{n\times n}$, $i = 1, 2$, are Hurwitz.

We have the following results.

Theorem 5.1. *System* (5.1) *with a pair of stable subsystems is globally exponentially stable for any switching rule if there exists a symmetric and positive definite matrix P such that both the following inequalities hold:*

(i) $PA_1 + A_1^T P < 0$,

(ii) $PA_2 + A_2^T P < 0$.

Proof. Consider the common Lyapunov function $V(x) = x^T Px$. Integrating by part $V(x)$ with respect to time t along the trajectories of the system (5.1), we have, for any $t > 0$,

$$V(x) \leq Kx_0^T Px_0 \exp\{-[\alpha\lambda_1 + (1-\alpha)\lambda_2]t\}, \tag{5.2}$$

where $K = \exp\{\alpha(1-\alpha)\lambda_2 T\}$, and λ_i is positive constant and satisfies $PA_i + A_i^T P + \lambda_i P < 0$, $i = 1, 2$. Note that for any $0 < \alpha < 1$, $\alpha\lambda_1 + (1-\alpha)\lambda_2 > 0$, and therefore, (5.2) concludes the proof. □

When the condition in Theorem 5.1 does not hold, we suggest the following claim.

Theorem 5.2. *Let*

$$\lambda_i = \sup_{P=P^T>0} \left\{\lambda : PA_i + A_i^T P + \lambda P < 0,\ \lambda > 0\right\}, \tag{5.3}$$

$$P_i = \arg\left(\sup_{P=P^T>0} \left\{\lambda : PA_i + A_i^T P + \lambda P < 0,\ \lambda > 0\right\}\right), \quad i = 1, 2. \tag{5.4}$$

Then, system (5.1) *is globally exponentially stable if the switching period T satisfies*

$$T > \frac{2\ln\beta}{\alpha\lambda_1 + (1-\alpha)\lambda_2}, \tag{5.5}$$

where $\beta = \max_{1\leq i\neq j\leq 2}\{\lambda_{\max}(P_i)/\lambda_{\min}(P_j)\}$.

Proof. Consider the multiple Lyapunov function

$$\begin{aligned} V_1(x) &= x^T P_1 x, \quad t \in [kT, kT + \alpha T), \\ V_2(x) &= x^T P_2 x, \quad t \in [kT + \alpha T, (k+1)T). \end{aligned} \tag{5.6}$$

Note that $P_i A_i + A_i^T P_i + \lambda_i P_i \leq 0$. We calculate the derivatives of V_i with respect to time t along the trajectories of the system (5.1) as follows:

$$\dot{V}_i(x) \leq -\lambda_i V_i(x), \quad \text{for } i = 1, 2. \tag{5.7}$$

Therefore, we have

(a) for any $t \in [kT, kT + \alpha T)$,

$$V(x) \leq V_1(0) \exp\left\{ -\left[\alpha\lambda_1 + (1-\alpha)\lambda_2 - \frac{2\ln\beta}{T} \right] kT \right\}, \tag{5.8}$$

(b) for any $t \in [kT + \alpha T, (k+1)T)$,

$$V(x) \leq \beta \exp\{-\alpha\lambda_1 T\} V_1(0) \exp\left\{ -\left[\alpha\lambda_1 + (1-\alpha)\lambda_2 - \frac{2\ln\beta}{T} \right] kT \right\}. \tag{5.9}$$

Hence, if $\alpha\lambda_1 + (1-\alpha)\lambda_2 - 2\ln\beta/T > 0$, that is, $T > 2\ln\beta/(\alpha\lambda_1 + (1-\alpha)\lambda_2)$, system (5.1) is globally exponentially stable. The proof is thus completed. □

5.2. Nonlinear Case

Consider again the nonlinear time-switched system (2.13) and the "linearization" system (2.14), but in this section we assume that both subsystems in (2.14) are robust Hurwitz-stable.

Arguing similarly with the previous subsection, we have the analogs of the results described by Theorems 5.1 and 5.2.

