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Research Article

Advanced Discrete Halanay-Type Inequalities: Stability of Difference Equations

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We derive new nonlinear discrete analogue of the continuous Halanay-type inequality. These inequalities can be used as basic tools in the study of the global asymptotic stability of the equilibrium of certain generalized difference equations.

1. Introduction

The investigation of stability of nonlinear difference equations with delays has attracted a lot of attention from many researchers such as Agarwal et al. [1–3], Baĭnov and Simeonov [4], Bay and Phat [5], Cooke and Ivanov [6], Gopalsamy [7], Liz et al. [8–10], Niamsup et al. [11, 12], Mohamad and Gopalsamy [13], Pinto and Trofimchuk [14], and references sited therein. In [15], Halanay proved an asymptotic formula for the solutions of a differential inequality involving the “maximum” functional and applied it in the stability theory of linear systems with delay. Such an inequality was called *Halanay inequality* in several works. Some generalizations as well as new applications can be found, for instance, in Agarwal et al. [2], Gopalsamy [7], Liz et al. [8–10], Niamsup et al. [11, 12], Mohamad and Gopalsamy [13], and Pinto and Trofimchuk [14]. In particular, in [2, 6, 10, 12, 13], the authors considered discrete Halanay-type inequalities to study some discrete version of functional differential equations.

In the following results of Liz et al. [10], authors showed that some discrete versions of these (maximum) inequalities can be applied to study the global asymptotic stability of a family of difference equations.

Theorem A. *Assume that (u, v) satisfies the system of inequalities*

$$\begin{gathered} \Delta u_n \leq -Au_n + B\tilde{u}_n + Cv_n + D\hat{v}_n, \quad n \geq 0, \\ v_n \leq Eu_n + F\tilde{u}_n, \quad n \geq 0, \end{gathered} \tag{1.1}$$

where $\Delta u_n = u_{n+1} - u_n$, $\tilde{u}_n = \max\{u_n, \ldots, u_{n-r}\}$, $\hat{v}_n = \max\{v_{n-1}, \ldots, v_{n-r}\}$, and $r > 0$ is a natural number. If $B, C, D, E, F \geq 0$, $FD + B > 0$, $E + F > 0$ and

$$B + (E + F)(C + D) < A \leq 1, \tag{1.2}$$

then there exist constants $K_1 \geq 0$, $K_2 \geq 0$, and $\lambda_0 \in (0, 1)$ such that

$$u_n \leq K_1 \lambda_0^n, \quad v_n \leq K_2 \lambda_0^n, \quad n \geq 0. \tag{1.3}$$

Moreover, λ_0 can be chosen as the smallest root in the interval $(0, 1)$ of equation $h(\lambda) = 0$, where

$$h(\lambda) = \lambda^{2r+1} - (1 - A + CE)\lambda^{2r} - (B + FC + ED)\lambda^r - FD. \tag{1.4}$$

By a simple use of Theorem A, authors also demonstrated the validity of the following statement, namely, Theorem B.

Theorem B. *Assume that f satisfies the following inequalities:*

$$\begin{gathered} \left| f(n, x_n, \ldots, x_{n-r}) \right| \leq \left\| (x_n, \ldots, x_{n-r}) \right\|_\infty, \quad \forall (x_n, \ldots, x_{n-r}) \in \mathbb{R}^{r+1}, \\ \left| f(n, x_n, \ldots, x_{n-r}) - x_n \right| \leq r \left\| (\Delta x_{n-1}, \ldots, \Delta x_{n-r}) \right\|_\infty, \quad \forall (x_n, \ldots, x_{n-r}) \in \mathbb{R}^{r+1}. \end{gathered} \tag{1.5}$$

If either

(a) $0 \leq a \leq 1 - b$, *and* $0 < br < 1$, *or*

(b) $a < 0$, *and* $0 < br < (a + b)(-a + b)^{-1}$

holds, then there exist $K > 0$ and $\lambda_0 \in (0, 1)$ such that for every solution $\{x_n\}$ of

$$\Delta x_n = -ax_n - bf(n, x_n, x_{n-1}, \ldots, x_{n-r}), \quad a > 0, \tag{1.6}$$

one has

$$\left| x_n \right| \leq \left(\max\left\{ \left| x_i \right| \right\} \right) \lambda_0^n, \quad n \geq 0, \tag{1.7}$$

where λ_0 can be calculated in the form established in Theorem A. As a consequence, the trivial solution of (1.6) is globally asymptotically stable.

The main aim of the present paper is to establish some new nonlinear retarded Halanay-type inequalities, which extend Theorem A, along with the derivation of new global stability conditions for nonlinear difference equations.

2. Halanay-Type Discrete Inequalities

Let $\mathbb{R}$ denote the set of all real numbers, $\mathbb{R}^+$ the set of positive real numbers, $\mathbb{R}^0$ the set of nonnegative real numbers, $\mathbb{Z}$ the set of integers, $\mathbb{Z}^+$ the set of positive integers, and $\mathbb{Z}^{-r} = \{z \in \mathbb{Z} : z \geq -r\}$. Consider the following nonlinear difference equation:

$$\Delta x_n = f(n, x_n, x_{n-1}, \ldots, x_{n-r}), \quad n \in \mathbb{Z}^+, \tag{2.1}$$

where $\Delta x_n = x_{n+1} - x_n$, and $f : \mathbb{N} \times \mathbb{R}^{r+1} \to \mathbb{R}$. Equation (2.1) is a generalized difference equation (see [3, Section 21] and [11]). The initial value problem for this equation requires the knowledge of the initial data $\{x_{-r}, x_{-r+1}, \ldots, x_0\}$. This vector is called the initial string in [6]. For every initial string, there exists a unique solution $\{x_n\}_{n \geq \mathbb{Z}^{-r}}$ of (2.1) that can be calculated using the explicit recurrence formula

$$x_{n+1} = x_n + f(n, x_n, x_{n-1}, \ldots, x_{n-r}), \quad n \in \mathbb{Z}^0. \tag{2.2}$$

In this section, we introduce new discrete inequalities which will be used to derive global stability conditions in the next section.

