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Research Article

A New Approximation Method for Solving Variational Inequalities and Fixed Points of Nonexpansive Mappings

Chakkrid Klin-eam and Suthep Suantai

Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, 50200, Thailand

Correspondence should be addressed to Suthep Suantai, scmti005@chiangmai.ac.th

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A new approximation method for solving variational inequalities and fixed points of nonexpansive mappings is introduced and studied. We prove strong convergence theorem of the new iterative scheme to a common element of the set of fixed points of nonexpansive mapping and the set of solutions of the variational inequality for the inverse-strongly monotone mapping which solves some variational inequalities. Moreover, we apply our main result to obtain strong convergence to a common fixed point of nonexpansive mapping and strictly pseudocontractive mapping in a Hilbert space.

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1. Introduction

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\theta(x) = \frac{1}{2}\langle Ax, x\rangle - \langle x, y\rangle \quad \forall x \in F(S), \tag{1.1}$$

where A is a linear bounded operator, $F(S)$ is the fixed point set of a nonexpansive mapping S, and y is a given point in H.

Let H be a real Hilbert space and C be a nonempty closed convex subset of H.

Recall that a mapping $S : C \to C$ is called *nonexpansive* if $\|Sx - Sy\| \le \|x - y\|$ for all $x, y \in C$. The set of all fixed points of S is denoted by $F(S)$, that is, $F(S) = \{x \in C : x = Sx\}$. A linear bounded operator A is *strongly positive* if there is a constant $\overline{\gamma} > 0$ with the property $\langle Ax, x\rangle \ge \overline{\gamma}\|x\|^2$ for all $x \in H$. A self-mapping $f : C \to C$ is a *contraction* on C if there is a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \le \alpha\|x - y\|$ for all $x, y \in C$. We use Π_C to denote the collection of all contractions on C. Note that each $f \in \Pi_C$ has a unique fixed point in C. A mapping B of C into H is called *monotone* if $\langle Bx - By, x - y\rangle \ge 0$ for all $x, y \in C$. The variational inequality problem is to find $x \in C$ such that

$$\langle Bx, y - x\rangle \ge 0 \quad \forall y \in C. \tag{1.2}$$

The set of solutions of the variational inequality is denoted by $VI(C, B)$. A mapping B of C to H is called *inverse-strongly monotone* if there exists a positive real number β such that

$$\langle x - y, Bx - By\rangle \ge \beta\|Bx - By\|^2 \quad \forall x, y \in C. \tag{1.3}$$

For such a case, B is β-inverse-strongly monotone. If B is a β-inverse-strongly monotone mapping of C to H, then it is obvious that B is $(1/\beta)$-Lipschitz continuous.

In 2000, Moudafi [1] introduced the viscosity approximation method for nonexpansive mapping and proved that if H is a real Hilbert space, the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in C$ is chosen arbitrarily:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Sx_n, \quad n \ge 0, \tag{1.4}$$

where $\{\alpha_n\} \subset (0,1)$ satisfies certain conditions, converges strongly to a fixed point of S (say $\overline{x} \in C$) which is the unique solution of the following variational inequality:

$$\langle (I - f)\overline{x}, x - \overline{x}\rangle \ge 0 \quad \forall x \in F(S). \tag{1.5}$$

In 2004, Xu [2] extended the results of Moudafi [1] to a Banach space. In 2006, Marino and Xu [3] introduced a general iterative method for nonexpansive mapping. They defined the sequence $\{x_n\}$ by the following algorithm:

$$x_0 \in C, \quad x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)Sx_n, \quad n \ge 0, \tag{1.6}$$

where $\{\alpha_n\} \subset (0,1)$ and A is a strongly positive linear bounded operator, and they proved that if $C = H$ and the sequence $\{\alpha_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.6) converges strongly to a fixed point of S (say $\overline{x} \in H$) which is the unique solution of the following variational inequality:

$$\langle (A - \gamma f)\overline{x}, x - \overline{x}\rangle \ge 0 \quad \forall x \in F(S), \tag{1.7}$$

which is the optimality condition for minimization problem $\min_{x\in C}(1/2)\langle Ax, x\rangle - h(x)$, where h is a potential function for γf (i.e., $h'(x) = \gamma f$ for all $x \in H$).

For finding a common element of the set of fixed points of nonexpansive mappings and the set of solution of the variational inequalities, Iiduka and Takahashi [4] introduced following iterative process:

$$x_0 \in C, \qquad x_{n+1} = \alpha_n u + (1-\alpha_n) S P_C (x_n - \lambda_n B x_n), \quad n \geq 0, \tag{1.8}$$

where P_C is the projection of H onto C, $u \in C$, $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < 2\beta$. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{\lambda_n\}$, the sequence $\{x_n\}$ generated by (1.8) converges strongly to a common element of the set of fixed points of nonexpansive mapping and the set of solutions of the variational inequality for an inverse strongly monotone mapping (say $\overline{x} \in C$) which solves the variational inequality

$$\langle \overline{x} - u, x - \overline{x} \rangle \geq 0 \quad \forall x \in F(S) \cap VI(C, B). \tag{1.9}$$

In 2007, Chen et al. [5] introduced the following iterative process: $x_0 \in C$,

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S P_C (x_n - \lambda_n B x_n), \quad n \geq 0, \tag{1.10}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < 2\beta$. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{\lambda_n\}$, the sequence $\{x_n\}$ generated by (1.10) converges strongly to a common element of the set of fixed points of nonexpansive mapping and the set of solutions of the variational inequality for an inverse strongly monotone mapping (say $\overline{x} \in C$) which solves the variational inequality

$$\left\langle (I - f)\overline{x}, x - \overline{x} \right\rangle \geq 0 \quad \forall x \in F(S) \cap VI(C, B). \tag{1.11}$$

In this paper, we modify the iterative methods (1.6) and (1.10) by purposing the following general iterative method:

$$x_0 \in C, \qquad x_{n+1} = P_C\left(\alpha_n \gamma f(x_n) + (I - \alpha_n A) S P_C (x_n - \lambda_n B x_n)\right), \quad n \geq 0, \tag{1.12}$$

where P_C is the projection of H onto C, f is a contraction, A is a strongly positive linear bounded operator, B is a β-inverse strongly monotone mapping, $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < 2\beta$.