Theorem 5.3. *Assume that, for $i = 1, 2$,*

$$\begin{aligned} \lambda_i &= \sup_{\substack{P=P^T>0 \\ \Sigma_i\in\Sigma^*}} \left\{ \lambda : P(C_i + E_i\Sigma_i F_i) + (C_i + E_i\Sigma_i F_i)^T P + \lambda P < 0,\ \lambda > 0 \right\}, \\ P_i &= \arg\left(\sup_{\substack{P=P^T>0 \\ \Sigma_i\in\Sigma^*}} \left\{ \lambda : P(C_i + E_i\Sigma_i F_i) + (C_i + E_i\Sigma_i F_i)^T P + \lambda P < 0,\ \lambda > 0 \right\} \right), \end{aligned} \tag{5.10}$$

exist. Then, the origin of system (2.13) *is globally exponentially stable if any one of the following conditions holds.*

(i) *There exists a symmetric and positive definite matrix P such that*

$$P(C_i + E_i\Sigma_i F_i) + (C_i + E_i\Sigma_i F_i)^T P < 0, \quad i = 1, 2, \tag{5.11}$$

is satisfied for any $\Sigma_i \in \Sigma^*$.

(ii) *The switching period T satisfies*

$$T > \frac{2\ln\beta}{\alpha\lambda_1 + (1-\alpha)\lambda_2}, \tag{5.12}$$

where $\beta = \max_{1\leq i\neq j\leq 2}\{\lambda_{\max}(P_i)/\lambda_{\min}(P_j)\}$.

Notice that, for any $\mu_i > 0$,

$$P(C_i + E_i\Sigma_i F_i) + (C_i + E_i\Sigma_i F_i)^T P \le PC_i + C_i^T P + \mu_i^{-1} P E_i E_i^T P + \mu_i F_i^T F_i. \tag{5.13}$$

Then, the following corollary is immediate.

Corollary 5.4. *System* (2.13) *is globally exponentially stable if any one of the following conditions holds.*

(i) *There exist a symmetric and positive definite matrix P and positive constants μ_i such that*

$$PC_i + C_i^T P + \mu_i^{-1} P E_i E_i^T P + \mu_i F_i^T F_i < 0, \quad i = 1, 2, \text{ hold}. \tag{5.14}$$

(ii) *If, for $i = 1, 2$, there exist symmetric and positive definite matrices P_i and positive constants λ_i satisfying*

$$\begin{gathered}\lambda_i = \sup_{\substack{P=P^T>0\\ \mu_i>0}} \left\{\lambda : PC_i + C_i^T P + \mu_i^{-1} P E_i E_i^T P + \mu_i F_i^T F_i + \lambda P < 0,\ \lambda > 0\right\},\\ P_i = \arg\left(\sup_{\substack{P=P^T>0\\ \mu_i>0}} \left\{\lambda : PC_i + C_i^T P + \mu_i^{-1} P E_i E_i^T P + \mu_i F_i^T F_i + \lambda P < 0,\ \lambda > 0\right\}\right),\end{gathered} \tag{5.15}$$

and further the switching period T and switching rate α satisfy $T > 2\ln\beta/(\alpha\lambda_1 + (1-\alpha)\lambda_2)$, where $\beta = \max_{1\le i\ne j\le 2}\{\lambda_{\max}(P_i)/\lambda_{\min}(P_j)\}$.

