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Research Article

Inequalities for the Polar Derivative of a Polynomial

M. Bidkham, M. Shakeri, and M. Eshaghi Gordji

Department of Mathematics, Faculty of Natural Sciences, Semnan University, Semnan 35195-363, Iran

Correspondence should be addressed to M. Bidkham, mdbidkham@gmail.com

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Let $p(z)$ be a polynomial of degree n and for any real or complex number α, and let $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ denote the polar derivative of the polynomial $p(z)$ with respect to α. In this paper, we obtain new results concerning the maximum modulus of a polar derivative of a polynomial with restricted zeros. Our results generalize as well as improve upon some well-known polynomial inequalities.

1. Introduction and Statement of Results

If $p(z)$ is a polynomial of degree n, then it is well known that

$$\max_{|z|=1}\left|p'(z)\right| \le n \max_{|z|=1}\left|p(z)\right|. \tag{1.1}$$

The above inequality, which is an immediate consequence of Bernstein's inequality applied to the derivative of a trigonometric polynomial, is best possible with equality holding if and only if $p(z)$ has all its zeros at the origin. If $p(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1}\left|p'(z)\right| \le \frac{n}{2} \max_{|z|=1}\left|p(z)\right|. \tag{1.2}$$

Inequality (1.2) was conjectured by Erdös and later proved by Lax [1]. If the polynomial $p(z)$ of degree n has all its zeros in $|z| < 1$, then it was proved by Turán [2] that

$$\max_{|z|=1}\left|p'(z)\right| \geq \frac{n}{2}\max_{|z|=1}\left|p(z)\right|. \tag{1.3}$$

Inequality (1.2) was generalized by Malik [3] who proved that if $p(z) \neq 0$ in $|z| < k,\ k \geq 1$, then

$$\max_{|z|=1}\left|p'(z)\right| \leq \frac{n}{1+k}\max_{|z|=1}\left|p(z)\right|. \tag{1.4}$$

For the class of polynomials having all its zeros in $|z| \leq k,\ k \geq 1$, Govil [4] proved that

$$\max_{|z|=1}\left|p'(z)\right| \geq \frac{n}{1+k^n}\max_{|z|=1}\left|p(z)\right|. \tag{1.5}$$

Inequality (1.5) is sharp and equality holds for $p(z) = z^n + k^n$. By considering a more general class of polynomials $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v,\ 1 \leq t \leq n$, not vanishing in $|z| < k$, $k \geq 1$, Gardner et al. [5] proved that

$$\max_{|z|=1}\left|p'(z)\right| \leq \frac{n}{1+s_0}\left\{\max_{|z|=1}\left|p(z)\right| - m\right\}, \tag{1.6}$$

where $m = \min_{|z|=k}|p(z)|$ and $s_0 = k^{t+1}\{((t/n)(|a_t|/(|a_0| - m))k^{t-1} + 1)/((t/n)(|a_t|/(|a_0| - m))k^{t+1} + 1)\}$.

Let $D_\alpha\{p(z)\}$ denote the polar derivative of the polynomial $p(z)$ of degree n with respect to the point α. Then

$$D_\alpha\{p(z)\} = np(z) + (\alpha - z)p'(z). \tag{1.7}$$

The polynomial $D_\alpha\{p(z)\}$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\left[\frac{D_\alpha\{p(z)\}}{\alpha}\right] = p'(z). \tag{1.8}$$

As an extension of (1.5), it was shown by Aziz and Rather [6] that if $p(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$, then for $|\alpha| \geq k$,

$$\max_{|z|=1}\left|D_\alpha p(z)\right| \geq n\left(\frac{|\alpha| - k}{1+k^n}\right)\max_{|z|=1}\left|p(z)\right|. \tag{1.9}$$

Inequality (1.9) was later sharpened by Dewan and Upadhye [7], who proved the following theorem.

Theorem A. *Let $p(z)$ be a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then for $|\alpha| \geq k$,*

$$\max_{|z|=1}\left|D_\alpha p(z)\right| \geq n(|\alpha|-k)\left\{\frac{1}{1+k^n}\max_{|z|=1}\left|p(z)\right| + \frac{1}{2k^n}\left(\frac{k^n-1}{k^n+1}\right)m\right\}, \tag{1.10}$$

where $m = \min_{|z|=k}|p(z)|$.

Recently, Dewan et al. [8] extented inequality (1.6) to the polar derivative of a polynomial and obtained the following result.