We note that when $A = I$ and $\gamma = 1$, the iterative scheme (1.12) reduces to the iterative scheme (1.10).

The purpose of this paper is twofold. First, we show that under some control conditions the sequence $\{x_n\}$ defined by (1.12) strongly converges to a common element of the set of fixed points of nonexpansive mapping and the set of solutions of the variational inequality for the inverse-strongly monotone mapping B in a real Hilbert space which solves some variational inequalities. Secondly, by using the first results, we obtain a strong convergence theorem for a common fixed point of nonexpansive mapping and strictly pseudocontractive mapping. Moreover, we consider the problem of finding a common element of the set of fixed points of nonexpansive mapping and the set of zeros of inverse-strongly monotone mapping.

2. Preliminaries

Let H be real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, C a nonempty closed convex subset of H. Recall that the metric (nearest point) projection P_C from a real Hilbert space H to a closed convex subset C of H is defined as follows: given $x \in H$, $P_C x$ is the only point in C with the property $\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}$. In what follows Lemma 2.1 can be found in any standard functional analysis book.

Lemma 2.1. *Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$, then*

(i) $y = P_C x$ *if and only if the inequality* $\langle x - y, y - z \rangle \geq 0$ *for all* $z \in C$,

(ii) P_C *is nonexpansive,*

(iii) $\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2$ *for all* $x, y \in H$,

(iv) $\langle x - P_C x, P_C x - y \rangle \geq 0$ *for all* $x \in H$ *and* $y \in C$.

Using Lemma 2.1, one can show that the variational inequality (1.2) is equivalent to a fixed point problem.

Lemma 2.2. *The point $u \in C$ is a solution of the variational inequality* (1.2) *if and only if u satisfies the relation $u = P_C(u - \lambda B u)$ for all $\lambda > 0$.*

We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and write $x_n \to x$ to indicate that $\{x_n\}$ converges strongly to x. It is well known that H satisfies the Opial's condition [6], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\| \tag{2.1}$$

holds for every $y \in H$ with $x \neq y$.

A set-valued mapping $T : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $u \in Tx$, and $v \in Ty$ imply $\langle x - y, u - v \rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is *maximal* if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, u) \in H \times H$, $\langle x - y, u - v \rangle \geq 0$ for every $(y, v) \in G(T)$ implies $u \in Tx$. Let B be an inverse-strongly monotone mapping of C to H and let $N_C v$ be normal cone to C at $v \in C$, that is, $N_C v = \{w \in H : \langle v - u, w \rangle \geq 0,\ \forall u \in C\}$, and define

$$Tv = \begin{cases} Bv + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \tag{2.2}$$

Then T is a maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, B)$ [7]. In the sequel, the following lemmas are needed to prove our main results.

Lemma 2.3 (see [8]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n) a_n + \delta_n$, $n \geq 0$, where $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n \to \infty} \delta_n / \gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \to \infty} a_n = 0$.

Lemma 2.4 (see [9]). *Let C be a closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. If a sequence $\{x_n\}$ in C is such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z = Tz$.*

Lemma 2.5 (see [3]). *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \le \|A\|^{-1}$, then $\|I - \rho A\| \le 1 - \rho\overline{\gamma}$.*

3. Main Results

In this section, we prove a strong convergence theorem for nonexpansive mapping and inverse strongly monotone mapping.

Theorem 3.1. *Let H be a real Hilbert space, let C be a closed convex subset of H, and let $B : C \to H$ be a β-inverse strongly monotone mapping, also let A be a strongly positive linear bounded operator of H into itself with coefficient $\overline{\gamma} > 0$ such that $\|A\| = 1$ and let $f : C \to C$ be a contraction with coefficient α $(0 < \alpha < 1)$. Assume that $0 < \gamma < \overline{\gamma}/\alpha$. Let S be a nonexpansive mapping of C into itself such that $\Omega = F(S) \cap VI(C, B) \neq \emptyset$. Suppose $\{x_n\}$ is the sequence generated by the following algorithm: $x_0 \in C$,*

$$x_{n+1} = P_C\big(\alpha_n \gamma f(x_n) + (I - \alpha_n A) S P_C(x_n - \lambda_n B x_n)\big) \tag{3.1}$$

for all $n = 0, 1, 2, \ldots$, where $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, 2\beta)$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\beta$,

$$\begin{aligned} &C1\text{: } \lim_{n\to 0}\alpha_n = 0, \qquad &&C2\text{: } \sum_{n=1}^{\infty}\alpha_n = \infty, \\ &C3\text{: } \sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty, \qquad &&C4\text{: } \sum_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty, \end{aligned} \tag{3.2}$$

then $\{x_n\}$ converges strongly to $q \in \Omega$, where $q = P_\Omega(\gamma f + I - A)(q)$ which solves the following variational inequality:

$$\big\langle (\gamma f - A)q, p - q \big\rangle \le 0 \quad \forall p \in \Omega. \tag{3.3}$$

Proof. First, we show the mapping $I - \lambda_n B$ is nonexpansive. Indeed, since B is a β-strongly monotone mapping and $0 < \lambda_n < 2\beta$, we have that for all $x, y \in C$,

$$\begin{aligned} \|(I - \lambda_n B)x - (I - \lambda_n B)y\|^2 &= \|(x - y) - \lambda_n(Bx - By)\|^2 \\ &= \|x - y\|^2 - 2\lambda_n\langle x - y, Bx - By\rangle + \lambda_n^2\|Bx - By\|^2 \\ &\le \|x - y\|^2 + \lambda_n(\lambda_n - 2\beta)\|Bx - By\|^2 \\ &\le \|x - y\|^2, \end{aligned} \tag{3.4}$$

which implies that the mapping $I - \lambda_n B$ is nonexpansive. Next, we show that the sequence $\{x_n\}$ is bounded. Put $y_n = P_C(x_n - \lambda_n x_n)$ for all $n \geq 0$. Let $u \in \Omega$, we have

$$\begin{aligned}
\|y_n - u\| &= \|P_C(x_n - \lambda_n B x_n) - P_C(u - \lambda_n B u)\| \\
&\leq \|(x_n - \lambda_n B x_n) - (u - \lambda_n B u)\| \\
&\leq \|(I - \lambda_n B)x_n - (I - \lambda_n B)u\| \\
&\leq \|x_n - u\|.
\end{aligned} \tag{3.5}$$

