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Research Article

On the Generalized B^m-Riesz Difference Sequence Space and β-Property

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We introduce the generalized Riesz difference sequence space $r^q(p, B^m)$ which is defined by $r^q(p, B^m) = \{x = (x_k) \in w : B^m x \in r^q(p)\}$ where $r^q(p)$ is the Riesz sequence space defined by Altay and Başar. We give some topological properties, compute the α_-, β_- duals, and determine the Schauder basis of this space. Finally; we study the characterization of some matrix mappings on this sequence space. At the end of the paper, we investigate some geometric properties of $r^q(p, B^m)$ and we have proved that this sequence space has property (β) for $p_k \geq 1$.

1. Introduction

Let w be the space of all real valued sequences. We write l_∞, c, c_0 for the sequence spaces of all bounded, convergent, and null sequences, respectively. Also by cs, l_1, and l_p, we denote the sequence spaces of all convergent, absolutely and p-absolutely, convergent series, respectively; where $1 < p < \infty$.

Let (q_k) be a sequence of positive numbers and

$$Q_n = \sum_{k=0}^{n} q_k, \quad (n \in \mathbb{N}). \tag{1.1}$$

Then the matrix $R^q = (r^q_{nk})$ of the Riesz mean (R, q_n), which is triangle limitation matrix, is given by

$$r^q_{nk} = \begin{cases} \dfrac{q_k}{Q_n}, & (0 \leq k \leq n), \\ 0, & (k > n). \end{cases} \tag{1.2}$$

It is well known that the matrix $R^q = (r^q_{nk})$ is regular if and only if $Q_n \to \infty$ as $n \to \infty$.

Altay and Başar [1, 2] introduced the Riesz sequence space $r^q(p)$, $r^q_\infty(p)$, $r^q_c(p)$, and $r^q_{c_0}(p)$ of nonabsolute type which is the set of all sequences whose R^q-transforms are in the space $l(p)$, $l_\infty(p)$, $c(p)$, and $c_0(p)$; respectively. Here and afterwards, $p=(p_k)$ will be used as a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M=\max\{1,H\}$ and $\mathcal{F}$ denotes the collection of all finite subsets of $\mathbb{N}$, where $\mathbb{N}=\{0,1,2,\ldots\}$. The Riesz sequence space introduced in [1] by Altay and Başar is

$$r^q(p)=\left\{x=(x_k)\in w:\sum_{k=0}^{\infty}\left|\frac{1}{Q_k}\sum_{j=0}^{k}q_jx_j\right|^{p_k}<\infty\right\};\quad \text{with } (0<p_k\le H<\infty). \tag{1.3}$$

The difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$ were first defined and studied by Kızmaz in [3] and studied by several authors, [4–9]. Başar and Altay [10] have studied the sequence space bv_p as the set of all sequences such that their Δ-transforms are in the space l_p; that is,

$$bv_p=\left\{x=(x_k)\in w:\sum_{k=0}^{\infty}|x_k-x_{k-1}|^p<\infty\right\},\quad 1\le p<\infty, \tag{1.4}$$

where Δ denotes the matrix $\Delta=(\Delta_{nk})$ defined by

$$\Delta_{nk}=\begin{cases}(-1)^{n-k}; & (n-1\le k\le n),\\ 0; & (k<n-1)\text{ or }(k>n).\end{cases} \tag{1.5}$$

The idea of difference sequences is generalized by Çolak and Et [11]. They defined the sequence spaces:

$$\Delta^m\lambda=\{x=(x_k)\in w:\Delta^m x\in\lambda\}, \tag{1.6}$$

where $m\in\mathbb{N}$, $\Delta^1x=x_k-x_{k+1}$, and $\Delta^m x=\Delta(\Delta^{m-1}x)$, where Δ^m denotes the matrix $\Delta^m=(\Delta^m_{nk})$ defined by

$$\Delta^m_{nk}=\begin{cases}(-1)^{n-k}\dbinom{m}{n-k}; & (\max\{0,n-m\}\le k\le n),\\ 0; & (0\le k<\max\{0,n-m\})\text{ or }(k>n),\end{cases} \tag{1.7}$$

for all $k,n\in\mathbb{N}$ and for any fixed $m\in\mathbb{N}$.

Recently, Başarir and Öztürk [12] introduced the Riesz difference sequence space as follows:

$$r^q(p,\Delta)=\{x=(x_k)\in w:\Delta x=(x_k-x_{k-1})\in r^q(p)\};\quad \text{with } (0<p_k\le H<\infty). \tag{1.8}$$

Başar and Altay defined the matrix $B = (b_{nk})$ which generalizes the matrix $\Delta = (\Delta_{nk})$. Now we define the matrix $B^m = (b^m_{nk})$ and if we take $r = 1$, $s = -1$, then it corresponds to the matrix $\Delta^m = (\Delta^m_{nk})$. We define

$$b^m_{nk} = \begin{cases} \binom{m}{n-k} r^{m-n+k} s^{n-k}; & (\max\{0, n-m\} \le k \le n), \\ 0; & (0 \le k < \max\{0, n-m\}) \text{ or } (k > n). \end{cases} \tag{1.9}$$

The results related to the matrix domain of the matrix B^m are more general and more comprehensive than the corresponding consequences of matrix domain of Δ^m.

Our main subject in the present paper is to introduce the generalized Riesz difference sequence space $r^q(p, B^m)$ which consists of all the sequences such that their B^m-transforms are in the space $r^q(p)$ and to investigate some topological and geometric properties with respect to paranorm on this space.

2. Basic Facts and Definitions

In this section we give some definitions and lemmas which will be frequently used.

Definition 2.1. Let λ and μ be two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A : \lambda \to \mu$ if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ; where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad (n \in \mathbb{N}). \tag{2.1}$$

By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (2.1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A-summable to α if Ax converges to α which is called as the A-limit of x.

Definition 2.2. For any sequence space λ, the matrix domain λ_A of an infinite matrix A is defined by

$$\lambda_A = \{x = (x_k) \in w : Ax \in \lambda\}. \tag{2.2}$$

Definition 2.3. If a sequence space λ paranormed by h contains a sequence (b_n) with the property that for every $x \in \lambda$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} h\left(x - \sum_{k=0}^{n} \alpha_k b_k\right) = 0, \tag{2.3}$$

then (b_n) is called a Schauder basis (or briefly basis) for λ. The series $\sum_{k=0}^{\infty} \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and is written as $x = \sum_{k=0}^{\infty} \alpha_k b_k$.

