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Research Article

Interpolation Functions of q-Extensions of Apostol's Type Euler Polynomials

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The main purpose of this paper is to present new q-extensions of Apostol's type Euler polynomials using the fermionic p-adic integral on $\mathbb{Z}_p$. We define the q-λ-Euler polynomials and obtain the interpolation functions and the Hurwitz type zeta functions of these polynomials. We define q-extensions of Apostol type's Euler polynomials of higher order using the multivariate fermionic p-adic integral on $\mathbb{Z}_p$. We have the interpolation functions of these q-λ-Euler polynomials. We also give (h, q)-extensions of Apostol's type Euler polynomials of higher order and have the multiple Hurwitz type zeta functions of these (h, q)-λ-Euler polynomials.

1. Introduction, Definitions, and Notations

After Carlitz [1] gave q-extensions of the classical Bernoulli numbers and polynomials, the q-extensions of Bernoulli and Euler numbers and polynomials have been studied by several authors. Many authors have studied on various kinds of q-analogues of the Euler numbers and polynomials (cf., [1–39]).T Kim [7–23] has published remarkable research results for q-extensions of the Euler numbers and polynomials and their interpolation functions. In [13], T Kim presented a systematic study of some families of multiple q-Euler numbers and polynomials. By using the q-Volkenborn integration on $\mathbb{Z}_p$, he constructed the p-adic q-Euler numbers and polynomials of higher order and gave the generating function of these numbers and the Euler q-ζ-function. In [20], Kim studied some families of multiple q-Genocchi and q-Euler numbers using the multivariate p-adic q-Volkenborn integral on $\mathbb{Z}_p$, and gave interesting identities related to these numbers. Recently, Kim [21] studied some families of q-Euler numbers and polynomials of Nölund's type using multivariate fermionic p-adic integral on $\mathbb{Z}_p$.

Many authors have studied the Apostol-Bernoulli polynomials, the Apostol-Euler polynomials, and their q-extensions (cf., [1, 6, 25, 27, 28, 33–41]). Choi et al. [6] studied some q-extensions of the Apostol-Bernoulli and the Apostol-Euler polynomials of order n, and multiple Hurwitz zeta function. In [24], Kim et al. defined Apostol's type q-Euler numbers and polynomials using the fermionic p-adic q-integral and obtained the generating functions of these numbers and polynomials, respectively. They also had the distribution relation for Apostol's type q-Euler polynomials and obtained q-zeta function associated with Apostol's type q-Euler numbers and Hurwitz type q-zeta function associated with Apostol's type q-Euler polynomials for negative integers.

In this paper, we will present new q-extensions of Apostol's type Euler polynomials using the fermionic p-adic integral on $\mathbb{Z}_p$, and then we give interpolation functions and the Hurwitz type zeta functions of these polynomials. We also give q-extensions of Apostol's type Euler polynomials of higher order using the multivariate fermionic p-adic integral on $\mathbb{Z}_p$.

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then one assumes $|q-1|_p < 1$.

Now we recall some q-notations. The q-basic natural numbers are defined by $[n]_q = (1-q^n)/(1-q)$ and the q-factorial by $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$. The q-binomial coefficients are defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{[n]_q[n-1]_q\cdots[n-k+1]_q}{[k]_q!} \quad \text{(see [20])}. \tag{1.1}$$

Note that $\lim_{q\to 1}\binom{n}{k}_q = \binom{n}{k} = n!/(n-k)!k!$, which is the binomial coefficient. The q-shift factorial is given by

$$(b;q)_0 = 1, \qquad (b;q)_k = (1-b)(1-bq)\cdots\left(1-bq^{k-1}\right). \tag{1.2}$$

Note that $\lim_{q\to 1}(b;q)_k = (1-b)^k$. It is well known that the q-binomial formulae are defined as

$$\begin{gathered}(b;q)_k = (1-b)(1-bq)\cdots\left(1-bq^{k-1}\right) = \sum_{i=0}^{k}\binom{k}{i}_q q^{\binom{i}{2}}(-1)^i b^i, \\ \frac{1}{(b;q)_k} = \sum_{i=0}^{\infty}\binom{k+i-1}{i}_q b^i, \quad \text{(see [20])}.\end{gathered} \tag{1.3}$$