Then, we have

$$\begin{aligned}
\|x_{n+1} - u\| &= \left\|P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A)Sy_n) - P_C(u)\right\| \\
&\leq \left\|\alpha_n(\gamma f(x_n) - Au) + (I - \alpha_n A)(Sy_n - u)\right\| \\
&\leq \alpha_n \left\|\gamma f(x_n) - Au\right\| + (1 - \alpha_n \overline{\gamma})\left\|y_n - u\right\| \\
&\leq \alpha_n \left\|\gamma f(x_n) - \gamma f(u)\right\| + \alpha_n \left\|\gamma f(u) - Au\right\| + (1 - \alpha_n \overline{\gamma})\left\|y_n - u\right\| \\
&\leq \alpha\gamma\alpha_n \|x_n - u\| + \alpha_n \left\|\gamma f(u) - Au\right\| + (1 - \alpha_n \overline{\gamma})\|x_n - u\| \\
&= (1 - (\overline{\gamma} - \gamma\alpha)\alpha_n)\|x_n - u\| + \alpha_n \left\|\gamma f(u) - Au\right\| \\
&= (1 - (\overline{\gamma} - \gamma\alpha)\alpha_n)\|x_n - u\| + (\overline{\gamma} - \gamma\alpha)\alpha_n \frac{\left\|\gamma f(u) - Au\right\|}{\overline{\gamma} - \gamma\alpha} \\
&\leq \max\left\{\|x_n - u\|, \frac{\left\|\gamma f(u) - Au\right\|}{\overline{\gamma} - \gamma\alpha}\right\}.
\end{aligned} \tag{3.6}$$

It follows from induction that

$$\|x_n - u\| \leq \max\left\{\|x_0 - u\|, \frac{\left\|\gamma f(u) - Au\right\|}{\overline{\gamma} - \gamma\alpha}\right\}, \quad n \geq 0. \tag{3.7}$$

Therefore, $\{x_n\}$ is bounded, so are $\{y_n\}$,$\{Sy_n\}$,$\{Bx_n\}$, and $\{f(x_n)\}$. Since $I - \lambda_n B$ is nonexpansive and $y_n = P_C(x_n - \lambda_n B x_n)$, we also have

$$\begin{aligned}
\|y_{n+1} - y_n\| &\leq \|(x_{n+1} - \lambda_{n+1} B x_{n+1}) - (x_n - \lambda_n B x_n)\| \\
&\leq \|(x_{n+1} - \lambda_{n+1} B x_{n+1}) - (x_n - \lambda_{n+1} B x_n)\| + |\lambda_n - \lambda_{n+1}|\|Bx_n\| \\
&\leq \|(I - \lambda_{n+1} B)x_{n+1} - (I - \lambda_{n+1} B)x_n\| + |\lambda_n - \lambda_{n+1}|\|Bx_n\| \\
&\leq \|x_{n+1} - x_n\| + |\lambda_n - \lambda_{n+1}|\|Bx_n\|.
\end{aligned} \tag{3.8}$$

So we obtain

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \left\|\left(P_C\left(\alpha_n\gamma f(x_n)+(I-\alpha_n A)Sy_n\right)\right)-\left(P_C\left(\alpha_{n-1}\gamma f(x_{n-1})+(I-\alpha_{n-1}A)Sy_{n-1}\right)\right)\right\| \\
&\le \big\|(I-\alpha_n A)\left(Sy_n-Sy_{n-1}\right)-(\alpha_n-\alpha_{n-1})ASy_{n-1} \\
&\qquad +\gamma\alpha_n\left(f(x_n)-f(x_{n-1})\right)+\gamma(\alpha_n-\alpha_{n-1})f(x_{n-1})\big\| \\
&\le (1-\alpha_n\overline{\gamma})\left\|y_n-y_{n-1}\right\|+|\alpha_n-\alpha_{n-1}|\left\|ASy_{n-1}\right\| \\
&\quad +\gamma\alpha\alpha_n\|x_n-x_{n-1}\|+\gamma|\alpha_n-\alpha_{n-1}|\left\|f(x_{n-1})\right\| \\
&\le (1-\alpha_n\overline{\gamma})\left[\|x_n-x_{n-1}\|+|\lambda_{n-1}-\lambda_n|\|Bx_{n-1}\|\right]+|\alpha_n-\alpha_{n-1}|\left\|ASy_{n-1}\right\| \\
&\quad +\gamma\alpha\alpha_n\|x_n-x_{n-1}\|+\gamma|\alpha_n-\alpha_{n-1}|\left\|f(x_{n-1})\right\| \\
&\le (1-\alpha_n\overline{\gamma})\|x_n-x_{n-1}\|+|\lambda_{n-1}-\lambda_n|\|Bx_{n-1}\|+|\alpha_n-\alpha_{n-1}|\left\|ASy_{n-1}\right\| \\
&\quad +\gamma\alpha\alpha_n\|x_n-x_{n-1}\|+\gamma|\alpha_n-\alpha_{n-1}|\left\|f(x_{n-1})\right\| \\
&= \left(1-(\overline{\gamma}-\gamma\alpha)\alpha_n\right)\|x_n-x_{n-1}\|+L|\lambda_{n-1}-\lambda_n|+M|\alpha_n-\alpha_{n-1}|,
\end{aligned} \tag{3.9}$$