Definition 2.4. For the sequence spaces λ and μ, define the set $S(\lambda,\mu)$ by

$$S(\lambda,\mu)=\{z=(z_k)\in w : xz=(x_k z_k)\in\mu\ \forall x\in\lambda\}. \tag{2.4}$$

With the notation of (2.2), the α-, β-, γ-duals of a sequence space λ, which are, respectively, denoted by $\lambda^\alpha, \lambda^\beta, \lambda^\gamma$, are defined by

$$\lambda^\alpha=S(\lambda,l_1),\qquad \lambda^\beta=S(\lambda,cs),\qquad \lambda^\gamma=S(\lambda,bs). \tag{2.5}$$

Now we give some lemmas which we need to prove our theorems.

Lemma 2.5 (see [13]). (i) *Let $1<p_k\le H<\infty$ for every $k\in\mathbb{N}$. Then $A\in(l(p):l_1)$ if and only if there exists an integer $K>1$ such that*

$$\sup_{K\in\mathcal{F}}\sum_{k=0}^{\infty}\left|\sum_{n\in K}a_{nk}K^{-1}\right|^{p'_k}<\infty. \tag{2.6}$$

(ii) *Let $0<p_k\le 1$ for every $k\in\mathbb{N}$. Then $A\in(l(p):l_1)$ if and only if*

$$\sup_{K\in\mathcal{F}}\sup_{k\in\mathbb{N}}\left|\sum_{n\in K}a_{nk}\right|^{p_k}<\infty. \tag{2.7}$$

Lemma 2.6 (see [14]). (i) *Let $1<p_k\le H<\infty$ for every $k\in\mathbb{N}$. Then $A\in(l(p):l_\infty)$ if and only if there exists an integer $K>1$ such that*

$$\sup_{n\in\mathbb{N}}\sum_{k=0}^{\infty}\left|a_{nk}^{-1}K^{-1}\right|^{p'_k}<\infty. \tag{2.8}$$

(ii) *Let $0<p_k\le 1$ for every $k\in\mathbb{N}$. Then $A\in(l(p):l_\infty)$ if and only if*

$$\sup_{n,k\in\mathbb{N}}|a_{nk}|^{p_k}<\infty. \tag{2.9}$$

Lemma 2.7 (see [14]). *Let $0<p_k\le H<\infty$ for every $k\in\mathbb{N}$. Then $A\in(l(p):c)$ if and only if* (2.8), (2.9) *hold, and*

$$\lim_{n\to\infty}a_{nk}=\beta_k\quad \text{for } k\in\mathbb{N} \tag{2.10}$$

also holds.

3. Some Topological Properties of Generalized B^m-Riesz Difference Sequence Space

Let us define the sequence $y = \{y_n(q)\}$, which will be used for the $(R^q B^m)$-transform of a sequence $x = (x_k)$, that is,

$$y_n(q) = (R^q B^m x)_n = \frac{1}{Q_n}\sum_{k=0}^{n-1}\left[\sum_{i=k}^{n}\binom{m}{i-k} r^{m-i+k} s^{i-k} q_i x_k\right] + \frac{r^m}{Q_n} q_n x_n. \tag{3.1}$$

After this, by $R^q B^m$, we denote the matrix $R^q B^m = (r_{nk}(m,q,r,s))$ defined by

$$r_{nk}(m,q,r,s) = \begin{cases} \dfrac{1}{Q_n}\displaystyle\sum_{k=0}^{n-1}\left[\sum_{i=k}^{n}\binom{m}{i-k} r^{m-i+k} s^{i-k} q_i\right], & (k < n), \\ \dfrac{r^m}{Q_n} q_n, & (k = n), \\ 0, & (k > n), \end{cases} \tag{3.2}$$

for all $n, k, m \in \mathbb{N}$. Then we define

$$\begin{aligned} r^q(p, B^m) &= \{x = (x_k) \in w : y_n(q) \in l(p)\} \\ &= \left\{x = (x_k) \in w : \sum_{k=0}^{\infty}\left|\frac{1}{Q_k}\sum_{n=0}^{k-1}\left[\sum_{i=n}^{k}\binom{m}{i-n} r^{m-i+n} s^{i-n} q_i x_n\right] + \frac{r^m}{Q_k} q_k x_k\right|^{p_k} < \infty\right\}. \end{aligned} \tag{3.3}$$

If we take $m = 1$, then we have

$$r^q(p, B) = \left\{x = (x_k) \in w : \sum_{k=0}^{\infty}\left|\frac{1}{Q_k}\left[\sum_{j=0}^{k-1}(q_j r + q_{j+1} s) x_j + q_k r x_k\right]\right|^{p_k} < \infty\right\}. \tag{3.4}$$

Here are some topological properties of the generalized Riesz difference sequence space.

Theorem 3.1. *The sequence space $r^q(p, B^m)$ is a complete linear metric space paranormed by*

$$g(x) = \left(\sum_{k=0}^{\infty}\left|\frac{1}{Q_k}\sum_{j=0}^{k-1}\left[\sum_{i=j}^{k}\binom{m}{i-j} r^{m-i+j} s^{i-j} q_i x_j\right] + \frac{r^m q_k}{Q_k} x_k\right|^{p_k}\right)^{1/M}, \tag{3.5}$$

where $H = \sup_k p_k$ and $M = \max(1, H)$.