Theorem B. *If $p(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu$, $1 \leq t \leq n$, is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for $|\alpha| \geq 1$,*

$$\max_{|z|=1}\left|D_\alpha p(z)\right| \leq \frac{n}{1+s_0}\left\{(|\alpha|+s_0)\max_{|z|=1}\left|p(z)\right| - (|\alpha|-1)m\right\}, \tag{1.11}$$

where $m = \min_{|z|=k}|p(z)|$ and $s_0 = k^{t+1}\{((t/n)(|a_t|/(|a_0|-m))k^{t-1}+1)/((t/n)(|a_t|/(|a_0|-m))k^{t+1}+1)\}$.

In this paper, we will first generalize Theorem A as well as improve upon the bound obtained in inequality (1.10) by involving some of the coefficients of $p(z)$. More precisely, we prove the following.

Theorem 1.1. *If $p(z) = \sum_{i=0}^{n} a_i z^i$ is a polynomial of degree $n \geq 3$ having all its zeros in $|z| \leq k$, $k \geq 1$, then for $|\alpha| \geq k$,*

$$\begin{aligned}
&\max_{|z|=1}\left|D_\alpha p(z)\right| \\
&\quad \geq n(|\alpha|-k)\Bigg[\frac{1}{k^n+1}\max_{|z|=1}\left|p(z)\right| + \frac{k^n-1}{2k^n(k^n+1)}m \\
&\qquad\qquad + \frac{2|a_{n-1}|}{k(k^n+1)(n+1)}\left(\frac{k^n-1}{n}-(k-1)\right) \\
&\qquad\qquad + \frac{2|a_{n-2}|}{(k^n+1)k^2}\left(\left(\frac{(k^n-1)-n(k-1)}{n(n-1)}\right)-\left(\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right)\right)\Bigg] \\
&\qquad + \frac{1}{k^{n-1}}\left[\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}\right]\left|(n-1)a_1+2\alpha a_2\right| \\
&\qquad + \frac{2}{k^{n-1}}\left(\frac{k^{n-1}-1}{n+1}\right)\left|na_0+\alpha a_1\right| + n\frac{(|\alpha|+k)}{2k^n}m
\end{aligned} \tag{1.12}$$

for $n > 3$ *and*

$$\max_{|z|=1}\left|D_\alpha p(z)\right| \geq n(|\alpha|-k)\left[\frac{1}{k^n+1}\max_{|z|=1}|p(z)| + \frac{k^n-1}{2k^3(k^n+1)}m + \frac{2|a_{n-1}|}{k(k^n+1)(n+1)}\left(\frac{k^n-1}{n}-(k-1)\right) + \frac{2k^{n-5}|a_{n-2}|}{(k^n+1)}\left(\frac{(k-1)^n}{n(n-1)}\right)\right] + \frac{k-1}{2k^2}((k+1)|na_0+\alpha a_1| + (k-1)|(n-1)a_1+2\alpha a_2|) + n\frac{(|\alpha|+k)}{2k^3}m \tag{1.13}$$

for $n = 3$, *where* $m = \min_{|z|=k}|p(z)|$.

Now it is easy to verify that if $k \geq 1$, then $(k^n-1)/n-(k-1) \geq 0$, $[(k^{n-1}-1)/(n-1)-(k^{n-3}-1)/(n-3)] \geq 0$ and $[(((k^n-1)-n(k-1))/n(n-1))-(((k^{n-2}-1)-(n-2)(k-1))/(n-2)(n-3))] \geq 0$ for $n > 3$. Hence for polynomial of degree $n \geq 3$, Theorem 1.1 is a refinement of Theorem A.

Dividing both sides of inequalities (1.12) and (1.13) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the following result.

Corollary 1.2. *If* $p(z) = \sum_{i=0}^{n} a_i z^i$ *is a polynomial of degree* $n \geq 3$ *having all its zeros in* $|z| \leq k$, $k \geq 1$, *then*

$$\max_{|z|=1}\left|p'(z)\right| \geq \frac{n}{k^n+1}\left[\max_{|z|=1}|p(z)| + m + \frac{2}{k(n+1)}\left(\frac{k^n-1}{n}-(k-1)\right)|a_{n-1}| + \frac{2}{k^2}\left(\frac{(k^n-1)-n(k-1)}{n(n-1)} - \frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right)|a_{n-2}|\right] \times \frac{2(k^{n-1}-1)}{k^{n-1}(n+1)}|a_1| + \frac{2}{k^{n-1}}\left[\frac{k^{n-1}-1}{n-1} - \frac{k^{n-3}-1}{n-3}\right]|a_2| \tag{1.14}$$

for $n > 3$ *and*

$$\max_{|z|=1}\left|p'(z)\right| \geq \frac{n}{k^n+1}\left[\max_{|z|=1}|p(z)| + m + \frac{2}{k(n+1)}\left(\frac{k^n-1}{n}-(k-1)\right)|a_{n-1}| + \frac{2}{k^2}\left(\frac{(k-1)^n}{n(n-1)}\right)|a_{n-2}|\right] + \frac{k-1}{2k^2}((k+1)|a_1| + 2(k-1)|a_2|) \tag{1.15}$$

for $n = 3$, *where* $m = \min_{|z|=k}|p(z)|$.