where $L=\sup\{\|Bx_{n-1}\|:n\in\mathbb{N}\}$, $M=\max\{\sup_{n\in\mathbb{N}}\|ASy_{n-1}\|,\ \sup_{n\in\mathbb{N}}\gamma\|f(x_{n-1})\|\}$. Since $\sum_{n=1}^{\infty}|\alpha_n-\alpha_{n-1}|<\infty$ and $\sum_{n=1}^{\infty}|\lambda_{n-1}-\lambda_n|<\infty$, by Lemma 2.3, we have $\|x_{n+1}-x_n\|\to 0$. For $u\in\Omega$ and $u=P_C(u-\lambda_n Bu)$, we have

$$\begin{aligned}
\|x_{n+1}-u\|^2 &= \left\|P_C\left(\alpha_n\gamma f(x_n)+(I-\alpha_n A)Sy_n\right)-P_C(u)\right\|^2 \\
&\le \left\|\alpha_n\left(\gamma f(x_n)-Au\right)+(I-\alpha_n A)\left(Sy_n-u\right)\right\|^2 \\
&\le \left(\alpha_n\left\|\gamma f(x_n)-Au\right\|+\|I-\alpha_n A\|\left\|Sy_n-u\right\|\right)^2 \\
&\le \left(\alpha_n\left\|\gamma f(x_n)-Au\right\|+(1-\alpha_n\overline{\gamma})\left\|y_n-u\right\|\right)^2 \\
&\le \alpha_n\left\|\gamma f(x_n)-Au\right\|^2+(1-\alpha_n\overline{\gamma})\left\|y_n-u\right\|^2 \\
&\quad +2\alpha_n(1-\alpha_n\overline{\gamma})\left\|\gamma f(x_n)-Au\right\|\left\|y_n-u\right\| \\
&\le \alpha_n\left\|\gamma f(x_n)-Au\right\|^2+(1-\alpha_n\overline{\gamma})\|(I-\lambda_n B)x_n-(I-\lambda_n B)u\|^2 \\
&\quad +2\alpha_n(1-\alpha_n\overline{\gamma})\left\|\gamma f(x_n)-Au\right\|\left\|y_n-u\right\| \\
&\le \alpha_n\left\|\gamma f(x_n)-Au\right\|^2+(1-\alpha_n\overline{\gamma})\left(\|x_n-u\|^2+\lambda_n(\lambda_n-2\beta)\|Bx_n-Bu\|^2\right) \\
&\quad +2\alpha_n(1-\alpha_n\overline{\gamma})\left\|\gamma f(x_n)-Au\right\|\left\|y_n-u\right\| \\
&\le \alpha_n\left\|\gamma f(x_n)-Au\right\|^2+\|x_n-u\|^2+(1-\alpha_n\overline{\gamma})a(b-2\beta)\|Bx_n-Bu\|^2 \\
&\quad +2\alpha_n(1-\alpha_n\overline{\gamma})\left\|\gamma f(x_n)-Au\right\|\left\|y_n-u\right\|.
\end{aligned} \tag{3.10}$$

So, we obtain

$$\begin{aligned}
&-(1-\alpha_n\overline{\gamma})a(b-2\beta)\|Bx_n-Bu\|^2\\
&\quad\le \alpha_n\|\gamma f(x_n)-Au\|^2+(\|x_n-u\|+\|x_{n+1}-u\|)(\|x_n-u\|-\|x_{n+1}-u\|)+\epsilon_n\\
&\quad\le \alpha_n\|\gamma f(x_n)-Au\|^2+\epsilon_n+\|x_n-x_{n+1}\|(\|x_n-u\|+\|x_{n+1}-u\|),
\end{aligned}\tag{3.11}$$

where $\epsilon_n=2\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n)-Au\|\|y_n-u\|$. Since $\alpha_n\to 0$ and $\|x_{n+1}-x_n\|\to 0$, we obtain that $\|Bx_n-Bu\|\to 0$ as $n\to\infty$. Further, by Lemma 2.1(iii), we have

$$\begin{aligned}
\|y_n-u\|^2&=\|P_C(x_n-\lambda_nBx_n)-P_C(u-\lambda_nBu)\|^2\\
&\le\langle(x_n-\lambda_nBx_n)-(u-\lambda_nBu),y_n-u\rangle\\
&=\frac{1}{2}\Big(\|(x_n-\lambda_nBx_n)-(u-\lambda_nBu)\|^2+\|y_n-u\|^2\\
&\qquad-\|(x_n-\lambda_nBx_n)-(u-\lambda_nBu)-(y_n-u)\|^2\Big)\\
&\le\frac{1}{2}\Big(\|x_n-u\|^2+\|y_n-u\|^2-\|(x_n-y_n)-\lambda_n(Bx_n-Bu)\|^2\Big)\\
&=\frac{1}{2}\Big(\|x_n-u\|^2+\|y_n-u\|^2-\|x_n-y_n\|^2\Big)\\
&\quad+\frac{1}{2}\Big(2\lambda_n\langle x_n-y_n,Bx_n-Bu\rangle-\lambda_n^2\|Bx_n-Bu\|^2\Big).
\end{aligned}\tag{3.12}$$

So, we obtain that

$$\|y_n-u\|^2\le\|x_n-u\|^2-\|x_n-y_n\|^2+2\lambda_n\langle x_n-y_n,Bx_n-Bu\rangle-\lambda_n^2\|Bx_n-Bu\|^2.\tag{3.13}$$