Proof. The linearity of $r^q(p,B^m)$ with respect to the co-ordinatewise addition and scalar multiplication follows from the inequalites which are satisfied for $u,v\in r^q(p,B^m)$ [15]:

$$\left(\sum_{k=0}^{\infty}\left|(R^qB^mu)_k+(R^qB^mv)_k\right|^{p_k}\right)^{1/M}\le\left(\sum_{k=0}^{\infty}\left|(R^qB^mu)_k\right|^{p_k}\right)^{1/M}+\left(\sum_{k=0}^{\infty}\left|(R^qB^mv)_k\right|^{p_k}\right)^{1/M}, \tag{3.6}$$

and for any $\alpha\in\mathbb{R}$ [16], we have

$$|\alpha|^{p_k}\le\max\left\{1,|\alpha|^M\right\}. \tag{3.7}$$

It is obvious that $g(\theta)=0$ and $g(-u)=g(u)$ for all $u\in r^q(p,B^m)$. Let $u_k,v_k\in r^q(p,B^m)$:

$$\begin{aligned} g(u+v) &= \left(\sum_{k=0}^{\infty}\left|\frac{1}{Q_k}\sum_{j=0}^{k-1}\left[\sum_{i=j}^{k}\binom{m}{i-j}r^{m-i+j}s^{i-j}q_i(u_j+v_j)\right]+\frac{r^mq_k}{Q_k}(u_k+v_k)\right|^{p_k}\right)^{1/M}\\ &\le\left(\sum_{k=0}^{\infty}\left|\frac{1}{Q_k}\sum_{j=0}^{k-1}\left[\sum_{i=j}^{k}\binom{m}{i-j}r^{m-i+j}s^{i-j}q_i(u_j)\right]+\frac{r^mq_k}{Q_k}u_k\right|^{p_k}\right)^{1/M}\\ &\quad+\left(\sum_{k=0}^{\infty}\left|\frac{1}{Q_k}\sum_{j=0}^{k-1}\left[\sum_{i=j}^{k}\binom{m}{i-j}r^{m-i+j}s^{i-j}q_i(v_j)\right]+\frac{r^mq_k}{Q_k}v_k\right|^{p_k}\right)^{1/M}, \end{aligned} \tag{3.8}$$

$$g(u+v)\le g(u)+g(v). \tag{3.9}$$

Again the inequalities (3.7) and (3.9) yield the subadditivity of g and

$$g(\alpha u)\le\max\{1,|\alpha|\}g(u). \tag{3.10}$$

Let $\{x^n\}$ be any sequence of the elements of the space $r^q(p,B^m)$ such that

$$g(x^n-x)\longrightarrow 0, \tag{3.11}$$

and (λ_n) also be any sequence of scalars such that $\lambda_n\to\lambda$. Then, since the inequality

$$g(x^n)\le g(x)+g(x^n-x) \tag{3.12}$$

holds by subadditivity of g, $\{g(x^n)\}$ is bounded, and thus we have

$$\begin{aligned}
&g(\lambda_n x^n - \lambda x)\\
&\quad = \left(\sum_{k=0}^{\infty}\left|\frac{1}{Q_k}\sum_{j=0}^{k-1}\left[\sum_{i=j}^{k}\binom{m}{i-j}r^{m-i+j}s^{i-j}q_i\left(\lambda_n x_j^n - \lambda x_j\right)\right] + \frac{r^m q_k}{Q_k}(\lambda_n x_k^n - \lambda x_k)\right|^{p_k}\right)^{1/M},\\
&\quad \le |\lambda_n - \lambda|^{1/M}\left(\sum_{k=0}^{\infty}\left|\frac{1}{Q_k}\sum_{j=0}^{k-1}\left[\sum_{i=j}^{k}\binom{m}{i-j}r^{m-i+j}s^{i-j}q_i\left(x_j^n\right)\right] + \frac{r^m q_k}{Q_k}x_k^n\right|^{p_k}\right)^{1/M}\\
&\qquad + |\lambda|^{1/M}\left(\sum_{k=0}^{\infty}\left|\frac{1}{Q_k}\sum_{j=0}^{k-1}\left[\sum_{i=j}^{k}\binom{m}{i-j}r^{m-i+j}s^{i-j}q_i\left(x_j^n - x_j\right)\right] + \frac{r^m q_k}{Q_k}(x_k^n - x_k)\right|^{p_k}\right)^{1/M},\\
&\quad \le |\lambda_n - \lambda|^{1/M}g(x^n) + |\lambda|^{1/M}g(x^n - x),
\end{aligned} \tag{3.13}$$

which tends to zero as $n \to \infty$. Hence the continuity of the scalar multiplication has shown. Finally; it is clear to say that g is a paranorm on the space $r^q(p, B^m)$.

Moreover; we will prove the completeness of the space $r^q(p, B^m)$. Let x^i be any Cauchy sequence in the space $r^q(p, B^m)$ where $x^i = \{x_k^i\} = \{x_0^i, x_1^i, \ldots\} \in r^q(p, B^m)$. Then, for a given $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that

$$g\left(x^i - x^j\right) < \varepsilon, \tag{3.14}$$

for all $i, j \ge n_0(\varepsilon)$. If we use the definition of g, we obtain for each fixed $k \in \mathbb{N}$ that

$$\left|\left(R^q B^m x^i\right)_k - \left(R^q B^m x^j\right)_k\right| \le \left[\sum_{k=0}^{\infty}\left|\left(R^q B^m x^i\right)_k - \left(R^q B^m x^j\right)_k\right|^{p_k}\right]^{1/M} < \varepsilon, \tag{3.15}$$

for $i, j \ge n_0(\varepsilon)$ which leads us to the fact that

$$\left\{\left(R^q B^m x^0\right)_k, \left(R^q B^m x^1\right)_k, \ldots\right\}, \tag{3.16}$$

is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, so we write $(R^q B^m x^i)_k \to (R^q B^m x)_k$ as $i \to \infty$. Hence by using these infinitely many limits $(R^q B^m x)_0, (R^q B^m x)_1, \ldots$, we define the sequence $\{(R^q B^m x)_0, (R^q B^m x)_1, \ldots\}$. Since (3.14) holds for each $p \in \mathbb{N}$ and $i, j \ge n_0(\varepsilon)$,

$$\sum_{k=0}^{p}\left|\left(R^q B^m x^i\right)_k - \left(R^q B^m x^j\right)_k\right|^{p_k} \le \left[g\left(x^i - x^j\right)\right]^M < \varepsilon^M. \tag{3.17}$$

Take any $i \geq n_0(\varepsilon)$, first let $j \to \infty$ in (3.17) and then $p \to \infty$, to obtain $g(x^i - x) \leq \varepsilon$. Finally, taking $\varepsilon = 1$ in (3.17) and letting $i \geq n_0(1)$, we have Minkowski's inequality for each $p \in \mathbb{N}$, that is,

$$\left[\sum_{k=0}^{p}\left|\left(R^q B^m x^i\right)_k\right|^{p_k}\right]^{1/M} \leq g\left(x^i - x\right) + g\left(x^i\right) \leq 1 + g\left(x^i\right), \tag{3.18}$$

which implies that $x \in r^q(p, B^m)$. Since $g(x^i - x) \leq \varepsilon$ for all $i \geq n_0(\varepsilon)$ it follows that $x^i \to x$ as $i \to \infty$, so $r^q(p, B^m)$ is complete. □

Theorem 3.2. *Let* $\sum_{i=j}^{k}\binom{m}{i-j} r^{m-i+j} s^{i-j} q_i \neq 0$ *for each* $k \in \mathbb{N}$. *Then the difference sequence space* $r^q(p, B^m)$ *is linearly isomorphic to the space* $l(p)$ *where* $0 < p_k \leq H < \infty$.