Since $\binom{-k}{l} = (-1)^l\binom{k+l-1}{l}$, it follows that

$$\frac{1}{(1-z)^k} = (1-z)^{-k} = \sum_{l=0}^{\infty}\binom{-k}{l}(-z)^l = \sum_{l=0}^{\infty}\binom{k+l-1}{l}z^l. \tag{1.4}$$

Hence it follows that

$$\frac{1}{(z;q)_k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n}_q z^n, \tag{1.5}$$

which converges to $1/(1-z)^k = \sum_{n=0}^{\infty} \binom{n+k-1}{n} z^n$ as $q \to 1$.

For a fixed odd positive integer d with $(p,d) = 1$, let

$$\begin{gathered}
X = X_d = \lim_{\overrightarrow{N}} \frac{\mathbb{Z}}{dp^N\mathbb{Z}}, \qquad X_1 = \mathbb{Z}_p, \\
X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\,\mathbb{Z}_p), \\
a + dp^N\mathbb{Z}_p = \left\{ x \in X \mid x \equiv a \left(\bmod\ dp^N\right) \right\},
\end{gathered} \tag{1.6}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. The distribution is defined by

$$\mu_q\left(a + dp^N\mathbb{Z}_p\right) = \frac{q^a}{\left[dp^N\right]_q}. \tag{1.7}$$

Let $\mathrm{UD}(\mathbb{Z}_p)$ be the set of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in \mathrm{UD}(\mathbb{Z}_p)$, the p-adic invariant q-integral is defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) = \lim_{N\to\infty} \frac{1}{\left[p^N\right]_q} \sum_{x=0}^{p^N-1} f(x) q^x. \tag{1.8}$$

The fermionic p-adic invariant q-integral on $\mathbb{Z}_p$ is defined as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{\left[p^N\right]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \tag{1.9}$$

where $[x]_{-q} = (1-(-q)^n)/(1+q)$. The fermionic p-adic integral on $\mathbb{Z}_p$ is defined as

$$I_{-1}(f) = \lim_{q\to 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-1}(x). \tag{1.10}$$

It follows that $I_{-1}(f_1) = -I_{-1}(f) + 2f(0)$, where $f_1(x) = f(x+1)$. For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$. we have

$$I_{-1}(f_n) = (-1)^n I_{-1}(f) + \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l). \tag{1.11}$$

For details, see [7–21].

The classical Euler numbers E_n and the classical Euler polynomials $E_n(x)$ are defined, respectively, as follows:

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \qquad \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.12}$$

It is known that the classical Euler numbers and polynomials are interpolated by the Euler zeta function and Hurwitz type zeta function, respectively, as follows:

$$\zeta_E(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \quad \zeta_E(s,x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}, \quad s \in \mathbb{C}, \quad (\text{see [10]}). \tag{1.13}$$

In Section 2, we define new q-extensions of Apostol's type Euler polynomials using the fermionic p-adic integral on $\mathbb{Z}_p$ which will be called the q-λ-Euler polynomials . Then we obtain the interpolation functions and the Hurwitz type zeta functions of these polynomials. In Section 3, we define q-extensions of Apostol's type Euler polynomials of higher order using the multivariate fermionic p-adic integral on $\mathbb{Z}_p$. We have the interpolation functions of these higher-order q-λ-Euler polynomials. In Section 4, we also give (h,q)-extensions of Apostol's type Euler polynomials of higher order and have the multiple Euler zeta functions of these (h,q)-λ-Euler polynomials.

2. q-Extensions of Apostol's Type Euler Polynomials

First, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. In $\mathbb{C}_p$, the q-Euler polynomials are defined by

$$E_{n,q}(x) = \int_{\mathbb{Z}_p} q^y [x+y]_q^n d\mu_{-1}(y), \tag{2.1}$$

and $E_{n,q}(0) = E_{n,q}$ are called the q-Euler numbers. Then it follows that

$$E_{n,q}(x) = \frac{2}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{1}{1+q^{l+1}}. \tag{2.2}$$

The generating functions of $E_{n,q}(x)$ are defined as

$$F_q(t,x) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} q^y e^{[x+y]_q t} d\mu_{-1}(y). \tag{2.3}$$