So, we have

$$\begin{aligned}
\|x_{n+1}-u\|^2&=\|P_C(\alpha_n\gamma f(x_n)+(I-\alpha_nA)Sy_n)-P_C(u)\|^2\\
&\le\|\alpha_n(\gamma f(x_n)-Au)+(I-\alpha_nA)(Sy_n-u)\|^2\\
&\le(\alpha_n\|\gamma f(x_n)-Au\|+\|I-\alpha_nA\|\|Sy_n-u\|)^2\\
&\le\Big(\alpha_n\|\gamma f(x_n)-Au\|^2+(1-\alpha_n\overline{\gamma})\|y_n-u\|\Big)^2\\
&\le\alpha_n\|\gamma f(x_n)-Au\|^2+(1-\alpha_n\overline{\gamma})\|y_n-u\|^2+2\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n)-Au\|\|y_n-u\|\\
&\le\alpha_n\|\gamma f(x_n)-Au\|^2+(1-\alpha_n\overline{\gamma})\|x_n-u\|^2-(1-\alpha_n\overline{\gamma})\|x_n-y_n\|^2\\
&\quad+2(1-\alpha_n\overline{\gamma})\lambda_n\langle(x_n-y_n,Bx_n-Bu\rangle-(1-\alpha_n\overline{\gamma})\lambda_n^2\|Bx_n-Bu\|^2\\
&\quad+2\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n)-Au\|\|y_n-u\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n \left\| \gamma f(x_n) - Au \right\|^2 + \|x_n - u\|^2 - (1 - \alpha_n \overline{\gamma}) \left\| x_n - y_n \right\|^2 \\
&\quad + 2(1 - \alpha_n \overline{\gamma}) \lambda_n \langle x_n - y_n, Bx_n - Bu \rangle - (1 - \alpha_n \overline{\gamma}) \lambda_n^2 \|Bx_n - Bu\|^2 \\
&\quad + 2\alpha_n (1 - \alpha_n \overline{\gamma}) \left\| \gamma f(x_n) - Au \right\| \left\| y_n - u \right\|,
\end{aligned}
\tag{3.14}
$$

which implies

$$
\begin{aligned}
(1 - \alpha_n \overline{\gamma}) \left\| x_n - y_n \right\|^2 \leq{}& \alpha_n \left\| \gamma f(x_n) - Au \right\|^2 + (\|x_n - u\| + \|x_{n+1} - u\|) \|x_n - x_{n+1}\| \\
&+ 2(1 - \alpha_n \overline{\gamma}) \lambda_n \langle x_n - y_n, Bx_n - Bu \rangle - (1 - \alpha_n \overline{\gamma}) \lambda_n^2 \|Bx_n - Bu\|^2 \\
&+ 2\alpha_n (1 - \alpha_n \overline{\gamma}) \left\| \gamma f(x_n) - Au \right\| \left\| y_n - u \right\|.
\end{aligned}
\tag{3.15}
$$

Since $\alpha_n \to 0$, $\|x_{n+1} - x_n\| \to 0$, and $\|Bx_n - Bu\| \to 0$, we obtain $\|x_n - y_n\| \to 0$ as $n \to \infty$. Next, we have

$$
\begin{aligned}
\left\| x_{n+1} - Sy_n \right\| &= \left\| P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A) Sy_n) - P_C(Sy_n) \right\| \\
&\leq \left\| \alpha_n \gamma f(x_n) + (I - \alpha_n A) Sy_n - Sy_n \right\| \\
&= \alpha_n \left\| \gamma f(x_n) + ASy_n \right\|.
\end{aligned}
\tag{3.16}
$$

Since $\alpha_n \to 0$ and $\{f(x_n)\}$, $\{ASy_n\}$ are bounded, we have $\|x_{n+1} - Sy_n\| \to 0$ as $n \to \infty$. Since

$$
\left\| x_n - Sy_n \right\| \leq \|x_n - x_{n+1}\| + \left\| x_{n+1} - Sy_n \right\|,
\tag{3.17}
$$

it implies that $\|x_n - Sy_n\| \to 0$ as $n \to \infty$. Since

$$
\begin{aligned}
\|x_n - Sx_n\| &\leq \left\| x_n - Sy_n \right\| + \left\| Sy_n - Sx_n \right\| \\
&\leq \left\| x_n - Sy_n \right\| + \left\| y_n - x_n \right\|,
\end{aligned}
\tag{3.18}
$$

we obtain that $\|x_n - Sx_n\| \to 0$ as $n \to \infty$. Moreover, from

$$
\left\| y_n - Sy_n \right\| \leq \left\| y_n - x_n \right\| + \left\| x_n - Sy_n \right\|,
\tag{3.19}
$$

it follows that $\|y_n - Sy_n\| \to 0$ as $n \to \infty$.

Observe that $P_\Omega(\gamma f + (I - A))$ is a contraction. Indeed, by Lemma 2.5, we have that $\|I - A\| \leq 1 - \overline{\gamma}$ and since $0 < \gamma < \overline{\gamma}/\alpha$, we have

$$\begin{aligned} \|P_\Omega(\gamma f + (I - A))x - P_\Omega(\gamma f + (I - A))y\| &\leq \|(\gamma f + (I - A))x - (\gamma f + (I - A))y\| \\ &\leq \gamma\|f(x) - f(y)\| + \|I - A\|\|x - y\| \\ &\leq \gamma\alpha\|x - y\| + (1 - \overline{\gamma})\|x - y\| \\ &= (1 - (\overline{\gamma} - \gamma\alpha))\|x - y\|. \end{aligned} \tag{3.20}$$

Then Banach's contraction mapping principle guarantees that $P_\Omega(\gamma f + (I - A))$ has a unique fixed point, say $q \in H$. That is, $q = P_\Omega(\gamma f + (I - A))(q)$. By Lemma 2.1(i), we obtain that $\langle(\gamma f - A)q, p - q\rangle \leq 0$ for all $p \in \Omega$. Choose a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - A)q, Sy_n - q\rangle = \lim_{k\to\infty}\langle(\gamma f - A)q, Sy_{n_k} - q\rangle. \tag{3.21}$$