Proof. For the proof of the theorem, we should show the existence of a linear bijection between the spaces $r^q(p, B^m)$ and $l(p)$ for $0 < p_k \leq H < \infty$. With the notation of

$$y_k = \frac{1}{Q_k}\sum_{j=0}^{k-1}\left[\sum_{i=j}^{k}\binom{m}{i-j} r^{m-i+j} s^{i-j} q_i x_j\right] + \frac{r^m}{Q_k} q_k x_k, \tag{3.19}$$

define the transformation T from $r^q(p, B^m)$ to $l(p)$ by $x \mapsto y = Tx$. However, T is a linear transformation, moreover; it is obviuos that $x = \theta$ whenever $Tx = \theta$ and hence T is injective.

Let $y \in l(p)$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{n=0}^{k-1}\left[\sum_{i=n}^{n+1}(-1)^{k-n}\frac{s^{k-i}}{r^{m+k-i}}\binom{m+k-i-1}{k-i}\frac{1}{q_i}Q_n y_n\right] + \frac{Q_k}{r^m q_k} y_k, \quad \text{for } k \in \mathbb{N}. \tag{3.20}$$

Then,

$$\begin{aligned} g(x) &= \left(\sum_{k=0}^{\infty}\left|\frac{1}{Q_k}\sum_{j=0}^{k-1}\left[\sum_{i=j}^{k}\binom{m}{i-j} r^{m-i+j} s^{i-j} q_i x_j\right] + \frac{r^m}{Q_k} q_k x_k\right|^{p_k}\right)^{1/M} \\ &= \left(\sum_{k=0}^{\infty}\left|\sum_{j=0}^{k}\delta_{kj} y_j\right|^{p_k}\right)^{1/M} \\ &= \left(\sum_{k=0}^{\infty}|y_k|^{p_k}\right)^{1/M} = g_1(y) < \infty, \end{aligned} \tag{3.21}$$

where

$$\delta_{kj} = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases} \tag{3.22}$$

and $g_1(y)$ is a paranorm on $l(p)$. Thus, we have that $x \in r^q(p, B^m)$. Consequently; T is surjective and is paranorm preserving. Hence, T is a linear bijection and this explains that the spaces $r^q(p, B^m)$ and $l(p)$ are linearly isomorphic. □

Now, the Schauder basis for the space $r^q(p, B^m)$ will be given in the following theorem.

Theorem 3.3. *Define the sequence $b^{(k)}(q) = \{b_n^{(k)}(q)\}_{n\in\mathbb{N}}$ of the elements of the space $r^q(p, B^m)$ for every fixed $k \in \mathbb{N}$ by*

$$b_n^{(k)}(q) = \begin{cases} \displaystyle\sum_{i=k}^{k+1} (-1)^{n-k} \frac{s^{n-i}}{r^{m+n-i}} \binom{m+n-i-1}{n-i} \frac{1}{q_i} Q_k, & (n > k), \\ \displaystyle\frac{Q_k}{r^m q_k}, & (n = k), \\ 0, & (k > n). \end{cases} \tag{3.23}$$

Then; the sequence $\{b^{(k)}(q)\}_{k\in\mathbb{N}}$ is a basis for the space $r^q(p, B^m)$ and any $x \in r^q(p, B^m)$ has a unique representation of the form

$$x = \sum_{k=0}^{\infty} \mu_k(q) b^k(q), \tag{3.24}$$

where $\mu_k(q) = (R^q B^m x)_k$ for all $k \in \mathbb{N}$ and $0 < p_k \le H < \infty$.

Proof. This can be easily obtained by [12, Theorem 5] so we omit the proof. □

Theorem 3.4. (i) *Let $1 < p_k \le H < \infty$ for every $k \in \mathbb{N}$. Define the set $Q_1(p)$ as follows:*

$$Q_1(p) = \bigcup_{K>1} \left\{ a = (a_k) \in w : \sup_{N\in\mathcal{F}} \sum_{k=0}^{\infty} \left| \sum_{n\in N} \left[\sum_{i=k}^{k+1} (-1)^{n-k} \frac{s^{n-i}}{r^{m+n-i}} \binom{m+n-i-1}{n-i} \frac{1}{q_i} a_n Q_k + \frac{a_n}{r^m q_n} Q_n \right] K^{-1} \right|^{p'_k} < \infty \right\}. \tag{3.25}$$

Then; $[r^q(p, B^m)]^\alpha = Q_1(p)$.

(ii) *Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Define the set $Q_2(p)$ by*

$$Q_2(p) = \bigcup_{K>1} \left\{ a = (a_k) \in w : \sup_{N \in \mathcal{F}} \sup_{k \in \mathbb{N}} \left| \sum_{n \in N} \left[\sum_{i=k}^{k+1} (-1)^{n-k} \frac{s^{n-i}}{r^{m+n-i}} \binom{m+n-i-1}{n-i} \frac{1}{q_i} a_n Q_k + \frac{a_n}{r^m q_n} Q_n \right] K^{-1} \right|^{p_k} < \infty \right\}. \tag{3.26}$$

Then; $[r^q(p, B^m)]^\alpha = Q_2(p)$.

