Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 381324, 8 pages
doi:10.1155/2009/381324

Research Article

Sums of Products of q-Euler Polynomials and Numbers

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Received 4 March 2009; Accepted 25 April 2009

Recommended by Vijay Gupta

We derive formulae for the sums of products of the q-Euler polynomials and numbers using the multivariate fermionic p-adic q-Volkenborn integral on $\mathbb{Z}_p$.

1. Introduction

The purpose of this paper is to derive formulae for the sums of products of the q-Euler polynomials and numbers, since many identities can be obtained from our sums of products of the q-Euler polynomials and numbers. In [1], Simsek evaluated the complete sums for the Euler numbers and polynomials and obtained some identities related to Euler numbers and polynomials from his complete sums, and Jang et al. [2] also considered the sums of products of Euler numbers. Kim [3] derived the sums of products of the q-Euler numbers using the fermionic p-adic q-Volkenborn integral. In this paper, we will evaluate the complete sum of the q-Euler polynomials and numbers using the fermionic p-adic q-Volkenborn integral on $\mathbb{Z}_p$. Assume that p is a fixed odd prime. Throughout this paper, the symbols $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks about q-extension, q is variously considered as an indeterminate, which is a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then one assumes $|q-1|_p < 1$. We use the notations

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}, \tag{1.1}$$

for all $x \in \mathbb{Z}_p$. Hence $\lim_{q\to 1}[x]_q = x$ for any x with $|x|_p \le 1$ (cf. [3–12]).

For a fixed odd positive integer d with $(p, d) = 1$, let

$$X = X_d = \lim_{\overrightarrow{N}} \mathbb{Z}/dp^N\mathbb{Z}, \qquad X_1 = \mathbb{Z}_p,$$
$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\,\mathbb{Z}_p), \tag{1.2}$$
$$a + dp^N\mathbb{Z}_p = \left\{ x \in X \mid \ x \equiv a \left(\bmod dp^N \right) \right\},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. The distribution on X is defined by

$$\mu_q\left(a + dp^N\mathbb{Z}_p\right) = \frac{q^a}{\left[dp^N\right]_q}. \tag{1.3}$$

We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$, if the difference quotients $F_f(x, y) = (f(x) - f(y))/(x - y)$ have a limit $l = f'(a)$ as $(x, y) \to (a, a)$.

For $f \in UD(\mathbb{Z}_p)$, the q-deformed bosonic p-adic integral is defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{\left[p^N\right]_q} \sum_{x=0}^{p^N-1} f(x)q^x \tag{1.4}$$

(see [12]). The fermionic p-adic q-measures on $\mathbb{Z}_p$ are defined as

$$\mu_{-q}\left(a + dp^N\mathbb{Z}_p\right) = \frac{(-q)^a}{\left[dp^N\right]_{-q}}, \tag{1.5}$$

and the q-deformed fermonic p-adic integral is defined by

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{\left[p^N\right]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \tag{1.6}$$

(see [6]), for $f \in UD(\mathbb{Z}_p)$. The fermionic p-adic integral on $\mathbb{Z}_p$ is defined as

$$I_{-1}(f) = \lim_{q\to 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x). \tag{1.7}$$

It follows that $I_{-1}(f_1) = -I_{-1}(f) + 2f(0)$, where $f_1(x) = f(x+1)$. If we take $f(x) = e^{tx}$, then the classical Euler polynomials $E_n(x)$ are defined by the generating function,

$$\frac{2}{e^t+1}e^{xt} = \sum_{x=0}^{\infty} E_n(x)\frac{t^n}{n!}, \tag{1.8}$$

and the Euler numbers E_n are defined as $E_n = E_n(0)$ (cf. [1–20]).

It is known that the q-Euler numbers $E_{n,q}$ are defined as

$$E_{n,q} = \int_{\mathbb{Z}_p} [x]_q^n d\mu_{-q}(x), \tag{1.9}$$

and the q-Euler polynomials $E_{n,q}(x)$ are defined as

$$E_{n,q}(x) = \int_{\mathbb{Z}_p} [x+t]_q^n d\mu_{-q}(t), \tag{1.10}$$

where $x \in \mathbb{C}$ with $|x|_p \leq 1$. We note that $\lim_{q\to1} E_{n,q} = E_n$ and $\lim_{q\to1} E_{n,q}(x) = E_n(x)$ (cf. [3, 6–12, 15–17]).