As $\{y_{n_k}\}$ is bounded, there exists a subsequence $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$ which converges weakly to p. We may assume without loss of generality that $y_{n_k} \rightharpoonup p$. Since $\|y_n - Sy_n\| \to 0$, we obtain $Sy_{n_k} \rightharpoonup p$. Since $\|x_n - Sx_n\| \to 0$, $\|x_n - y_n\| \to 0$ and by Lemma 2.4, we have $p \in F(S)$. Next, we show that $p \in VI(C, B)$. Let

$$Tv = \begin{cases} Bv + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C, \end{cases} \tag{3.22}$$

where $N_C v$ is normal cone to C at $v \in C$, that is, $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0,\ \forall u \in C\}$. Then T is a maximal monotone. Let $(v, w) \in G(T)$. Since $w - Bv \in N_C v$ and $y_n \in C$, we have $\langle v - y_n, w - Bv\rangle \geq 0$. On the other hand, by Lemma 2.1(iv) and from $y_n = P_C(x_n - \lambda_n Bx_n)$, we have

$$\langle v - y_n, y_n - (x_n - \lambda_n Bx_n)\rangle \geq 0, \tag{3.23}$$

and hence $\langle v - y_n, (y_n - x_n)/\lambda_n + Bx_n\rangle \geq 0$. Therefore, we have

$$\begin{aligned} \langle v - y_{n_k}, w\rangle &\geq \langle v - y_{n_k}, Bv\rangle \\ &\geq \langle v - y_{n_k}, Bv\rangle - \left\langle v - y_{n_k}, \frac{y_{n_k} - x_{n_k}}{\lambda_n} + Bx_{n_k}\right\rangle \\ &= \left\langle v - y_{n_k}, Bv - Bx_{n_k} - \frac{y_{n_k} - x_{n_k}}{\lambda_n}\right\rangle \\ &= \langle v - y_{n_k}, Bv - By_{n_k}\rangle + \langle v - y_{n_k}, By_{n_k} - Bx_{n_k}\rangle - \left\langle v - y_{n_k}, \frac{y_{n_k} - x_{n_k}}{\lambda_n}\right\rangle \\ &\geq \langle v - y_{n_k}, By_{n_k} - Bx_{n_k}\rangle - \left\langle v - y_{n_k}, \frac{y_{n_k} - x_{n_k}}{\lambda_n}\right\rangle. \end{aligned} \tag{3.24}$$

This implies $\langle v - p, w\rangle \geq 0$ as $k \to \infty$. Since T is maximal monotone, we have $p \in T^{-1}0$ and hence $p \in VI(C, B)$. We obtain that $p \in \Omega$. It follows from the variational inequality $\langle (\gamma f - A)q, p - q\rangle \leq 0$ for all $p \in \Omega$ that

$$\limsup_{n\to\infty}\langle (\gamma f - A)q, Sy_n - q\rangle = \lim_{k\to\infty}\langle (\gamma f - A)q, Sy_{n_k} - q\rangle = \langle (\gamma f - A)q, p - q\rangle \leq 0. \tag{3.25}$$

Finally, we prove $x_n \to q$. By using (3.5) and together with Schwarz inequality, we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|P_C(\alpha_n\gamma f(x_n) + (I - \alpha_n A)Sy_n) - P_C(q)\|^2 \\
&\leq \|\alpha_n(\gamma f(x_n) - Aq) + (I - \alpha_n A)(Sy_n - q)\|^2 \\
&\leq \|(I - \alpha_n A)(Sy_n - q)\|^2 + \alpha_n^2\|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\alpha_n\langle (I - \alpha_n A)(Sy_n - q), \gamma f(x_n) - Aq\rangle \\
&\leq (1 - \alpha_n\overline{\gamma})^2\|y_n - q\|^2 + \alpha_n^2\|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\alpha_n\langle Sy_n - q, \gamma f(x_n) - Aq\rangle - 2\alpha_n^2\langle A(Sy_n - q), \gamma f(x_n) - Aq\rangle \\
&\leq (1 - \alpha_n\overline{\gamma})^2\|x_n - q\|^2 + \alpha_n^2\|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\alpha_n\langle Sy_n - q, \gamma f(x_n) - \gamma f(q)\rangle + 2\alpha_n\langle Sy_n - q, \gamma f(q) - Aq\rangle \\
&\quad - 2\alpha_n^2\langle A(Sy_n - q), \gamma f(x_n) - Aq\rangle \\
&\leq (1 - \alpha_n\overline{\gamma})^2\|x_n - q\|^2 + \alpha_n^2\|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\alpha_n\|Sy_n - q\|\|\gamma f(x_n) - \gamma f(q)\| + 2\alpha_n\langle Sy_n - q, \gamma f(q) - Aq\rangle \\
&\quad - 2\alpha_n^2\langle A(Sy_n - q), \gamma f(x_n) - Aq\rangle \\
&\leq (1 - \alpha_n\overline{\gamma})^2\|x_n - q\|^2 + \alpha_n^2\|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\gamma\alpha\alpha_n\|y_n - q\|\|x_n - q\| + 2\alpha_n\langle Sy_n - q, \gamma f(q) - Aq\rangle \\
&\quad - 2\alpha_n^2\langle A(Sy_n - q), \gamma f(x_n) - Aq\rangle \\
&\leq (1 - \alpha_n\overline{\gamma})^2\|x_n - q\|^2 + \alpha_n^2\|\gamma f(x_n) - Aq\|^2 \\
&\quad + 2\gamma\alpha\alpha_n\|x_n - q\|^2 + 2\alpha_n\langle Sy_n - q, \gamma f(q) - Aq\rangle \\
&\quad - 2\alpha_n^2\langle A(Sy_n - q), \gamma f(x_n) - Aq\rangle \\
&\leq \left((1 - \alpha_n\overline{\gamma})^2 + 2\gamma\alpha\alpha_n\right)\|x_n - q\|^2 \\
&\quad + \alpha_n\Big(2\langle Sy_n - q, \gamma f(x_n) - Aq\rangle + \alpha_n\|\gamma f(x_n) - Aq\|^2 \\
&\qquad + 2\alpha_n\|A(Sy_n - q)\|\|\gamma f(x_n) - Aq\|)
\end{aligned}$$

$$= (1 - 2(\overline{\gamma} - \gamma\alpha)\alpha_n)\|x_n - q\|^2$$
$$+ \alpha_n\Big(2\langle Sy_n - q, \gamma f(q) - Aq\rangle + \alpha_n\|\gamma f(x_n) - Aq\|^2$$
$$+2\alpha_n\|A(Sy_n - q)\|\|\gamma f(x_n) - Aq\| + \alpha_n\overline{\gamma}^2\|x_n - q\|^2\Big). \tag{3.26}$$