6. Intermittent Control with Time Duration

Intermittent control is a straightforward engineering approach to process control of any type. As a special form of switching control, intermittent control is also divided into two classes: state-dependent switching rule and time-switching rule. The former implies that the control operation is activated only when the states enter the certain region which is often pregiven; while the later activates the control only in some finite time intervals; the system evolves freely when the time goes out of those intervals. Therefore, these intermittent control systems are open-loop. Intermittent control has been used for a variety of purposes in such engineering fields as manufacturing, transportation, air-quality control, and communication, and so on. In the chaos control context, Carr and Schwartz [22–24] demonstrated a method to control unstable steady states in high-dimensional flows using the duration of time for which feedback control was applied as an addition parameter. Starrett [25] proposed the so-called "occasional bang-bang" method to stabilize a periodic saddle point in a strange attractor. The main idea of occasional bang-bang is to apply control only at regular intervals and for fixed durations. An application of intermittent control in air-quality control was reported in [26] where the authors reported a kind of intermittent control system which was designed to achieve ambient air quality standards constantly by varying emission rates in response to changing atmospheric dispersion conditions. Recently, the authors in [27] proposed a scheme to decouple neighboring qubits in quantum computers through bang-bang pulse control and demonstrated that two similar sequence of pulses with different time intervals not only suppress decoherence but entirely or selectively decouple two neighboring qubits. The authors in [27–31] suggested to controlling the evolution of a system by using strong, short pulses as a new means for quantum error prevention. Also, the authors in [32] discussed chaotic synchronization by using intermittent control.

An extreme case of intermittent control is impulsive control which has been gained increasing interest and intensively researched [33, 34]. The prominent characteristic of impulsive control is that the states of controlled system will "jump" at certain discrete time moments, namely, the control is with zero duration of time. Because the states of controlled systems are changed directly, impulsive control is an effective approach when the states are observable, but it seems to be invalid when the states of controlled systems are unobservable.

Our interest focuses on the class of intermittent control with time duration, namely, the control is activated in certain nonzero time intervals, and off in other time intervals. Specifically, the control law is of the form

$$u(t) = \begin{cases} u_1(t), & t \in [t_k, t_k + \tau), \\ 0, & t \in [t_k + \tau, t_{k+1}). \end{cases} \tag{6.1}$$

Consider the nonlinear system described by

$$\begin{gathered} \dot{x}(t) = Ax(t) + Bf(x(t)) + u(t), \\ y(t) = Cx(t), \\ x(t_0) = x_0, \end{gathered} \tag{6.2}$$

where $x = [x_1, x_2, \dots, x_n]^T \in R^n$ denotes the state vector, $A = (a_{ij}) \in R^{n\times n}$ and $B = (b_{ij}) \in R^{n\times n}$ are constant matrices, and the nonlinear function, $f(x) = [f_1(x_1), \dots, f_n(x_n)]^T : R^n \to R^n$, is continuous with $f(0) = 0$, and satisfies the Lipstchitz condition with the Liptchiz constant α_i, namely, $|f_i(x_i)| \leq \alpha_i |x_i|$, $i = 1, 2, \dots, n$. $y(t)$ presents the output of the system with the coefficient matrix $C = (c_{ij}) \in R^{m\times n}$, and $u(t)$ is the external input with the form of (6.1). For analytical simplification, we assume in this paper the input $u(t)$ is periodical switching with the fixed duration of time. Specifically, we take $u(t)$, in the sequel, as the form

$$u(t) = k(t)y(t), \tag{6.3}$$

with

$$\begin{gathered} k(t) = \begin{cases} K_{n\times m}, & 0 \leq t < \alpha T, \\ 0, & \alpha T \leq t < T, \ 0 < \alpha < 1, \end{cases} \\ k(t + T) = k(t). \end{gathered} \tag{6.4}$$

Our object is to find the appropriate K, α, and T such that system (6.2) is stabilized exponentially at the origin under the periodically intermittent control (6.3) with (6.4).