Let $r \in \mathbb{N}$. We consider the q-Euler numbers $E_{n,q}^{(r)}$ of order r defined by

$$E_{n,q}^{(r)} = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [x_1 + \cdots + x_r]_q^n \, d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r), \tag{1.11}$$

and the q-Euler polynomials $E_{n,q}^{(r)}(x)$ of order r defined by

$$E_{n,q}^{(r)}(x) = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [x + x_1 + \cdots + x_r]_q^n \, d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r). \tag{1.12}$$

(see [5–7]). In Section 2, we evaluate the following multivariate fermionic p-adic q-integral :

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [N_1(x_1+\alpha_1) + \cdots + N_r(x_r+\alpha_r)]_q^n \, d\mu_{-q^{N_1}}(x_1) \cdots d\mu_{-q^{N_r}}(x_r), \tag{1.13}$$

for any elements $\alpha_1, \ldots, \alpha_r \in \mathbb{C}_p$, $n \in \mathbb{N}$ and distinct odd positive integers $N_1, \ldots, N_r$. We have the formulae for the complete sum of the products of q-Euler polynomials related to the higher order q-Euler polynomials using the fermionic p-adic q-Volkenborn integral on $\mathbb{Z}_p$. We also obtain the formulae for the q-Euler numbers.

In [21–24], Khrennikov introduced other theories of p-adic distributions which were recently generated in p-adic mathematical physics, both bosonic and fermionic: Khrennikov tried to build a p-adic picture of reality based on the field of p-adic numbers $\mathbb{Q}_p$ and corresponding analysis (a particular case of so-called non-Archimedean analysis). He showed that many problems of the description of reality with the aid of real numbers are induced by unlimited application of the Archimedean axiom. This axiom means that the physical observation can be measured with an infinite exactness. The results connected with an infinite exactness of measurements appear all the time in the formalisms of quantum mechanics and quantum field theories, which have the real continuum as one of their foundations. In particular, the author explains that the famous EPR paradox is nothing other than a result of using ideal real elements corresponding to an infinite exactness of measurement of the position and the momentum of a quantum particle. From the author's point of view, the EPR paradox is only a new form of Zeno's ancient paradox of Achilles and the tortoise. Both of these paradoxes are connected with the notion of an infinitely deep and infinitely divisible real continuum (see [21, 22]). In [23, 24], Khrennikov outlines both the p-adic frequency model and a measure-theoretic approach. The latter is understood in the sense of non-Archimedean integration theory where measures have only additive property, not σ-additive property, and satisfy a condition of the boundedness. Analogues of the laws of large numbers including the central limit theorem are given. They studied a possible statistical interpretation of group-valued probabilities as well as nontraditional probabilistic models in physics and the cognitive sciences.

2. Sums of Products of q-Euler Polynomials and Numbers

Let $\alpha_1, \cdots, \alpha_r \in \mathbb{C}_p$, $n \in \mathbb{N}$, and let $N_1, \ldots, N_r$ be distinct odd positive integers. Let N be the least common multiple of $N_1, \ldots, N_r$.

Now we evaluate the multivariate fermionic p-adic q-integral

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [N_1(x_1+\alpha_1)+\cdots+N_r(x_r+\alpha_r)]_q^n d\mu_{-q^{N_1}}(x_1)\cdots d\mu_{-q^{N_r}}(x_r). \tag{2.1}$$