Since $\{x_n\}$, $\{f(x_n)\}$ and $\{Sy_n\}$ are bounded, we can take a constant $\eta > 0$ such that

$$\eta \geq \|\gamma f(x_n) - Aq\|^2 + 2\alpha_n\|A(Sy_n - q)\|\|\gamma f(x_n) - Aq\| + \alpha_n\overline{\gamma}^2\|x_n - q\|^2 \tag{3.27}$$

for all $n \geq 0$. It then follows that

$$\|x_{n+1} - q\|^2 \leq (1 - 2(\overline{\gamma} - \gamma\alpha)\alpha_n)\|x_n - q\|^2 + \alpha_n\beta_n, \tag{3.28}$$

where $\beta_n = 2\langle Sy_n - q, \gamma f(q) - Aq\rangle + \eta\alpha_n$. By $\limsup_{n\to\infty}\langle(\gamma f - A)q, Sy_n - q\rangle \leq 0$, we get $\limsup_{n\to\infty}\beta_n \leq 0$. By applying Lemma 2.3 to (3.28), we can conclude that $x_n \to q$. This completes the proof □

Taking $A = I$ and $\gamma = 1$ in Theorem 3.1, we get the results of Chen et al. [5]

Corollary 3.2 (see [5, Proposition 3.1]). *Let H be a real Hilbert space, let C be a closed convex subset of H, and let $B : C \to H$ be a β-inverse strongly monotone mapping. Let $f : C \to C$ be a contraction with coefficient α $(0 < \alpha < 1)$ and let S be a nonexpansive mapping of C into itself such that $\Omega = F(S) \cap VI(C, B) \neq \emptyset$. Suppose $\{x_n\}$ is a sequence generated by the following algorithm: $x_0 \in C$,*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)SP_C(x_n - \lambda_n Bx_n) \tag{3.29}$$

for all $n = 0, 1, 2, \ldots$, where $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0, 2\beta)$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\beta$,

$$\text{C1: } \lim_{n\to 0}\alpha_n = 0, \qquad \text{C2: } \sum_{n=1}^{\infty}\alpha_n = \infty,$$
$$\text{C3: } \sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty, \qquad \text{C4: } \sum_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty, \tag{3.30}$$

then $\{x_n\}$ converges strongly to $q \in \Omega$, which is the unique solution in the Ω to the following variational inequality:

$$\langle(f - I)q, p - q\rangle \leq 0 \quad \forall p \in \Omega. \tag{3.31}$$

Taking $A = I$, $\gamma = 1$ and $f \equiv u \in C$ is a constant in Theorem 3.1, we get the results of Iiduka and Takahashi [4].

Corollary 3.3 (see [5, Theorem 3.1]). *Let H be a real Hilbert space, let C be a closed convex subset of H, and let $B : C \to H$ be a β-inverse strongly monotone mapping. Let $f : C \to C$ be a contraction with coefficient α $(0 < \alpha < 1)$ and let S be a nonexpansive mapping of C into itself such that $\Omega = F(S) \cap VI(C,B) \neq \emptyset$. Suppose $\{x_n\}$ is a sequence generated by the following algorithm: $x_0, u \in C$,*

$$x_{n+1} = \alpha_n u + (1-\alpha_n) SP_C(x_n - \lambda_n B x_n) \tag{3.32}$$

for all $n = 0, 1, 2, \ldots$, where $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0, 2\beta)$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a,b]$ for some a, b with $0 < a < b < 2\beta$,

$$\begin{aligned} &\text{C1: } \lim_{n\to 0} \alpha_n = 0, && \text{C2: } \sum_{n=1}^{\infty} \alpha_n = \infty, \\ &\text{C3: } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, && \text{C4: } \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \end{aligned} \tag{3.33}$$

then $\{x_n\}$ converges strongly to $q \in \Omega$, which is the unique solution in the Ω to the following variational inequality:

$$\langle u - q, p - q \rangle \leq 0 \quad \forall p \in \Omega. \tag{3.34}$$

4. Applications

In this section, we apply the iterative scheme (1.12) for finding a common fixed point of nonexpansive mapping and strictly pseudocontractive mapping and also apply Theorem 3.1 for finding a common fixed point of nonexpansive mapping and inverse strongly monotone mapping. Recall that a mapping $T : C \to C$ is called *strictly pseudocontractive* if there exists k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \quad \forall x, y \in C. \tag{4.1}$$

If $k = 0$, then T is nonexpansive. Put $B = I - T$, where $T : C \to C$ is a strictly pseudocontractive mapping with k. Then B is $((1-k)/2)$-inverse-strongly monotone. Actually, we have, for all $x, y \in C$,

$$\|(I - B)x - (I - B)y\|^2 \leq \|x - y\|^2 + k\|Bx - By\|^2. \tag{4.2}$$

On the other hand, since H is a real Hilbert space, we have

$$\|(I - B)x - (I - B)y\|^2 = \|x - y\|^2 + \|Bx - By\|^2 - 2\langle x - y, Bx - By \rangle. \tag{4.3}$$

Hence, we have

$$\langle x - y, Bx - By \rangle \geq \frac{1-k}{2}\|Bx - By\|^2. \tag{4.4}$$

Using Theorem 3.1, we firse prove a strongly convergence theorem for finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping.