Note that the controlled system (6.2) with (6.3)-(6.4) can be rewritten as

$$\begin{aligned}\dot{x}(t) &= (A + KC)x(t) + Bf(x(t)), \quad t \in [kT, kT + \alpha T), \\ \dot{x}(t) &= Ax(t) + Bf(x(t)), \quad t \in [kT + \alpha T, (k+1)T), \\ x(t_0) &= x_0, \quad k = 0, 1, 2, \ldots.\end{aligned} \tag{6.5}$$

According to (2.8), it is not difficult to obtain $C_1 = A + KC$, $C_2 = A$ and $H_1 = H_2 = |B|L$, where $|B| = (|b_{ij}|)_{n\times n}$ and $L = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$. Because $H_1 = H_2 = |B|L$, both the equalities $E_1 = E_2$ and $F_1 = F_2$ hold. For notational simplification, let us define $H \equiv H_1 = H_2$, $E \equiv E_1 = E_2$, and $F \equiv F_1 = F_2$. From Corollary 4.7, we have the following result.

Theorem 6.1. *There exist a matrix $Q \in R^{n\times m}$, and a symmetric, positive definite matrix P, and positive constants λ_1, λ_2, and μ such that*

(i) $QC + C^T Q^T + (\lambda_1 + \lambda_2)P \le 0$,

(ii) $PA + A^T P + \mu^{-1} PEE^T P + \mu F^T F - \lambda_2 P \le 0$,

(iii) $1 > \alpha > \lambda_2/(\lambda_1 + \lambda_2)$.

Then, the origin of system (6.5) *is exponentially stable for any $T > 0$, and the corresponding control gain matrix is determined by $K = P^{-1}Q$.*

Proof. We only need to show that the conditions in Corollary 4.7 are satisfied. The proof is trivial, and therefore omitted here. □

To end this section, we consider a special external input of the form $u(t) = \overline{k}(t)y(t)$ where

$$\overline{k}(t) = \begin{cases} k_0, & t \in [kt, kT + \alpha T), \\ 0, & t \in [kt + \alpha T, (k+1)T), \quad k = 1, 2, \ldots, \end{cases} \tag{6.6}$$

in which k_0 is a constant scalar. Then, from Theorem 6.1, the following corollary is immediate.

Corollary 6.2. *Let $\lambda_0 = \inf_{\mu>0}\{\lambda_{\max}(A + A^T + \mu^{-1}EE^T + \mu F^T F)\}$, and $\lambda_c = \lambda_{\max}(C + C^T)$. The origin of controlled system* (6.5) *is exponentially stable for any $T > 0$ if $1 > \alpha > -\lambda_0/\lambda_c k_0$.*

7. Illustrating Examples

In this section, we will give two examples to show the validity of the proposed results.

Example 7.1. Consider system (2.13) with

$$\begin{aligned} A_1 &= \begin{bmatrix} -0.5 & 1 \\ 100 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & -100 \\ -0.5 & -1 \end{bmatrix}, \qquad B_1 = B_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \\ f(x) &= g(x) = \left[\frac{1}{2}(|x_1 + 1| - |x_1 - 1|), 0\right]^T. \end{aligned} \tag{7.1}$$

Both subsystems are unstable, as shown in Figure 1(a). Note that

$$C_1 = \begin{bmatrix} -1 & 1 \\ 100 & -2 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} -1.5 & -100 \\ -0.5 & -1.5 \end{bmatrix},$$
$$E_1 E_1^T = E_2 E_2^T = F_1^T F_1 = F_2^T F_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}. \tag{7.2}$$

Taking $P = I$ (identity matrix) and $q_1 = q_2 = 1$, we have, when $\alpha = 0.5$, $\Omega_1 = \left[\begin{smallmatrix} -1.5 & 0.25 \\ 0.25 & -2 \end{smallmatrix}\right]$ with the eigenvalues -2.104 and -1.396. Hence, it follows from Theorem 3.1 that when $\alpha = 0.5$ the system in this example is globally exponentially stable for some small enough $T > 0$, as shown in Figure 1(b).