By the definition of the multivariate p-adic q-integral, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [N_1(x_1+\alpha_1)+\cdots+N_r(x_r+\alpha_r)]_q^n d\mu_{-q^{N_1}}(x_1)\cdots d\mu_{-q^{N_r}}(x_r) \\
&\quad = \lim_{\rho\to\infty} \frac{1}{[N/N_1 p^\rho]_{-q^{N_1}} \cdots [N/N_r p^\rho]_{-q^{N_r}}} \\
&\qquad \times \sum_{x_1=0}^{(N/N_1)p^\rho-1} \cdots \sum_{x_r=0}^{(N/N_r)p^\rho-1} [N_1(x_1+\alpha_1)+\cdots+N_r(x_r+\alpha_r)]_q^n (-q)^{N_1x_1+\cdots+N_rx_r} \\
&\quad = \lim_{\rho\to\infty} \frac{[N_1]_{-q}[N_2]_{-q}\cdots[N_r]_{-q}}{[N]_{-q}^r [p^\rho]_{-q^N}^r} \\
&\qquad \times \sum_{i_1=0}^{N/N_1-1} \cdots \sum_{i_r=0}^{N/N_r-1} \sum_{n_1,\cdots,n_r=0}^{p^\rho-1} (-q)^{N_1(i_1+(N/N_1)n_1)+\ldots+N_r(i_r+(N/N_r)n_r)} \\
&\qquad \times \left[N_1\left(i_1+\frac{N}{N_1}n_1+\alpha_1\right)+\ldots+N_r\left(i_r+\frac{N}{N_r}n_r+\alpha_r\right)\right]_q^n,
\end{aligned} \tag{2.2}$$

where $\sum_{n_1,\ldots,n_r=0}^{p^\rho-1} = \sum_{n_1=0}^{p^\rho-1}\cdots\sum_{n_r=0}^{p^\rho-1}$. The second equality in (2.2) is satisfied by the equation

$$\frac{1}{[(N/N_j)p^\rho]_{-q^{N_j}}} = \frac{1+q^{N_j}}{1+q^{Np^\rho}} = \frac{[N_j]_{-q}}{[N]_{-q}[p^\rho]_{-q^N}}, \quad \text{for } 1 \le j \le r. \tag{2.3}$$

We easily see that

$$\begin{aligned} &\left[N_1\left(i_1 + \frac{N}{N_1}n_1 + \alpha_1\right) + \cdots + N_r\left(i_r + \frac{N}{N_r}n_r + \alpha_r\right)\right]_q^n \\ &\quad = [N]_q^n\left[\frac{N_1}{N}(i_1+\alpha_1) + \cdots + \frac{N_r}{N}(i_r+\alpha_r) + n_1 + \cdots + n_r\right]_{q^N}^n. \end{aligned} \tag{2.4}$$

From (2.2), (2.4), and the definition of the q-Euler polynomials, we derive the following equations:

$$\begin{aligned} &\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[N_1(x_1+\alpha_1)+\cdots+N_r(x_r+\alpha_r)]_q^n\, d\mu_{-q^{N_1}}(x_1)\cdots d\mu_{-q^{N_r}}(x_r) \\ &= \frac{[2]_q^r}{[2]_{q^N}^r}[N]_q^n[N_1]_{-q}[N_2]_{-q}\cdots[N_r]_{-q}\sum_{i_1=0}^{N/N_1-1}\cdots\sum_{i_r=0}^{N/N_r-1}(-1)^{i_1+\cdots+i_r}q^{N_1i_1+\cdots+N_ri_r} \\ &\quad\times\lim_{\rho\to\infty}\frac{1}{[p^\rho]_{-q^N}^r}\sum_{n_1,\ldots,n_r=0}^{p^\rho-1}\left[\frac{N_1}{N}(i_1+\alpha_1)+\cdots+\frac{N_r}{N}(i_r+\alpha_r)+n_1+\cdots+n_r\right]_{q^N}^n \\ &\quad\times(-q)^{N(n_1+\cdots+n_r)} \\ &= \frac{[2]_q^r}{[2]_{q^N}^r}[N]_q^n[N_1]_{-q}[N_2]_{-q}\cdots[N_r]_{-q}\sum_{i_1=0}^{N/N_1-1}\cdots\sum_{i_r=0}^{N/N_r-1}(-1)^{i_1+\cdots+i_r}q^{N_1i_1+\cdots+N_ri_r} \\ &\quad\times E_{n,q^N}^{(r)}\left(\frac{N_1}{N}(i_1+\alpha_1)+\cdots+\frac{N_r}{N}(i_r+\alpha_r)\right). \end{aligned} \tag{2.5}$$

Therefore we have the following theorem.