These inequalities are sharp and equality holds for the polynomial $p(z) = z^n + k^n$.

If we take $k = 1$ in the previous Theorem, we get a result, which was proved by Aziz and Dawood [9].

Next we consider a class of polynomial having no zeros in $|z| < k$, where $k \geq 1$ and prove the following generalization of Theorem B.

Theorem 1.3. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu,\ 1 \leq \mu \leq n$, is a polynomial of degree n having no zeros in $|z| < k,\ k \geq 1$, then for $0 < r \leq R \leq k$ and $|\alpha| \geq R$,*

$$\max_{|z|=R}\left|D_\alpha p(z)\right| \leq \frac{n}{1+s_0'}\left[\left(\frac{|\alpha|}{R}+s_0'\right)\exp\left\{n\int_r^R A_t dt\right\}\max_{|z|=r}\left|p(z)\right| + \left(s_0'+1-\left(\frac{|\alpha|}{R}+s_0'\right)\exp\left\{n\int_r^R A_t dt\right\}\right)m\right], \tag{1.16}$$

where

$$A_t = \frac{(\mu/n)\left(\left|a_\mu\right|/(|a_0|-m)\right)k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+(\mu/n)\left(\left|a_\mu\right|/(|a_0|-m)\right)\left(k^{\mu+1}t^\mu+k^{2\mu}t\right)},$$
$$s_0' = \left(\frac{k}{R}\right)^{\mu+1}\left\{\frac{(\mu/n)\left(\left|a_\mu\right|Rk^{\mu-1}/(|a_0|-m)\right)+1}{(\mu/n)\left(\left|a_\mu\right|k^{\mu+1}/(|a_o|-m)R\right)+1}\right\}, \tag{1.17}$$
$$m = \min_{|z|=k}\left|p(z)\right|.$$

Remark 1.4. For $R = r = 1$ Theorem 1.3 reduces to Theorem B.

Remark 1.5. Dividing the two sides of (1.16) by $|\alpha|$ and letting $|\alpha| \to \infty$, we obtain a result of Chanam and Dewan [10].

2. Lemmas

For the proofs of these theorems we need the following lemmas.

Lemma 2.1. *If $p(z)$ has all its zeros in $|z| \leq 1$, then for every $|\alpha| \geq 1$,*

$$\max_{|z|=1}\left|D_\alpha p(z)\right| \geq \frac{n}{2}\left\{(|\alpha|-1)\max_{|z|=1}\left|p(z)\right| + (|\alpha|+1)m\right\}, \tag{2.1}$$

where $m = \min_{|z|=1}|p(z)|$.

This lemma is due to Aziz and Rather [6].

Lemma 2.2. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \le k$, where $k \ge 1$, then*

$$\max_{|z|=k}|p(z)| \ge \frac{2k^n}{1+k^n}\max_{|z|=1}|p(z)|. \tag{2.2}$$

Inequality (2.2) *is best possible and equality holds for $p(z) = z^n + k^n$.*

This lemma is according to Aziz [11].

Lemma 2.3. *If $p(z)$ is a polynomial of degree n, then for $R \ge 1$,*

$$\begin{aligned}\max_{|z|=R}|p(z)| \le{}& R^n\max_{|z|=1}|p(z)| - \frac{2(R^n-1)}{n+2}|p(0)| \\ &- \left[\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right]|p'(0)|\end{aligned} \tag{2.3}$$

if $n > 2$, and

$$\max_{|z|=R}|p(z)| \le R^2\max_{|z|=1}|p(z)| - \frac{(R-1)}{2}\left[(R+1)|p(0)| + (R-1)|p'(0)|\right] \tag{2.4}$$

if $n = 2$.

This lemma is according to Dewan et al. [12].