Proof. (i) Let $a = (a_k) \in w$. We easily derive with the notation

$$y_k = \frac{1}{Q_k} \sum_{j=0}^{k-1} \left[\sum_{i=j}^{k} \binom{m}{i-j} r^{m-i+j} s^{i-j} q_i x_j \right] + \frac{1}{Q_k} q_k x_k, \tag{3.27}$$

and the matrix $U = (u_{nk})$ which is defined by

$$u_{nk} = \begin{cases} \displaystyle\sum_{i=k}^{k+1} (-1)^{n-k} \frac{s^{n-i}}{r^{m+n-i}} \binom{m+n-i-1}{n-i} \frac{1}{q_i} a_n Q_k, & (0 \leq k \leq n-1), \\ \dfrac{a_n Q_n}{r^m q_n}, & (k = n), \\ 0, & (k > n), \end{cases} \tag{3.28}$$

for all $k, n \in \mathbb{N}$, thus, by using the method in [1],[12] we deduce that $ax = (a_n x_n) \in l_1$ whenever $x = (x_k) \in r^q(p, B^m)$ if and only if $Uy \in l_1$ whenever $y = (y_k) \in l(p)$. From Lemma 2.5(i), we obtain the desired result that

$$\left[r^q(p, B^m) \right]^\alpha = Q_1(p). \tag{3.29}$$

(ii) This is easily obtained by proceeding as in the proof of (i), above by using the second part of Lemma 2.5. So we omit the detail. □

Theorem 3.5. (i) *Let* $1 < p_k \leq H < \infty$ *for every* $k \in \mathbb{N}$. *Define the set* $Q_3(p)$ *as follow:*

$$Q_3(p) = \bigcup_{K>1}\left\{a=(a_k)\in w : \sum_{k=0}^{\infty}\left|\left[\left(\frac{a_k}{r^m q_k}+\sum_{i=k}^{k+1}(-1)^{n-k}\frac{s^{n-i}}{r^{m+n-i}}\binom{m+n-i-1}{n-i}\frac{1}{q_i}\sum_{j=k+1}^{n}a_j\right)Q_k\right]K^{-1}\right|^{p'_k} < \infty\right\}. \tag{3.30}$$

Then; $[r^q(p,B^m)]^{\beta} = Q_3(p) \cap cs$.

(ii) *Let* $0 < p_k \leq 1$ *for every* $k \in \mathbb{N}$. *Define the set* $Q_4(p)$ *by*

$$Q_4(p) = \left\{a = (a_k) \in w : \sup_{k\in\mathbb{N}}\left|\left[\left(\frac{a_k}{r^m q_k}+\sum_{i=k}^{k+1}(-1)^{n-k}\frac{s^{n-i}}{r^{m+n-i}}\binom{m+n-i-1}{n-i}\frac{1}{q_i}\sum_{j=k+1}^{n}a_j\right)Q_k\right]\right|^{p_k} < \infty\right\}. \tag{3.31}$$

Then; $[r^q(p,B^m)]^{\beta} = Q_4(p) \cap cs$.

Proof. (i) If we take the matrix $T = (t_{nk})$ by

$$t_{nk} = \begin{cases}\left(\frac{a_k}{r^m q_k}+\sum_{i=k}^{k+1}(-1)^{n-k}\frac{s^{n-i}}{r^{m+n-i}}\binom{m+n-i-1}{n-i}\frac{1}{q_i}\sum_{j=k+1}^{n}a_j\right)Q_k, & (0 \leq k \leq n),\\ 0, & (k > n),\end{cases} \tag{3.32}$$

for $k, n \in \mathbb{N}$ and if we carry out the method which is used in [1, 12], we get that $ax = (a_n x_n) \in cs$ whenever $x = (x_k) \in r^q(p,B^m)$ if and only if $Ty \in c$ whenever $y = (y_k) \in l(p)$. Hence we deduce from Lemma 2.7 that

$$\sum_{k=0}^{\infty}\left|\left[\left(\frac{a_k}{r^m q_k}+\sum_{i=k}^{k+1}(-1)^{n-k}\frac{s^{n-i}}{r^{m+n-i}}\binom{m+n-i-1}{n-i}\frac{1}{q_i}\sum_{j=k+1}^{n}a_j\right)Q_k\right]K^{-1}\right|^{p'_k} < \infty, \tag{3.33}$$

and $\lim_n t_{nk}$ exists which is shown that

$$[r^q(p,B^m)]^{\beta} = Q_3(p) \cap cs. \tag{3.34}$$

(ii) This may be obtained in the similar way as in the proof of (i) above by using the second part of Lemmas 2.6 and 2.7. So we omit the detail. □

Now we will characterize the matrix mappings from the space $r^q(p, B^m)$ to the space l_∞. It can be proved by applying the method in [1, 12]. So we omit the proof.

Theorem 3.6. (i) *Let $1 < p_k \le H < \infty$ for every $k \in \mathbb{N}$. Then $A \in (r^q(p, B^m); l_\infty)$ if and only if there exists an integer $K > 1$ such that*

$$Q(K) = \sup_{n\in\mathbb{N}} \sum_{k=0}^{\infty} \left| \left[\left(\frac{a_{nk}}{r^m q_k} + \sum_{i=k}^{k+1} (-1)^{n-k} \frac{s^{n-i}}{r^{m+n-i}} \binom{m+n-i-1}{n-i} \frac{1}{q_i} \sum_{j=k+1}^{n} a_{nj} \right) Q_k \right] K^{-1} \right|^{p'_k} < \infty,$$

$$\{a_{nk}\}_{k\in\mathbb{N}} \in cs, \tag{3.35}$$

for each $n \in \mathbb{N}$.

(ii) *Let $0 < p_k \le 1$ for every $k \in \mathbb{N}$. Then $A \in (r^q(p, B^m); l_\infty)$ if and only if*

$$\sup_{n,k\in\mathbb{N}} \left| \left[\left(\frac{a_{nk}}{r^m q_k} + \sum_{i=k}^{k+1} (-1)^{n-k} \frac{s^{n-i}}{r^{m+n-i}} \binom{m+n-i-1}{n-i} \frac{1}{q_i} \sum_{j=k+1}^{n} a_{nj} \right) Q_k \right] \right|^{p_k} < \infty, \tag{3.36}$$

$$\{a_{nk}\}_{k\in\mathbb{N}} \in cs\ ,$$

for each $n \in \mathbb{N}$.

4. β-Property of Generalized Riesz Difference Sequence Space

In the previous section; we show that the sequence space $r^q(p, B^m)$, which is the space of all real sequences $x = (x_n)$ such that $\sum_{k=0}^{\infty} |(R^q B^m x)_k|^{p_k} < \infty$, is a complete paranormed space. It is paranormed by $g(x) = (\sum_{k=0}^{\infty} |(R^q B^m x)_k|^{p_k})^{1/M}$ for all $x = (x_n) \in r^q(p, B^m)$, where $M = \max\{1, H\}; H = \sup_k p_k$. We recall that a paranormed space is total if $g(x) = 0$ implies $x = 0$. Every total paranormed space becomes a linear metric space with the metric given by $d(x, y) = g(x - y)$. It is clear that $r^q(p, B^m)$ is a total paranormed space.