Theorem 2.1. *Let $r \in \mathbb{N}, n \in \mathbb{N}$, and let $\alpha_1, \ldots, \alpha_r \in \mathbb{C}_p$. Let $N_1, \ldots, N_r$ be distinct odd positive integers, and let N be the least common multiple of $N_1, \ldots, N_r$. Then we have*

$$\begin{aligned} &\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[N_1(x_1+\alpha_1)+\cdots+N_r(x_r+\alpha_r)]_q^n\, d\mu_{-q^{N_1}}(x_1)\cdots d\mu_{-q^{N_r}}(x_r) \\ &= \frac{[2]_q^r}{[2]_{q^N}^r}[N]_q^n[N_1]_{-q}[N_2]_{-q}\cdots[N_r]_{-q}\sum_{i_1=0}^{N/N_1-1}\cdots\sum_{i_r=0}^{N/N_r-1}(-1)^{i_1+\cdots+i_r}q^{N_1i_1+\cdots+N_ri_r} \\ &\quad\times E_{n,q^N}^{(r)}\left(\frac{N_1}{N}(i_1+\alpha_1)+\cdots+\frac{N_r}{N}(i_r+\alpha_r)\right). \end{aligned} \tag{2.6}$$

By using the multinomial theorem, we can obtain the following Theorem 2.2. Theorem 2.2 is important to derive the main results of our paper.

Theorem 2.2. *Let $r \in \mathbb{N}, n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_r \in \mathbb{C}_p$. Let $N_1, \ldots, N_r$ be distinct odd positive integers. Then we have*

$$\begin{aligned}
&[N_1(x_1+\alpha_1)+\cdots+N_r(x_r+\alpha_r)]_q^n \\
&\quad= \sum_{\substack{i_1,\ldots,i_r\geq 0\\ i_1+\cdots+i_r=n}} \sum_{k_1=0}^{n-i_1} \sum_{k_2=0}^{n-i_1-i_2} \cdots \sum_{k_{r-1}=0}^{n-i_1-\cdots-i_{r-1}} \\
&\quad\times \binom{n}{i_1,\cdots,i_r}\binom{n-i_1}{k_1}\binom{n-i_1-i_2}{k_2}\cdots\binom{n-i_1-i_2-\cdots-i_{r-1}}{k_{r-1}} \\
&\quad\times (q-1)^{k_1+\cdots+k_{r-1}}[N_1]_q^{i_1+k_1}\cdots[N_{r-1}]_q^{i_{r-1}+k_{r-1}}[N_r]_q^{i_r} \\
&\quad\times [x_1+\alpha_1]_{q^{N_1}}^{i_1+k_1}\cdots[x_{r-1}+\alpha_{r-1}]_{q^{N_{r-1}}}^{i_{r-1}+k_{r-1}}[x_r+\alpha_r]_{q^{N_r}}^{i_r}.
\end{aligned} \tag{2.7}$$

By Theorem 2.2, we obtain

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[N_1(x_1+\alpha_1)+\cdots+N_r(x_r+\alpha_r)]_q^n d\mu_{-q^{N_1}}(x_1)\cdots d\mu_{-q^{N_r}}(x_r) \\
&\quad= \sum_{\substack{i_1,\ldots,i_r\geq 0\\ i_1+\cdots+i_r=n}} \sum_{k_1=0}^{n-i_1} \sum_{k_2=0}^{n-i_1-i_2} \cdots \sum_{k_{r-1}=0}^{n-i_1-\cdots-i_{r-1}} \binom{n}{i_1,\ldots,i_r} \\
&\quad\times \binom{n-i_1}{k_1}\binom{n-i_1-i_2}{k_2}\cdots\binom{n-i_1-\cdots-i_{r-1}}{k_{r-1}} \\
&\quad\times (q-1)^{k_1+\cdots+k_{r-1}}[N_1]_q^{i_1+k_1}\cdots[N_{r-1}]_q^{i_{r-1}+k_{r-1}}[N_r]_q^{i_r} \\
&\quad\times E_{i_1+k_1,q^{N_1}}(\alpha_1)\cdots E_{i_{r-1}+k_{r-1},q^{N_{r-1}}}(\alpha_{r-1})E_{i_r,q^{N_r}}(\alpha_r).
\end{aligned} \tag{2.8}$$

Hence we have the complete sum for q-Euler polynomials as follows.