By (2.3), the interpolation functions of the q-Euler polynomials $E_{n,q}(x)$ are obtained as follows:

$$\begin{aligned} F_q(t,x) &= \sum_{n=0}^{\infty} \frac{2}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \left(\frac{q^{lx}}{1+q^{l+1}} \right) \frac{t^n}{n!} \\ &= 2\sum_{m=0}^{\infty} (-1)^m q^m \sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{(x+m)l} \frac{t^n}{n!} \\ &= 2\sum_{m=0}^{\infty} (-1)^m q^m \sum_{n=0}^{\infty} [x+m]_q^n \frac{t^n}{n!} \\ &= 2\sum_{m=0}^{\infty} (-1)^m q^m e^{[x+m]_q t}. \end{aligned} \tag{2.4}$$

Thus, we have the following theorem.

Theorem 2.1. *Assume $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. Then one has*

$$F_q(t,x) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} = 2\sum_{m=0}^{\infty} (-1)^m q^m e^{[x+m]_q t}. \tag{2.5}$$

Differentiating $F_q(t,x)$ at $x = 0$ shows that

$$E_{n,q}(x) = \left. \frac{d^n F_q(t,x)}{dt^n} \right|_{t=0} = 2\sum_{m=0}^{\infty} (-1)^m q^m [x+m]_q^n. \tag{2.6}$$

In $\mathbb{C}$, we assume that $q \in \mathbb{C}$ with $|q| < 1$. The q-Euler polynomials $E_{n,q}(x)$ are defined by

$$2\sum_{m=0}^{\infty} (-1)^m q^m e^{[x+m]_q t} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \tag{2.7}$$

By (2.7)*, we have*

$$\begin{aligned} E_{n,q}(x) &= 2\sum_{m=0}^{\infty} (-1)^m q^m [x+m]_q^n \\ &= \frac{2}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{1}{1+q^{l+1}}. \end{aligned} \tag{2.8}$$

For $s \in \mathbb{C}$, the Hurwitz type zeta functions for the q-Euler polynomials $E_{n,q}(x)$ are given as

$$\zeta_{q,E}(s,x) = \sum_{m=0}^{\infty} \frac{(-1)^m q^m}{[x+m]_q^s}, \qquad x \neq 0, -1, -2, \ldots. \tag{2.9}$$

For $k \in \mathbb{Z}_+$, we have from (2.9) that

$$\zeta_{q,E}(-k,x) = \sum_{m=0}^{\infty} [x+m]_q^k (-1)^m q^m = E_{k,q}(x). \tag{2.10}$$

Now we give new q-extensions of Apostol's type Euler polynomials. For $n \in \mathbb{N}$, let $\mathbb{C}_{p^n} = \{\omega \mid \omega^{p^n} = 1\}$ be the cyclic group of order p^n. Let T_p be the p-adic locally constant space defined by

$$T_p = \bigcup_{n\geq 1} \mathbb{C}_{p^n} = \lim_{n\to\infty} \mathbb{C}_{p^n}. \tag{2.11}$$

First, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. For $\lambda \in T_p$, we define q-Euler polynomials of Apostol's type using the fermionic p-adic integral as follows:

$$E_{n,q,\lambda}(x) = \int_{\mathbb{Z}_p} q^y \lambda^y [x+y]_q^n \, d\mu_{-1}(y), \tag{2.12}$$

and we will call them the q-λ-Euler polynomials. Then $E_{n,q,\lambda}(0) = E_{n,q,\lambda}$ are defined as the q-λ-Euler numbers. From (2.12), we have

$$E_{n,q,\lambda}(x) = \frac{2}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{1}{1+\lambda q^{l+1}}. \tag{2.13}$$

Let $F_{q,\lambda}(t,x) = \sum_{n=0}^{\infty} E_{n,q,\lambda}(x)(t^n/n!)$. From (2.12), we easily derive

$$F_{q,\lambda}(t,x) = \int_{\mathbb{Z}_p} q^y \lambda^y e^{[x+y]_q t} \, d\mu_{-1}(y). \tag{2.14}$$

On the other hand, we have

$$\begin{aligned} \int_{\mathbb{Z}_p} q^y \lambda^y e^{[x+y]_q t} \, d\mu_{-1}(y) &= \sum_{n=0}^{\infty} \frac{2}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{1}{1+\lambda q^{l+1}} \frac{t^n}{n!} \\ &= 2\sum_{m=0}^{\infty} (-1)^m q^m \lambda^m \sum_{n=0}^{\infty} [x+m]_q^n \frac{t^n}{n!}. \end{aligned} \tag{2.15}$$

From (2.14) and (2.15), we obtain the following theorem.