Lemma 2.4. *If $p(z)$ is a polynomial of degree $n \ge 3$ having no zeros in $|z| < 1$ and $m = \min_{|z|=1}|p(z)|$, then for $R \ge 1$,*

$$\begin{aligned}\max_{|z|=R}|p(z)| \le{}& \left(\frac{R^n+1}{2}\right)\max_{|z|=1}|p(z)| - \left(\frac{R^n-1}{2}\right)m - |p'(0)|\frac{2}{(n+1)}\left[\frac{R^n-1}{n} - (R-1)\right] \\ &- |p''(0)|\left[\left(\frac{(R^n-1)-n(R-1)}{n(n-1)}\right) - \left(\frac{(R^{n-2}-1)-(n-2)(R-1)}{(n-2)(n-3)}\right)\right]\end{aligned} \tag{2.5}$$

if $n > 3$, and

$$\begin{aligned}\max_{|z|=R}|p(z)| \le{}& \left(\frac{R^n+1}{2}\right)\max_{|z|=1}|p(z)| - \left(\frac{R^n-1}{2}\right)m \\ &- |p'(0)|\frac{2}{n+1}\left[\frac{(R^n-1)}{n} - (R-1)\right] \\ &- |p''(0)|\frac{(R-1)^n}{n(n-1)}\end{aligned} \tag{2.6}$$

if $n = 3$.

This result is according to Dewan et al. [13].

Lemma 2.5. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$ *is a polynomial of degree* n *such that* $p(z) \neq 0$ *in* $|z| < k$, $k > 0$, *then for* $0 < r \leq R \leq k$,

$$\max_{|z|=R}|p(z)| \leq \exp\left\{n\int_r^R \frac{(\mu/n)(|a_\mu|/(|a_0|-m))k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+(\mu/n)(|a_\mu|/(|a_0|-m))(k^{\mu+1}t^\mu+k^{2\mu}t)}dt\right\}\max_{|z|=r}|p(z)|$$
$$+\left[1-\exp\left\{n\int_r^R \frac{(\mu/n)(|a_\mu|/(|a_0|-m))k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+(\mu/n)(|a_\mu|/(|a_0|-m))(k^{\mu+1}t^\mu+k^{2\mu}t)}dt\right\}\right]m, \tag{2.7}$$

where $m = \min_{|z|=k}|p(z)|$.

Lemma 2.5 is according to Chanam and Dewan [10].

3. Proof of the Theorems

Proof of Theorem 1.1. By hypothesis that the polynomial $p(z)$ has all its zeros in $|z| \leq k$, where $k \geq 1$, therefore all the zeros of the polynomial $G(z) = p(kz)$ lie in $|z| \leq 1$. Applying Lemma 2.1 to the polynomial $G(z)$ and noting that $|\alpha|/k \geq 1$, we get

$$\max_{|z|=1}|D_{\alpha/k}G(z)| \geq \frac{n}{2}\left[\left(\frac{|\alpha|}{k}-1\right)\max_{|z|=1}|G(z)| + \left(\frac{|\alpha|}{k}+1\right)\min_{|z|=1}|G(z)|\right], \tag{3.1}$$

that is,

$$\max_{|z|=k}|D_\alpha p(z)| \geq \frac{n}{2}\left[\left(\frac{|\alpha|-k}{k}\right)\max_{|z|=k}|p(z)| + \left(\frac{|\alpha|+k}{k}\right)m\right]. \tag{3.2}$$

The polynomial $p(z)$ is of degree $n > 3$ and so $D_\alpha p(z)$ is the polynomial of degree $n-1$, where $n-1 > 2$, hence applying Lemma 2.3 to the polynomial $D_\alpha p(z)$, we get for $k \geq 1$

$$\max_{|z|=k}|D_\alpha p(z)| \leq k^{n-1}\max_{|z|=1}|D_\alpha p(z)| - \frac{2(k^{n-1}-1)}{n+1}|na_0+\alpha a_1|$$
$$-\left[\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}\right]|(n-1)a_1+2\alpha a_2|. \tag{3.3}$$

Combining (3.2) and (3.3), we get for $k \geq 1$

$$\begin{aligned}\max_{|z|=1}\left|D_\alpha p(z)\right| \geq \frac{n}{2}&\left[\left(\frac{|\alpha|-k}{k^n}\right)\max_{|z|=k}|p(z)|+\left(\frac{|\alpha|+k}{k^n}\right)m\right]\\&+\frac{2(k^{n-1}-1)}{k^{n-1}(n+1)}|na_0+\alpha a_1|\\&+\frac{1}{k^{n-1}}\left[\left(\frac{k^{n-1}-1}{n-1}\right)-\left(\frac{k^{n-3}-1}{n-3}\right)\right]|(n-1)a_1+2\alpha a_2|.\end{aligned}\tag{3.4}$$