Theorem 2.3. *Let $r \in \mathbb{N}, n \in \mathbb{N}$, and let $\alpha_1, \ldots, \alpha_r \in \mathbb{C}_p$. Let $N_1, \ldots, N_r$ be distinct odd positive integers and N be the least common multiple of $N_1, \ldots, N_r$. Then we have*

$$\begin{aligned}
&\sum_{\substack{i_1,\ldots,i_r\geq 0\\ i_1+\cdots+i_r=n}} \sum_{k_1=0}^{n-i_1} \sum_{k_2=0}^{n-i_1-i_2} \cdots \sum_{k_{r-1}=0}^{n-i_1-\cdots-i_{r-1}} \binom{n}{i_1,\cdots,i_r}\binom{n-i_1}{k_1}\binom{n-i_1-i_2}{k_2}\cdots \\
&\quad\times \binom{n-i_1-\cdots-i_{r-1}}{k_{r-1}}(q-1)^{k_1+\cdots+k_{r-1}}[N_1]_q^{i_1+k_1}\cdots[N_{r-1}]_q^{i_{r-1}+k_{r-1}}[N_r]_q^{i_r} \\
&\quad\times E_{i_1+k_1,q^{N_1}}(\alpha_1)\cdots E_{i_{r-1}+k_{r-1},q^{N_{r-1}}}(\alpha_{r-1})E_{i_r,q^{N_r}}(\alpha_r)
\end{aligned}$$

$$= \frac{[2]_q^r}{[2]_{q^N}^r}[N]_q^n[N_1]_{-q}[N_2]_{-q}\cdots[N_r]_{-q}\sum_{i_1=0}^{(N/N_1)-1}\cdots\sum_{i_r=0}^{(N/N_r)-1}(-1)^{i_1+\cdots+i_r}q^{N_1 i_1+\cdots+N_r i_r}$$
$$\times E_{n,q^N}^{(r)}\left(\frac{N_1}{N}(i_1+\alpha_1)+\cdots+\frac{N_r}{N}(i_r+\alpha_r)\right). \tag{2.9}$$

When $\alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$ in Theorem 2.3, we obtain the following formula involving the q-Euler numbers.

Corollary 2.4. *Let $r \in \mathbb{N}, n \in \mathbb{N}$, and $\alpha_1, \ldots, \alpha_r \in \mathbb{C}_p$. Let $N_1, \ldots, N_r$ be distinct odd positive integers, and let N be the least common multiple of $N_1, \ldots, N_r$. Then we have*

$$\sum_{\substack{i_1,\ldots,i_r\geq 0\\ i_1+\cdots+i_r=n}}\sum_{k_1=0}^{n-i_1}\sum_{k_2=0}^{n-i_1-i_2}\cdots\sum_{k_{r-1}=0}^{n-i_1-\cdots-i_{r-1}}\binom{n}{i_1,\ldots,i_r}$$
$$\times\binom{n-i_1}{k_1}\binom{n-i_1-i_2}{k_2}\cdots\binom{n-i_1-\cdots-i_{r-1}}{k_{r-1}}$$
$$\times(q-1)^{k_1+\cdots+k_{r-1}}[N_1]_q^{i_1+k_1}\cdots[N_{r-1}]_q^{i_{r-1}+k_{r-1}}[N_r]_q^{i_r}$$
$$\times E_{i_1+k_1,q^{N_1}}\cdots E_{i_{r-1}+k_{r-1},q^{N_{r-1}}}E_{i_r,q^{N_r}}$$
$$= \frac{[2]_q^r}{[2]_{q^N}^r}[N]_q^n[N_1]_{-q}[N_2]_{-q}\cdots[N_r]_{-q}\sum_{i_1=0}^{N/N_1-1}\cdots\sum_{i_r=0}^{N/N_r-1}(-1)^{i_1+\cdots+i_r}q^{N_1 i_1+\cdots+N_r i_r}$$
$$\times E_{n,q^N}^{(r)}\left(\frac{N_1 i_1+\cdots+N_r i_r}{N}\right). \tag{2.10}$$

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