Theorem 4.1. *Let H be a real Hilbert space, let C be a closed convex subset of H, and let A be a strongly positive linear bounded operator of H into itself with coefficient $\overline{\gamma} > 0$ such that $\|A\| = 1$, so let $f : C \to C$ be a contraction with coefficient α $(0 < \alpha < 1)$. Assume that $0 < \gamma < \overline{\gamma}/\alpha$. Let S be a nonexpansive mapping of C into itself and let T be a strictly pseudocontractive mapping of C into itself with β such that $F(S) \cap F(T) \neq \emptyset$. Suppose $\{x_n\}$ is a sequence generated by the following algorithm:*

$$x_0 \in C, \qquad x_{n+1} = P_C\big(\alpha_n \gamma f(x_n) + (I - \alpha_n A) S((1 - \lambda_n)x_n - \lambda_n T x_n)\big) \tag{4.5}$$

for all $n = 0, 1, 2, \ldots$, where $\{\alpha_n\} \subset [0,1)$ and $\{\lambda_n\} \subset [0, 1 - \beta)$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 1 - \beta$,

$$\begin{aligned} &\text{C1: } \lim_{n \to 0} \alpha_n = 0, && \text{C2: } \sum_{n=1}^{\infty} \alpha_n = \infty, \\ &\text{C3: } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, && \text{C4: } \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \end{aligned} \tag{4.6}$$

then $\{x_n\}$ converges strongly to $q \in F(S) \cap F(T)$, such that

$$\big\langle (\gamma f - A)q, p - q \big\rangle \leq 0 \quad \forall p \in F(S) \cap F(T). \tag{4.7}$$

Proof. Put $B = I - T$, then B is $((1-k)/2)$-inverse-strongly monotone and $F(T) = VI(C, B)$ and $P_C(x_n - \lambda_n B x_n) = (1 - \lambda_n)x_n + \lambda_n T x_n$. So by Theorem 3.1, we obtain the desired result. □

Taking $A = I$ and $\gamma = 1$ in Theorem 4.1, we get the results of Chen et al. [5]

Corollary 4.2 (see [5, Theorem 4.1]). *Let H be a real Hilbert space and let C be a closed convex subset of H. Let $f : C \to C$ be a contraction with coefficient α $(0 < \alpha < 1)$, let S be a nonexpansive mapping of C into itself, and let T be a strictly pseudocontractive mapping of C into itself with β such that $F(S) \cap F(T) \neq \emptyset$. Suppose $\{x_n\}$ is a sequence generated by the following algorithm:*

$$x_0 \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S((1 - \lambda_n)x_n - \lambda_n T x_n) \tag{4.8}$$

for all $n = 0, 1, 2, \ldots$, where $\{\alpha_n\} \subset [0,1)$ and $\{\lambda_n\} \subset [0, 1 - \beta)$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 1 - \beta$,

$$\begin{aligned} &\text{C1: } \lim_{n \to 0} \alpha_n = 0, && \text{C2: } \sum_{n=1}^{\infty} \alpha_n = \infty, \\ &\text{C3: } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, && \text{C4: } \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \end{aligned} \tag{4.9}$$

then $\{x_n\}$ converges strongly to $q \in F(S) \cap F(T)$, such that

$$\langle (f - I)q, p - q \rangle \leq 0 \quad \forall p \in F(S) \cap F(T). \tag{4.10}$$

Theorem 4.3. *Let H be a real Hilbert space, A a strongly positive linear bounded operator of H into itself with coefficient $\overline{\gamma} > 0$ such that $\|A\| = 1$ and let $f : H \to H$ be a contraction with coefficient α $(0 < \alpha < 1)$. Assume that $0 < \gamma < \overline{\gamma}/\alpha$. Let S be a nonexpansive mapping of H into itself and B a β-inverse strongly monotone mapping of H into itself such that $F(S) \cap B^{-1}0 \neq \emptyset$. Suppose $\{x_n\}$ is a sequence generated by the following algorithm:*

$$x_0 \in H, \quad x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) S(x_n - \lambda_n B x_n) \tag{4.11}$$

for all $n = 0, 1, 2, \ldots$, where $\{\alpha_n\} \subset [0, 1)$ and $\{\lambda_n\} \subset [0, 2\beta)$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\beta$,

$$\begin{aligned} &\text{C1: } \lim_{n \to 0} \alpha_n = 0, && \text{C2: } \sum_{n=1}^{\infty} \alpha_n = \infty, \\ &\text{C3: } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, && \text{C4: } \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \end{aligned} \tag{4.12}$$

then $\{x_n\}$ converges strongly to $q \in F(S) \cap B^{-1}0$, such that

$$\langle (\gamma f - A)q, p - q \rangle \leq 0 \quad \forall p \in F(S) \cap B^{-1}0. \tag{4.13}$$

Proof. We have $B^{-1}0 = VI(H, B)$. So putting $P_H = I$, by Theorem 3.1, we obtain the desired result. □

Taking $A = I$ and $\gamma = 1$ in Theorem 4.3, we get the results of Chen et al. [5]

Corollary 4.4 (see [2, Theorem 4.2]). *Let H be a real Hilbert space. Let f be a contractive mapping of H into itself with coefficient α $(0 < \alpha < 1)$ and S a nonexpansive mapping of H into itself and B a β-inverse strongly monotone mapping of H into itself such that $F(S) \cap B^{-1}0 \neq \emptyset$. Suppose $\{x_n\}$ is a sequence generated by the following algorithm:*

$$x_0 \in H, \quad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S(x_n - \lambda_n B x_n) \tag{4.14}$$

for all $n = 0, 1, 2, \ldots$, where $\{\alpha_n\} \subset [0, 1)$ and $\{\lambda_n\} \subset [0, 2\beta)$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\beta$,

$$\begin{aligned} &\text{C1: } \lim_{n \to 0} \alpha_n = 0, && \text{C2: } \sum_{n=1}^{\infty} \alpha_n = \infty, \\ &\text{C3: } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, && \text{C4: } \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \end{aligned} \tag{4.15}$$

then $\{x_n\}$ *converges strongly to* $q \in F(S) \cap B^{-1}0$*, such that*

$$\langle (f - I)q, p - q \rangle \leq 0 \quad \forall p \in F(S) \cap B^{-1}0. \tag{4.16}$$

Remark 4.5. By taking $A = I$, $\gamma = 1$, and $f \equiv u \in C$ in Theorems 4.1 and 4.3, we can obtain Theorems 4.1 and 4.2 in [4], respectively.

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