Example 7.2. Consider system (2.13) with

$$A_1 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 & 0.5 \\ -0.4 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},$$
$$f(x) = \left[\frac{1}{2}(|x_1 + 1| - |x_1 - 1|), 0\right]^T, \qquad g(x) = \left[0, \frac{1}{2}(|x_1 + 1| - |x_1 - 1|)\right]^T. \tag{7.3}$$

Based on Corollary 4.2, solving the linear matrix inequalities by LMI ToolBox involved in the engineering software MATLAB, we obtain

$$\lambda_1 = 0.3092, \qquad \lambda_2 = 1.675 \quad \text{with} \quad \mu_1 = 17.190433, \qquad \mu_2 = 44.332956,$$
$$P_1 = \begin{bmatrix} 11.3808 & 2.14237 \\ 2.14237 & 5.92472 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 39.37458 & -3.427225 \\ -3.427225 & 89.11437 \end{bmatrix}. \tag{7.4}$$

Therefore, $\beta = 17.235$ and an estimated region of Ω is

$$\Omega = \left\{ (\alpha, t) : 1.9842\alpha - 1.675 - \frac{5.6939}{T} > 0,\ T > 0,\ \alpha < 1 \right\}, \tag{7.5}$$

which covers the whole region above the curve $1.9842\alpha - 1.675 - 5.6939/T = 0$ with $T > 0$ and $\alpha < 1$.

We also obtain the switching law by using Corollary 4.6. Solving condition (i) in Corollary 4.6, we get the maximum $\lambda_1 = 0.3092$ and the corresponding P; then solving the condition (ii) yields $\lambda_2 = 9.8519$; finally, we calculate the feasible interval of switching rate, $\alpha \in (0.96057, 1)$. It follows from Corollary 4.6 that the system described in Example 7.2 is globally exponentially stable for any switching period $T > 0$, if the switching rate satisfies $1 > \alpha > 0.96057$. For the simulation, we choose $\alpha = 0.965$ and $T = 1$. The convergence behavior of this system is shown in Figure 2.

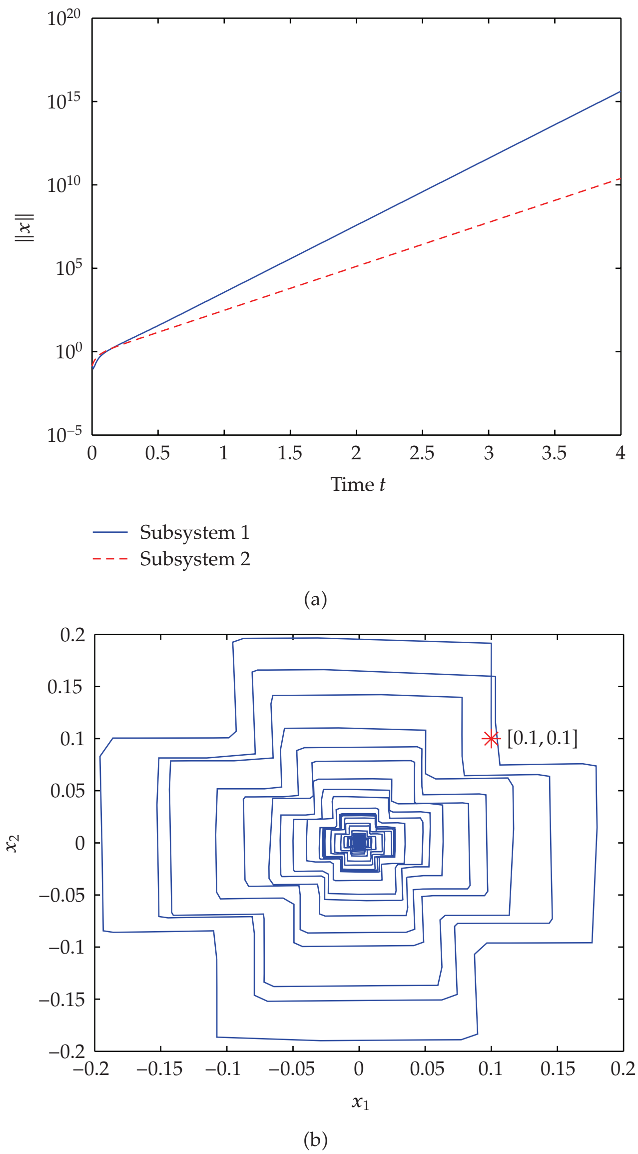

Figure 1: (a) Time response curves of norm of solution vectors of two subsystems in Example 7.1. (b) Phase diagram of the switched system in Example 7.1. The switching period $T = 0.02$, switching rate $\alpha = 0.5$, the initial value $x(0) = [0.1, 0.1]$.