Theorem 2.2. *Assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. For $\lambda \in T_p$, let $F_{q,\lambda}(t,x) = \sum_{n=0}^{\infty} E_{n,q,\lambda}(x)(t^n/n!)$. Then one has*

$$F_{q,\lambda}(t,x) = \int_{\mathbb{Z}_p} q^y \lambda^y e^{[x+y]_q t} \, d\mu_{-1}(y) = 2\sum_{m=0}^{\infty} (-1)^m q^m \lambda^m e^{[x+m]_q t}. \tag{2.16}$$

In $\mathbb{C}$, we assume that $q \in \mathbb{C}$ with $|q| < 1$. Let $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. We define the q-λ-Euler polynomials $E_{n,q,\lambda}(x)$ to be satisfied the following equation:

$$F_{q,\lambda}(t,x) = 2\sum_{m=0}^{\infty}(-1)^m q^y \lambda^y e^{[x+m]_q t} = \sum_{n=0}^{\infty} E_{n,q,\lambda}(x)\frac{t^n}{n!}. \tag{2.17}$$

When we differentiate both sides of (2.17) *at $t = 0$, we have*

$$\left.\frac{d^n F_{q,\lambda}(t,x)}{dt^n}\right|_{t=0} = 2\sum_{m=0}^{\infty}(-1)^m q^m \lambda^m [x+m]_q^n = E_{n,q,\lambda}(x). \tag{2.18}$$

Hence we have the interpolation functions of the q-λ-Euler polynomials as follows:

$$E_{n,q,\lambda}(x) = 2\sum_{m=0}^{\infty}(-1)^m q^m \lambda^m [x+m]_q^n. \tag{2.19}$$

For $s \in \mathbb{C}$, we define the Hurwitz type zeta function of the q-λ-Euler polynomials as

$$\zeta_{q,E,\lambda}(s,x) = 2\sum_{m=0}^{\infty}\frac{(-1)^m q^m \lambda^m}{[m+x]_q^s}, \tag{2.20}$$

where $x \neq 0, -1, -2, \ldots$. For $k \in \mathbb{Z}_+$, we have

$$\zeta_{q,E,\lambda}(-k,x) = 2\sum_{m=0}^{\infty}(-1)^m q^m \lambda^m [x+m]_q^k = E_{k,q,\lambda}(x). \tag{2.21}$$

3. q-Extensions of Apostol's Type Euler Polynomials of Higher Order

In this section, we give the q-extension of Apostol's type Euler polynomials of higher order using the multivariate fermionic p-adic integral.

First, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. Let $\lambda \in T_p$. We define the q-λ-Euler polynomials of order r as follows:

$$E_{n,q}^{(r)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{y_1+\cdots+y_r}\left[x+y_1+\cdots+y_r\right]_q^n \lambda^{y_1+\cdots+y_r} d\mu_{-1}(y_1)\cdots d\mu_{-1}(y_r). \tag{3.1}$$

Note that $E_{n,q,\lambda}^{(r)}(0) = E_{n,q,\lambda}^{(r)}$ are called the q-λ-Euler number of order r. Using the multivariate fermionic p-adic integral, we obtain from (3.1) that

$$E_{n,q,\lambda}^{(r)}(x) = \frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{1}{(1+\lambda q^{l+1})^r}. \tag{3.2}$$

Let $F^{(r)}_{q,\lambda}(t,x)$ be the generating functions of $E^{(r)}_{n,q,\lambda}(x)$ defined by

$$F^{(r)}_{q,\lambda}(t,x)=\sum_{n=0}^{\infty}E^{(r)}_{n,q,\lambda}(x)\frac{t^n}{n!}. \tag{3.3}$$