In this section, we investigate some geometric properties of $r^q(p, B^m)$. First we give the definition of the property (β) in a paranormed space and we will use the method in [17] to prove the property (β). Consequently, we obtain that $r^q(p, B^m)$ has property (β) for $p_k \ge 1$.

From here, for a sequence $x = (x_n) \in r^q(p, B^m)$ and for $i \in \mathbb{N}$, we use the notation $x_{|i} = (x(1), x(2), \ldots, x(i), 0, 0, \ldots)$ and $x_{|\mathbb{N}-i} = (0, 0, \ldots, 0, x(i+1), x(i+2), \ldots)$.

Now we give the definition of the property (β) in a linear metric space.

Definition 4.1. A linear metric space (X, d) is said to have the property (β) if for each $\varepsilon > 0$ and $r > 0$, there exists $\delta > 0$ such that for each element $x \in B(0, r)$ and each sequence (x_n) in $B(0, r)$ with $d(x_n, x_m) \ge \varepsilon$ for all $m \ne n$, there is an index k for which $d((x + x_k)/2, 0) \le r - \delta$.

Lemma 4.2. *If* $\liminf_{k\to\infty} p_k > 0$*, then for any* $L > 0$ *and* $\varepsilon > 0$*, and for any* $u, v \in r^q(p, B^m)$*, there exists* $\delta = \delta(\varepsilon, L) > 0$ *such that*

$$d^M(u+v,0) < d^M(u,0) + \varepsilon, \tag{4.1}$$

whenever $d^M(u,0) \le L$ *and* $d^M(v,0) \le \delta$*.*

Proof. Let $\varepsilon > 0$ and $L > 0$ be given. Let $0 < \alpha_0 < \liminf_{k\to\infty} p_k$ and $\alpha_0 < 1$, there exists $k_0 \in \mathbb{N}$ such that $0 < \alpha_0 \le p_k$ for all $k \ge k_0$. Let $\alpha = \min\{p_k : k = 1, 2, \ldots, k_0; \alpha_0\}$. Thus $\alpha \le p_k$ for all $k \in \mathbb{N}$. There exists $K_0 \ge 2$ such that

$$d^M(2u,0) \le K_0 d^M(u,0), \tag{4.2}$$

for all $u \in r^q(p, B^m)$. Set $\beta = (2^\alpha \varepsilon / 2K_0 L)^{1/\alpha}$. There exists $K_1 \ge K_0$ such that

$$d^M\left(\frac{2}{\beta}u, 0\right) \le K_1 d^M(u,0), \tag{4.3}$$

for all $u \in r^q(p, B^m)$. Set $\delta = (2^\alpha \varepsilon / 2\beta^\alpha K_1)$. Assume that $d^M(u,0) \le L$ and $d^M(v,0) \le \delta$. We recall that $x_{|i} = (x(1), x(2), \ldots, x(i), 0, 0, \ldots)$ and $x_{|\mathbb{N}-i} = (0, 0, \ldots, 0, x(i+1), x(i+2), \ldots)$. With these notations, let $A = \{k \in \mathbb{N} - i : p_k < 1\}$ and $C = \{k \in \mathbb{N} - i : p_k \ge 1\}$. By using convexity of the function $f(t) = |t|^{p_k}$ for all $p_k \ge 1$ and the fact that $(a+b)^{p_k} \le a^{p_k} + b^{p_k}$ for $p_k < 1$ and $0 < \beta^{p_k} < \beta^\alpha$ where $\beta \in (0,1)$ and $k \in \mathbb{N}$, we have

$$\begin{aligned}
d^M(u+v,0) &= d^M\left[(1-\beta)u + \beta\left(u + \beta^{-1}v\right), 0\right] \\
&= \sum_{i=0}^{\infty} \left| R^q B^m \left[(1-\beta)u(i) + \beta\left(u(i) + \beta^{-1}v(i)\right)\right]\right|^{p_i} \\
&\le \sum_{i=0}^{\infty} \left| R^q B^m [(1-\beta)u(i)] + R^q B^m \left[\beta\left(u(i) + \beta^{-1}v(i)\right)\right]\right|^{p_i} \\
&= \sum_{i\in A} \left| R^q B^m [(1-\beta)u(i)] + R^q B^m \left[\beta\left(u(i) + \beta^{-1}v(i)\right)\right]\right|^{p_i} \\
&\quad + \sum_{i\in C} \left| R^q B^m [(1-\beta)u(i)] + R^q B^m \left[\beta\left(u(i) + \beta^{-1}v(i)\right)\right]\right|^{p_i} \\
&\le (1-\beta)\sum_{i\in A} |R^q B^m u(i)|^{p_i} + \sum_{i\in A} \left| R^q B^m \beta\left[u(i) + \beta^{-1}v(i)\right]\right|^{p_i} \\
&\quad + (1-\beta)\sum_{i\in C} |R^q B^m u(i)|^{p_i} + \sum_{i\in C} \left| R^q B^m \beta\left[u(i) + \beta^{-1}v(i)\right]\right|^{p_i} \\
&\le \sum_{i\in A} |R^q B^m u(i)|^{p_i} + \beta^\alpha \sum_{i\in A} \left| R^q B^m \left[u(i) + \beta^{-1}v(i)\right]\right|^{p_i} \\
&\quad + \sum_{i\in C} |R^q B^m u(i)|^{p_i} + \beta^\alpha \sum_{i\in C} \left| R^q B^m \left[u(i) + \beta^{-1}v(i)\right]\right|^{p_i}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=0}^{\infty}\left|R^q B^m u(i)\right|^{p_i} + \beta^{\alpha}\sum_{i=0}^{\infty}\left|R^q B^m\left[u(i)+\beta^{-1}v(i)\right]\right|^{p_i}\\
&\leq d^M(u,0) + \beta^{\alpha}\sum_{i=0}^{\infty}\left|2^{-1}\left(2R^q B^m\left[u(i)+\beta^{-1}v(i)\right]\right)\right|^{p_i}\\
&\leq d^M(u,0) + \beta^{\alpha}\sum_{i\in A}\left|2^{-1}\left(2R^q B^m\left[u(i)+\beta^{-1}v(i)\right]\right)\right|^{p_i}\\
&\quad + \beta^{\alpha}\sum_{i\in C}\left|2^{-1}\left(2R^q B^m\left[u(i)+\beta^{-1}v(i)\right]\right)\right|^{p_i}\\
&\leq d^M(u,0) + \beta^{\alpha}\sum_{i\in A}\left|2^{-1}\left[\left(2R^q B^m u(i)\right)+\left(2R^q B^m \beta^{-1}v(i)\right)\right]\right|^{p_i}\\
&\quad + \beta^{\alpha}\sum_{i\in C}\left|2^{-1}\left[\left(2R^q B^m u(i)\right)+\left(2R^q B^m \beta^{-1}v(i)\right)\right]\right|^{p_i}\\
&\leq d^M(u,0) + \beta^{\alpha}\sum_{i\in A}\left|2^{-1}\left[2R^q B^m u(i)\right]\right|^{p_i}\\
&\quad + \beta^{\alpha}\sum_{i\in A}\left|2^{-1}\left[2R^q B^m \beta^{-1}v(i)\right]\right|^{p_i}\\
&\quad + \left(\frac{1}{2}\beta\right)^{\alpha}\sum_{i\in C}\left|2R^q B^m u(i)\right|^{p_i} + \left(\frac{1}{2}\beta\right)^{\alpha}\sum_{i\in C}\left|2R^q B^m \beta^{-1}v(i)\right|^{p_i}\\
&\leq d^M(u,0) + \left(\frac{1}{2}\beta\right)^{\alpha}\sum_{i=0}^{\infty}\left|2R^q B^m u(i)\right|^{p_i}\\
&\quad + \left(\frac{1}{2}\beta\right)^{\alpha}\sum_{i=0}^{\infty}\left|2R^q B^m \beta^{-1}v(i)\right|^{p_i}\\
&\leq d^M(u,0) + \frac{1}{2^{\alpha}}\frac{2^{\alpha}\varepsilon}{2K_0L}d^M(2u,0) + \frac{1}{2^{\alpha}}\beta^{\alpha}d^M\left(2\beta^{-1}v,0\right)\\
&\leq d^M(u,0) + \frac{\varepsilon}{2} + \frac{1}{2^{\alpha}}\beta^{\alpha}K_1\frac{2^{\alpha}\varepsilon}{2\beta^{\alpha}K_1},\\
d^M(u+v,0) &\leq d^M(u,0)+\varepsilon.
\end{aligned}
\tag{4.4}
$$