8. Conclusions

Globally exponential stability of a class of periodically time-switched systems with two nonlinear subsystems has been investigated in this paper. Based on the stability property of subsystems, we divided the considered systems into three subclasses, namely, system composed of a pair of unstable subsystems; system composed of both stable and unstable subsystems; system composed of a pair of stable subsystems. A least one sufficient condition guaranteeing global exponential stability of each of three subclasses was derived by different

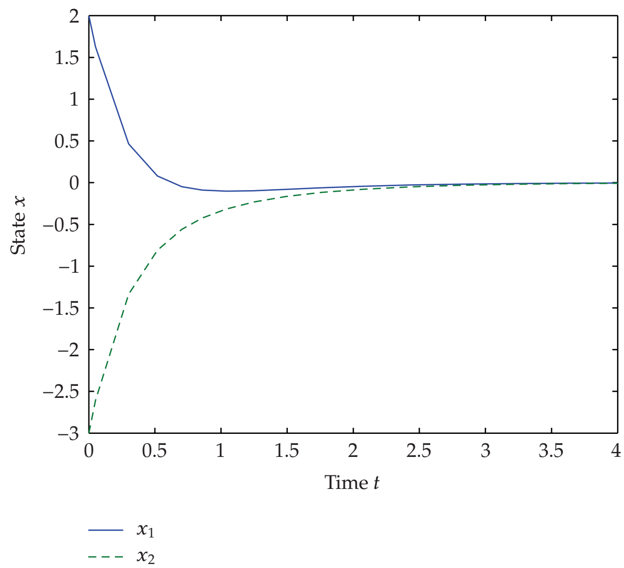

Figure 2: Time response curves of states of the system in Example 7.2.

method including the "averaged" system approach, multiple and single Lyapunov function, and robust analysis of linear time-variant systems. The periodically intermittent control design problem was also addressed. This paper focuses on only the neural-type Lipstchitz nonlinearity, therefore, the further work may deal with the general nonlinear subsystems. Also, the effect of short and strong control strength on the reduction of the control cost in the presence of output noise seems to be an interesting topic.

Appendix

A Proof of Lemma 2.1

For any $A \in N[\underline{A}, \overline{A}]$, it is easy to see that there exist real constants $\varepsilon_{ij}, |\varepsilon_{ij}| \leq 1$, such that

$$A = A_0 + \sum_{i,j=1}^{n} \varepsilon_{ij} H_{ij}, \tag{A.1}$$

where $H_{ij} = (\overline{h}_{kl}) \in R^{n\times n}$ satisfies

$$\overline{h}_{kl} = \begin{cases} h_{1,i,j}, & k = i,\ l = j, \\ 0, & \text{otherwise}. \end{cases} \tag{A.2}$$

Since $\operatorname{rank}(H_{ij}) \leq 1$, it can be decomposed as

$$H_{ij} = \sqrt{h_{1,i,j}}e_i \times \sqrt{h_{1,i,j}}e_j^T, \tag{A.3}$$

where e_i is the ith column-vector of $n \times n$ identity matrix.