By (2.12) and (3.3), we have

$$\begin{aligned}F^{(r)}_{q,\lambda}(t,x)&=2^r\sum_{n=0}^{\infty}\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\sum_{m=0}^{\infty}\binom{r+m-1}{m}(-1)^m\lambda^m q^{(l+1)m}\frac{t^n}{n!}\\&=2^r\sum_{m=0}^{\infty}\binom{r+m-1}{m}(-1)^m\lambda^m q^m\sum_{n=0}^{\infty}\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{l(x+m)}\frac{t^n}{n!}\\&=2^r\sum_{m=0}^{\infty}\binom{r+m-1}{m}(-1)^m\lambda^m q^m\sum_{n=0}^{\infty}[x+m]_q^n\frac{t^n}{n!}.\end{aligned} \tag{3.4}$$

Thus we have the following theorem.

Theorem 3.1. *Assume that $q\in\mathbb{C}_p$ with $|1-q|_p<1$. For $r\in\mathbb{N}$ and $\lambda\in T_p$, let $F^{(r)}_{q,\lambda}(t,x)=\sum_{n=0}^{\infty}E^{(r)}_{n,q,\lambda}(x)(t^n/n!)$. Then one has*

$$\begin{aligned}F^{(r)}_{q,\lambda}(t,x)&=2^r\sum_{m=0}^{\infty}\binom{r+m-1}{m}(-1)^m\lambda^m q^m e^{[x+m]_q t},\\E^{(r)}_{n,q,\lambda}(x)&=2^k\sum_{m=0}^{\infty}\binom{r+m-1}{m}(-1)^m\lambda^m q^m[x+m]_q^n.\end{aligned} \tag{3.5}$$

In $\mathbb{C}$, we assume that $q\in\mathbb{C}$ with $|q|<1$ and $\lambda\in\mathbb{C}$ with $\lambda=e^{2\pi i/f}$ for $f\in\mathbb{N}$. We define the q-λ-Euler polynomial $E^{(r)}_{n,q,\lambda}(x)$ of order k as follows:

$$\begin{aligned}F^{(r)}_{q,\lambda}(t,x)&=2^r\sum_{m=0}^{\infty}\binom{r+m-1}{m}(-1)^m\lambda^m q^m e^{[x+m]_q t}\\&=\sum_{n=0}^{\infty}E^{(r)}_{n,q,\lambda}(x)\frac{t^n}{n!}.\end{aligned} \tag{3.6}$$

From (3.6), we have

$$\left.\frac{d^kF^{(r)}_{q,\lambda}(t,x)}{dt^k}\right|_{t=0}=E^{(r)}_{k,q,\lambda}(x)=2^r\sum_{m=0}^{\infty}\binom{r+m-1}{m}(-1)^m\lambda^m q^m[x+m]_q^k. \tag{3.7}$$

For $s\in\mathbb{C}$, we define the multiple Hurwitz type zeta functions for q-λ-Euler polynomials as

$$\zeta^{(r)}_{q,E,\lambda}(s,x)=2^r\sum_{m=0}^{\infty}\binom{r+m-1}{n}\frac{(-1)^m\lambda^m q^m}{[m+x]_q^s}, \tag{3.8}$$

where $x \neq 0, -1, -2, \ldots$. *In the special case* $s = -k$ *with* $k \in \mathbb{Z}_+$, *we have*

$$\zeta_{q,E,\lambda}^{(r)}(-k, x) = E_{k,q,\lambda}^{(r)}(x). \tag{3.9}$$

4. (h, q)-Extension of Apostol's Type Euler Polynomials of Higher Order

In this section, we give the (h, q)-extension of q-λ-Euler polynomials of higher order using the multivariate fermionic p-adic integral.

Assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. For $h \in \mathbb{Z}$, we define (h, q)-λ-Euler polynomials of order r as follows:

$$\begin{aligned} E_{n,q,\lambda}^{(h,r)}(x) &= \int_{\mathbb{Z}_p} q^{\sum_{j=1}^{r}(h-j+1)y_j} \lambda^{\sum_{j=1}^{r} y_j} [x + y_1 + \cdots + y_r]_q^n d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r) \\ &= \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l q^{lx}}{\prod_{i=1}^{r}\left(1 + \lambda q^{h-r+l+i}\right)}. \end{aligned} \tag{4.1}$$

Note that $E_{n,q,\lambda}^{(h,r)}(0) = E_{n,q,\lambda}^{(h,r)}$ are called the (h, q)-λ-Euler numbers.