Since the polynomial $p(z)$ has all zeros in $|z| \leq k$, $k \geq 1$, the polynomial $q(z) = z^n p(1/z)$ has no zero in $|z| < 1/k$, hence the polynomial $q(z/k)$ has all its zeros in $|z| \geq 1$, therefore on applying Lemma 2.4 to the polynomial $q(z/k)$, we get

$$\begin{aligned}\max_{|z|=k\geq 1}\left|q\left(\frac{z}{k}\right)\right| \leq \left(\frac{k^n+1}{2}\right)\max_{|z|=1}\left|q\left(\frac{z}{k}\right)\right|-\left(\frac{k^n-1}{2}\right)\min_{|z|=1}\left|q\left(\frac{z}{k}\right)\right|\\-\frac{2|a_{n-1}|}{(n+1)k}\left[\frac{k^n-1}{n}-(k-1)\right]\\-\frac{2|a_{n-2}|}{k^2}\left[\left(\frac{(k^n-1)-n(k-1)}{n(n-1)}\right)-\left(\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right)\right].\end{aligned}\tag{3.5}$$

Since $\max_{|z|=1}|q(z/k)| = (1/k^n)\max_{|z|=k}|p(z)|$ (and similarly for the minima), (3.5) is equivalent to

$$\begin{aligned}\max_{|z|=k}|p(z)| \geq \left(\frac{2k^n}{k^n+1}\right)\max_{|z|=1}|p(z)|+\left(\frac{k^n-1}{k^n+1}\right)m\\+\frac{4k^{n-1}|a_{n-1}|}{(k^n+1)(n+1)}\left[\frac{k^n-1}{n}-(k-1)\right]\\+\frac{4k^{n-2}|a_{n-2}|}{k^n+1}\left[\left(\frac{(k^n-1)-n(k-1)}{n(n-1)}\right)-\left(\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right)\right].\end{aligned}\tag{3.6}$$

Combining (3.4) and (3.6) we get the desired result. This completes the proof of inequality (1.12). The proof of the Theorem in the case $n = 3$ follows along the same lines as the proof of (1.12) but instead of inequalities (2.3) and (2.5), we use inequalities (2.4) and (2.6), respectively. □

Proof of Theorem 1.3. By hypothesis that the polynomial $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, has no zero in $|z| < k$, where $k \geq 1$, therefore the polynomial $F(z) = p(Rz)$ has no zero in $|z| \leq k/R$, where $k/R \geq 1$. Since $|\alpha/R| \geq 1$, using Theorem B we have

$$\max_{|z|=1}|D_{\alpha/R}[F(z)]| = \max_{|z|=R}\left|D_\alpha\left[p(z)\right]\right| \leq \frac{n}{1+s_0'}\left\{\left(\frac{|\alpha|}{R}+s_0'\right)\max_{|z|=R}|p(z)|-\left(\frac{|\alpha|}{R}-1\right)m\right\},\tag{3.7}$$

where $m = \min_{|z|=k/R}|F(z)| = \min_{|z|=k}|p(z)|$ and

$$s_0' = \left(\frac{k}{R}\right)^{\mu+1}\left\{\frac{(\mu/n)\left(\left|a_\mu\right|Rk^{\mu-1}/(|a_0|-m)\right)+1}{(\mu/n)\left(\left|a_\mu\right|k^{\mu+1}/(|a_o|-m)R\right)+1}\right\}. \tag{3.8}$$

Using Lemma 2.5 in the previous inequality, we get

$$\begin{aligned}
&\max_{|z|=R}\left|D_\alpha p(z)\right| \\
&\leq \frac{n}{1+s_0'}\left[\left(\frac{|\alpha|}{R}+s_0'\right)\exp\left\{n\int_r^R \frac{(\mu/n)\left(\left|a_\mu\right|/(|a_0|-m)\right)k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+(\mu/n)\left(\left|a_\mu\right|/(|a_0|-m)\right)\left(k^{\mu+1}t^\mu+k^{2\mu}t\right)}dt\right\}\max_{|z|=r}|p(z)|\right. \\
&\qquad + \left(s_0'+1-\left(\frac{|\alpha|}{R}+s_0'\right)\right. \\
&\qquad\qquad \left.\left.\times\exp\left\{n\int_r^R \frac{(\mu/n)\left(\left|a_\mu\right|/(|a_0|-m)\right)k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+(\mu/n)\left(\left|a_\mu\right|/(|a_0|-m)\right)\left(k^{\mu+1}t^\mu+k^{2\mu}t\right)}dt\right\}\right)m\right].
\end{aligned} \tag{3.9}$$

This completes the proof of the theorem. □

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