From (A.1) and (A.3), we have

$$A = A_0 + \sum_{i,j=1}^{n} \varepsilon_{ij} \times \sqrt{h_{1,i,j}} e_i \times \sqrt{h_{1,i,j}} e_j^T. \tag{A.4}$$

From (4.1) and (4.2), we can see that there exists a $\Sigma_A \in \Sigma^*$ such that $A = A_0 + E_A \Sigma_A F_A$, that is, $A \in M[\underline{A}, \overline{A}]$. Hence, $N[\underline{A}, \overline{A}] \subseteq M[\underline{A}, \overline{A}]$. Noting that the process above is inverse, we can also derive the relation $N[\underline{A}, \overline{A}] \supseteq M[\underline{A}, \overline{A}]$. Therefore, $N[\underline{A}, \overline{A}] = M[\underline{A}, \overline{A}]$. Similarly, one can show $M[\underline{B}, \overline{B}] = N[\underline{B}, \overline{B}]$. The proof is thus completed.

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References

[1] M. Wicks, P. Peleties, and R. Decarlo, “Switched controller synthesis for the quadratic stabilization of a pair of unstable linear systems,” *European Journal of Control*, vol. 4, no. 2, pp. 140–147, 1998.

[2] D. Liberzon and A. S. Morse, “Basic problems in stability and design of switched systems,” *IEEE Control Systems Magazine*, vol. 19, no. 5, pp. 59–70, 1999.

[3] A. S. Morse, “Supervisory control of families of linear set-point controllers. I. Exact matching,” *IEEE Transactions on Automatic Control*, vol. 41, no. 10, pp. 1413–1431, 1996.

[4] M. S. Branicky, “Multiple Lyapunov functions and other analysis tools for switched and hybrid systems,” *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 475–482, 1998.

[5] D. Cheng, “Stabilization of planar switched systems,” *Systems & Control Letters*, vol. 51, no. 2, pp. 79–88, 2004.

[6] Z. G. Li, C. Y. Wen, and Y. C. Soh, “Stabilization of a class of switched systems via designing switching laws,” *IEEE Transactions on Automatic Control*, vol. 46, no. 4, pp. 665–670, 2001.

[7] Z. Ji, L. Wang, and G. Xie, “New results on quadratic stabilization of switched linear systems with polytopic uncertainties,” *IMA Journal of Mathematical Control and Information*, vol. 22, no. 4, pp. 441–452, 2005.

[8] X. Xu and P. J. Antsaklis, “Stabilization of second-order LTI switched systems,” *International Journal of Control*, vol. 73, no. 14, pp. 1261–1279, 2000.

[9] D. Liberzon, *Switching in Systems and Control*, Systems & Control: Foundations & Applications, Birkhäuser, Boston, Mass, USA, 2003.

[10] H. Xu, X. Liu, and K. L. Teo, “Robust H_∞ stabilisation with definite attenuance of an uncertain impulsive switched system,” *The ANZIAM Journal*, vol. 46, no. 4, pp. 471–484, 2005.

[11] H. Xu, X. Liu, and K. L. Teo, “Delay independent stability criteria of impulsive switched systems with time-invariant delays,” *Mathematical and Computer Modelling*, vol. 47, no. 3-4, pp. 372–379, 2008.

[12] H. Xu, X. Liu, and K. L. Teo, “A LMI approach to stability analysis and synthesis of impulsive switched systems with time delays,” *Nonlinear Analysis: Hybrid Systems*, vol. 2, no. 1, pp. 38–50, 2008.

[13] C. Y.-F. Ho, B. W.-K. Ling, Y.-Q. Liu, P. K.-S. Tam, and K.-L. Teo, “Optimal PWM control of switched-capacitor DC-DC power converters via model transformation and enhancing control techniques,” *IEEE Transactions on Circuits and Systems I*, vol. 55, no. 5, pp. 1382–1391, 2008.

[14] J. P. Hespanha and A. S. Morse, “Stability of switched systems with average dwell-time,” in *Proceedings of the 38th IEEE Conference on Decision and Control (CDC ’99)*, vol. 3, pp. 2655–2660, Phoenix, Ariz, USA, December 1999.

[15] B. Hu, X. Xu, A. N. Michel, and P. J. Antsaklis, “Stability analysis for a class of nonlinear switched system,” in *Proceedings of the 38th IEEE Conference on Decision and Control (CDC ’99)*, vol. 5, pp. 4374–4379, Phoenix, Ariz, USA, December 1999.