□

Lemma 4.3. *If* $\lim\inf_{n\to\infty}p_n > 0$, *then for any* $x \in r^q(p,B^m)$, *there exists* $k_0 \in \mathbb{N}$ *and* $\theta \in (0,1)$ *such that*

$$
d^M\left(\frac{x_{|\mathbb{N}-k}}{2},0\right) \leq \frac{(1-\theta)}{2}d^M\left(x_{|\mathbb{N}-k},0\right) \tag{4.5}
$$

for all $k \in \mathbb{N}$ *with* $k \geq k_0$.

Proof. Let α be a real number such that $1 < \alpha < \liminf_{n\to\infty} p_n$. Then there exists $k_0 \in \mathbb{N}$ such that $\alpha \le p_k$ for all $k \ge k_0$. Let $\theta \in (0,1)$ be a real number such that $(1/2)^\alpha < (1-\theta)/2$. Then for each $x \in r^q(p, B^m)$ and $k \ge k_0$, we have

$$\begin{aligned} d^M\left(\frac{x_{|\mathbb{N}-k}}{2}, 0\right) &= \sum_{i=k+1}^{\infty} \left| \frac{R^q B^m x(i)}{2} \right|^{p_i} \\ &\le \left(\frac{1}{2}\right)^{\alpha} \sum_{i=k+1}^{\infty} |R^q B^m x(i)|^{p_i} \\ &\le \frac{(1-\theta)}{2} \sum_{i=k+1}^{\infty} |R^q B^m x(i)|^{p_i} \\ &= \frac{(1-\theta)}{2} d^M\left(x_{|\mathbb{N}-k}, 0\right). \end{aligned} \tag{4.6}$$

□

Theorem 4.4. *If $p_k \ge 1$, then $r^q(p, B^m)$ has property (β).*

Proof. Let $\varepsilon > 0$ and $(x_n) \subset B(0,r)$ with $d(x_n, x_m) \ge \varepsilon$ for $m \ne n$. Take $0 < \varepsilon_0 < \varepsilon^M$. There exists $\delta > 0$ such that $\varepsilon^M - \delta \ge \varepsilon_0$. Let $x \in B(0,r)$. Since for each $j \in \mathbb{N}$, $(x_n(j))_{n=1}^{\infty}$ is bounded, by using the diagonal method, we have that for each $q \in \mathbb{N}$, we can find a subsequence (x_{n_a}) of (x_n) such that $x_{n_a}(j)$ converges for all $j \in \mathbb{N}$ with $1 \le j \le q$. Since $(x_{n_a}(j))$ is Cauchy sequence for all $1 \le j \le q$, there exists $t_q \in \mathbb{N}$ such that

$$\sum_{k=0}^{q} |(R^q B^m x_{n_a}(k)) - (R^q B^m x_{n_b}(k))|^{p_k} = \sum_{k=0}^{q} |R^q B^m (x_{n_a}(k) - x_{n_b}(k))|^{p_k} < \delta, \tag{4.7}$$

for all $n_a, n_b \ge t_q$. Then we see that

$$\begin{gathered} \varepsilon < d(x_{n_a}, x_{n_b}) = \left(\sum_{k=0}^{\infty} |R^q B^m (x_{n_a}(k) - x_{n_b}(k))|^{p_k} \right)^{1/M}, \\ \varepsilon^M \le \sum_{k=0}^{q} |R^q B^m (x_{n_a}(k) - x_{n_b}(k))|^{p_k} + \sum_{k=q+1}^{\infty} |R^q B^m (x_{n_a}(k) - x_{n_b}(k))|^{p_k}, \\ \varepsilon^M \le \delta + \sum_{k=q+1}^{\infty} |R^q B^m (x_{n_a}(k) - x_{n_b}(k))|^{p_k}. \end{gathered} \tag{4.8}$$