When $h = r$, the (h, q)-λ-Euler polynomials are

$$\begin{aligned} E_{n,q,\lambda}^{(r,r)}(x) &= \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{1}{(1+\lambda q^{k+l}) \cdots (1+\lambda q^{l+1})} \\ &= \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{1}{(-\lambda q^{l+1}; q)_r} \\ &= \sum_{m=0}^{\infty} \binom{r+m-1}{m}_q (-1)^m \lambda^m q^m \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{l(x+m)} \\ &= 2^r \sum_{m=0}^{\infty} \binom{r+m-1}{m}_q (-1)^m \lambda^m q^m [x+m]_q^n, \end{aligned} \tag{4.2}$$

where $\binom{r+m-1}{m}_q$ is the Gaussian binomial coefficient. From (4.2), we obtain the following theorem.

Theorem 4.1. *Assume that* $q \in \mathbb{C}_p$ *with* $|1-q|_p < 1$. *For* $r \in \mathbb{N}$ *and* $\lambda \in T_p$, *let* $F_{q,\lambda}^{(r,r)}(t, x) = \sum_{n=0}^{\infty} E_{n,q,\lambda}^{(r,r)}(x)(t^n/n!)$. *Then one has*

$$F_{q,\lambda}^{(r,r)}(t, x) = 2^r \sum_{m=0}^{\infty} \binom{r+m-1}{m}_q (-1)^m \lambda^m q^m e^{[x+m]_q t}. \tag{4.3}$$

In $\mathbb{C}$, *assume that* $q \in \mathbb{C}$ *with* $|q| < 1$ *and* $\lambda \in \mathbb{C}$ *with* $|\lambda| < 1$. *Then we can define* (h, q)-λ-*Euler polynomials* $E_{n,q,\lambda}^{(r,r)}(x)$ *for* $h = r$ *as follows:*

$$F_{q,\lambda}^{(r,r)}(t, x) = 2^r \sum_{m=0}^{\infty} \binom{r+m-1}{m}_q (-1)^m \lambda^m q^m e^{[x+m]_q t} = \sum_{n=0}^{\infty} E_{n,q,\lambda}^{(r,r)}(x) \frac{t^n}{n!}. \tag{4.4}$$

Differentiating both sides of (4.4) *at* $t = 0$, *we have*

$$\left. \frac{d^k F_{q,\lambda}^{(r,r)}(t, x)}{dt^k} \right|_{t=0} = 2^r \sum_{m=0}^{\infty} \binom{r+m-1}{m}_q (-1)^m \lambda^m q^m [x+m]_q^k = E_{k,q,\lambda}^{(r,r)}(x). \tag{4.5}$$

From (4.5), *we have*

$$2^r \sum_{m=0}^{\infty} \binom{r+m-1}{m}_q (-1)^m \lambda^m q^m e^{[x+m]_q t} = \sum_{n=0}^{\infty} E_{n,q,\lambda}^{(r,r)}(x) \frac{t^n}{n!}. \tag{4.6}$$

Then we have

$$E_{k,q,\lambda}^{(r,r)}(x) = 2^r \sum_{m=0}^{\infty} \binom{r+m-1}{m}_q (-1)^m \lambda^m q^m [x+m]_q^k. \tag{4.7}$$

For $s \in \mathbb{C}$, *we define the Hurwitz type zeta function of* q-λ-*Euler polynomials of order* r *as*

$$\zeta_{q,E,\lambda}^{(r,r)}(x, s) = 2^r \sum_{m=0}^{\infty} \binom{r+m-1}{m}_q \frac{(-1)^m \lambda^m q^m}{[m+x]_q^s}, \tag{4.8}$$

where $x \neq 0, -1, -2, \ldots$.

From (4.4) *and* (4.8), *we easily see that*

$$\zeta_{q,\lambda}^{(r,r)}(x, -k) = E_{k,q,\lambda}^{(r,r)}(x), \quad k \in \mathbb{N}. \tag{4.9}$$

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