[16] G. Zhai, B. Hu, K. Yasuda, and A. N. Michel, “Stability analysis of switched systems with stable and unstable subsystems: an average dwell time approach,” in *Proceedings of the American Control Conference*, vol. 1, pp. 200–204, Chicago, Ill, USA, 2000.

[17] C. D. Li, X. F. Liao, and R. Zhang, “Global robust asymptotical stability of multidelayed interval neural networks: an LMI approach,” *Physics Letters A*, vol. 328, no. 6, pp. 452–462, 2004.

[18] F. X. Wu, Z. K. Shi, and Z. X. Zhou, “Robust stabilization of linear time-variant interval systems,” in *Proceedings of the 3rd World Conference on Intelligent Control and Automation*, pp. 3415–3418, Hefei, China, July 2000.

[19] E. N. Sanchez and J. P. Perez, “Input-to-state stability (ISS) analysis for dynamic neural networks,” *IEEE Transactions on Circuits and Systems I*, vol. 46, no. 11, pp. 1395–1398, 1999.

[20] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15 of *SIAM Studies in Applied Mathematics*, SIAM, Philadelphia, Pa, USA, 1994.

[21] A. van der Schaft and H. Schumacher, *An Introduction to Hybrid Dynamical Systems*, vol. 251 of *Lecture Notes in Control and Information Sciences*, Springer, London, UK, 2000.

[22] T. W. Carr and I. B. Schwartz, “Controlling unstable steady states using system parameter variation and control duration,” *Physical Review E*, vol. 50, no. 5, pp. 3410–3415, 1994.

[23] T. W. Carr and I. B. Schwartz, “Controlling the unstable steady state in a multimode laser,” *Physical Review E*, vol. 51, no. 5, pp. 5109–5111, 1995.

[24] T. W. Carr and I. B. Schwartz, “Controlling high-dimensional unstable steady states using delay, duration and feedback,” *Physica D*, vol. 96, no. 1–4, pp. 1–25, 1996.

[25] J. Starrett, “Control of chaos by occasional bang-bang,” *Physical Review E*, vol. 67, no. 3, Article ID 036203, 4 pages, 2003.

[26] T. L. Montgomery, J. W. Frey, and W. B. Norris, “Intermittent control systems for SO_2,” *Environmental Science and Technology*, vol. 9, no. 6, pp. 528–532, 1975.

[27] Y. Zhang, Z.-W. Zhou, and G.-C. Guo, “Decoupling neighboring qubits in quantum computers through bang-bang pulse control,” *Physics Letters A*, vol. 327, no. 5-6, pp. 391–396, 2004.

[28] L. Viola and S. Lloyd, “Dynamical suppression of decoherence in two-state quantum systems,” *Physical Review A*, vol. 58, no. 4, pp. 2733–2744, 1998.

[29] L. Viola, E. Knill, and S. Lloyd, “Dynamical decoupling of open quantum systems,” *Physical Review Letters*, vol. 82, no. 12, pp. 2417–2421, 1999.

[30] L.-M. Duan and G.-C. Guo, “Suppressing environmental noise in quantum computation through pulse control,” *Physics Letters A*, vol. 261, no. 3-4, pp. 139–144, 1999.

[31] P. Zanardi, “Symmetrizing evolutions,” *Physics Letters A*, vol. 258, no. 2-3, pp. 77–82, 1999.

[32] T. Huang, C. Li, and X. Liu, “Synchronization of chaotic systems with delay using intermittent linear state feedback,” *Chaos*, vol. 18, no. 3, Article ID 033122, 8 pages, 2008.

[33] T. Yang, *Impulsive Control Theory*, vol. 272 of *Lecture Notes in Control and Information Sciences*, Springer, Berlin, Germany, 2001.

[34] T. Yang, *Impulsive Systems and Control: Theory and Applications*, Nova, Huntington, NY, USA, 2001.