Therefore, for each $q \in \mathbb{N}$, there exists $t_q \in \mathbb{N}$ such that

$$d^M\left(x_{n_a|\mathbb{N}-q}, x_{n_b|\mathbb{N}-q}\right) \ge \varepsilon^M - \delta \ge \varepsilon_0, \tag{4.9}$$

for all $n_a, n_b \geq t_q$. Hence, there is a sequence of positive integers $(\sigma_q)_{q=1}^{\infty}$ with $\sigma_1 < \sigma_2 < \cdots$ such that

$$d^M\left(x_{\sigma_q|\mathbb{N}-q}, 0\right) = \sum_{k=q+1}^{\infty}\left|R^q B^m\left(x_{\sigma_q}(k)\right)\right|^{p_k} \geq \frac{\varepsilon_0}{2}, \tag{4.10}$$

for all $q \in \mathbb{N}$. By Lemma 4.3, there exists $q_0 \in \mathbb{N}$ and $\theta \in (0,1)$ such that

$$d^M\left(\frac{u_{|\mathbb{N}-q}}{2}, 0\right) \leq \frac{(1-\theta)}{2} d^M\left(u_{|\mathbb{N}-q}, 0\right), \tag{4.11}$$

for all $u \in r^q(p, B^m)$ and $q \geq q_0$. Let δ_0 be a real number corresponding to Lemma 4.2 with

$$\varepsilon = \frac{\theta}{4} \cdot \frac{\varepsilon_0}{2}, \tag{4.12}$$

and $L = r^M$, that is

$$d^M(u+v, 0) < d^M(u, 0) + \frac{\theta}{4} \cdot \frac{\varepsilon_0}{2}, \tag{4.13}$$

whenever $d^M(u,0) \leq r^M$ and $d^M(v,0) \leq \delta_0$. Since $x \in B(0,r)$, we have that $d^M(x,0) \leq r^M$. Let $q \geq q_0$ be such that

$$d^M\left(x_{|\mathbb{N}-q}, 0\right) \leq \delta_0. \tag{4.14}$$

Put $u = x_{\sigma_q|\mathbb{N}-q}$ and $v = x_{|\mathbb{N}-q}$. Then

$$\begin{aligned} d^M\left(\frac{u}{2}, 0\right) &= d^M\left(\frac{x_{\sigma_q|\mathbb{N}-q}}{2}, 0\right) = \sum_{k=q+1}^{\infty}\left|R^q B^m\left(x_{\sigma_q}(k)\right)\right|^{p_k} < r^M, \\ d^M\left(\frac{v}{2}, 0\right) &= d^M\left(x_{|\mathbb{N}-q}, 0\right) = \sum_{k=q+1}^{\infty}\left|R^q B^m x(k)\right|^{p_k} < \delta_0. \end{aligned} \tag{4.15}$$

Hence;

$$\begin{aligned}
d^M\left(\frac{u+v}{2},0\right) &= \sum_{k=q+1}^{\infty}\left|\frac{R^qB^m\left(x_{\sigma_q}(k)+x(k)\right)}{2}\right|^{p_k} \\
&\le \sum_{k=q+1}^{\infty}\left|\frac{R^qB^m x_{\sigma_q}(k)+R^qB^m x(k)}{2}\right|^{p_k} \\
&\le d^M\left(\frac{u}{2},0\right)+\frac{\theta}{4}\cdot\frac{\varepsilon_0}{2} \\
&\le \frac{(1-\theta)}{2}d^M(u,0)+\frac{\theta}{4}\cdot\frac{\varepsilon_0}{2},
\end{aligned} \tag{4.16}$$

$$d^M\left(\frac{u+v}{2},0\right) = \frac{(1-\theta)}{2}\sum_{k=q+1}^{\infty}\left|R^qB^m x_{\sigma_q}(k)\right|^{p_k}+\frac{\theta}{4}\cdot\frac{\varepsilon_0}{2}. \tag{4.17}$$

By using (4.17) and convexity of the function $f(t)=|t|^{p_k}$, $k\in\mathbb{N}$, we have

$$\begin{aligned}
d^M\left(\frac{x+x_{\sigma_q}}{2},0\right) &= \sum_{k=0}^{\infty}\left|\frac{R^qB^m\left(x_{\sigma_q}(k)+x(k)\right)}{2}\right|^{p_k} \\
&= \sum_{k=0}^{\infty}\left|\frac{R^qB^m x_{\sigma_q}(k)+R^qB^m x(k)}{2}\right|^{p_k} \\
&\le \sum_{k=0}^{q}\left|\frac{R^qB^m x_{\sigma_q}(k)+R^qB^m x(k)}{2}\right|^{p_k} \\
&\quad + \sum_{k=q+1}^{\infty}\left|\frac{R^qB^m x_{\sigma_q}(k)+R^qB^m x(k)}{2}\right|^{p_k} \\
&\le \frac{1}{2}\sum_{k=0}^{q}|R^qB^m x(k)|^{p_k}+\frac{1}{2}\sum_{k=0}^{q}\left|R^qB^m x_{\sigma_q}(k)\right|^{p_k} \\
&\quad + \frac{(1-\theta)}{2}\sum_{k=q+1}^{\infty}\left|R^qB^m x_{\sigma_q}(k)\right|^{p_k}+\frac{\theta}{4}\cdot\frac{\varepsilon_0}{2} \\
&\le \frac{1}{2}\sum_{k=0}^{q}|R^qB^m x(k)|^{p_k}+\frac{1}{2}\sum_{k=0}^{\infty}\left|R^qB^m x_{\sigma_q}(k)\right|^{p_k} \\
&\quad - \frac{\theta}{2}\sum_{k=q+1}^{\infty}\left|R^qB^m x_{\sigma_q}(k)\right|^{p_k}+\frac{\theta}{4}\cdot\frac{\varepsilon_0}{2} \\
&\le \frac{r^M}{2}+\frac{r^M}{2}-\frac{\theta}{2}\cdot\frac{\varepsilon_0}{2}+\frac{\theta}{4}\cdot\frac{\varepsilon_0}{2} \\
&\le r^M-\frac{\theta}{4}\cdot\frac{\varepsilon_0}{2}.
\end{aligned} \tag{4.18}$$

Hence $d^M((x+x_{\sigma_q})/2,0) \leq (r^M - (\theta/4)\cdot(\varepsilon_0/2))^{1/M}$. So this implies that

$$d^M\left(\left(x+x_{\sigma_q}\right)/2,0\right) \leq r-\delta \tag{4.19}$$

for some $\delta > 0$. Finally; we can say that the sequence space $r^q(p, B^m)$ has property (β). □

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