Theorem 2.1. *Let* $a_i, b_i, c_i, d_i, e_i, f_i \in \mathbb{R}_0^+$, $\sum_{i=0}^r d_i e > 0$, $h_i \in \mathbb{Z}^0$ $(i = 0, \ldots, r)$, $0 = h_0 < h_1 < \cdots < h_r$; $h_r \in \mathbb{Z}^+$, *and*

$$b + (c+d)(e+f) < a \leq 1, \tag{2.3}$$

where $a = \sum_{i=0}^r a_i$, $b = \sum_{i=0}^r b_i$, $c = \sum_{i=0}^r c_i$, $d = \sum_{i=0}^r d_i$, $e = \sum_{i=0}^r e_i$, *and* $f = \sum_{i=0}^r f_i$. *Also, let* $\{u_n, v_n\}_{n \in \mathbb{Z}^{-h_r}}$ *be a sequence of nonnegative real numbers satisfying the system of inequalities*

$$\begin{gathered}\Delta u_n \leq \sum_{i=0}^r \left(-a_i u_n + b_i u_{n-h_i}^p + c_i v_n + d_i v_{n-h_i} \right), \quad n \in \mathbb{Z}^0, \\ v_n \leq \sum_{i=0}^r \left(e_i u_n + f_i u_{n-h_i}^p \right), \quad n \in \mathbb{Z}^0,\end{gathered} \tag{2.4}$$

where $p \geq 0$ *is a constant. Then there exist constants* $K_1 \geq 0$, $K_2 \geq 0$, *and* $\lambda_0 \in (0,1)$ *such that*

$$u_n \leq K_1 \lambda_0^n, \quad v_n \leq K_2 \lambda_0^n, \quad n \in \mathbb{Z}^0, \tag{2.5}$$

where $K_1 = \max_{0 \leq i \leq r}\{u_{-h_i}, \alpha^{-1} v_{-h_i}\}$, *and* $K_2 = \alpha K_1$ *with* $\alpha = e + \sum_{i=0}^r f_i \lambda_0^{-n+(n-h_i)p}$. *Moreover,* λ_0 *can be chosen as the smallest root in the interval* $(0,1)$ *of equation* $g(\lambda) = 0$, *where*

$$\begin{aligned} g(\lambda) = \lambda - (1 - a + ce) - \sum_{i=0}^r (b_i + c f_i) \lambda^{(n-h_i)p-n} \\ - \sum_{i=0}^r d_i e \lambda^{-h_i} - \sum_{i=0}^r \left(d_i \sum_{j=0}^r f_j \lambda^{(n-h_j-h_i)p-n} \right) \end{aligned} \tag{2.6}$$

with $n \in \mathbb{Z}^0$.

Proof. Let $\{x_n, y_n\}_{n\in\mathbb{Z}^{-h_r}}$ be a sequence of nonnegative real numbers satisfying the system of inequalities

$$\begin{aligned} x_n &= (1-a)^n x_0 + \sum_{j=0}^{n-1} (1-a)^{n-j-1} \\ &\quad \times \sum_{i=0}^{r} \left(- a_i x_j + b_i x_{j-h_i}^{p} + c_i y_j + d_i y_{j-h_i} \right), \\ y_n &= \sum_{i=0}^{r} \left(e_i x_n + f_i x_{n-h_i}^{p} \right), \end{aligned} \tag{2.7}$$

where $n \in \mathbb{Z}^0$. Since $(1-a) \geq 0$, it is easy to prove by induction that if $u_n \leq x_n$ and $v_n \leq y_n$ for $n = -h_r, \ldots, 0$, then $u_n \leq x_n$ and $v_n \leq y_n$ for all $n \in \mathbb{Z}^0$.

On the other hand, the system (2.7) is equivalent to

$$\begin{aligned} \Delta x_n &= \sum_{i=0}^{r} \left(- a_i x_n + b_i x_{n-h_i}^{p} + c_i y_n + d_i y_{n-h_i} \right), \\ y_n &= \sum_{i=0}^{r} \left(e_i x_n + f_i x_{n-h_i}^{p} \right), \end{aligned} \tag{2.8}$$

where $n \in \mathbb{Z}^0$. Next we prove, under the assumptions of the theorem, that there exists a solution $\{x_n, y_n\}_{n\in\mathbb{Z}^{-h_r}}$ to system (2.8) in the form $x_n = \lambda_0^n$, $y_n = \alpha\lambda_0^n$ with $\alpha > 0$, $\lambda_0 \in (0,1)$. Indeed, such $\{x_n, y_n\}_{n\in\mathbb{Z}^{-h_r}}$ is a solution of (2.8) if and only if

$$\begin{aligned} \lambda_0^{n+1} &= (1-a)\lambda_0^n + \sum_{i=0}^{r} \left(b_i \lambda_0^{(n-h_i)p} + c_i \alpha \lambda_0^n + d_i \left(\alpha \lambda_0^{n-h_i} \right) \right), \quad n \in \mathbb{Z}^0, \\ \alpha\lambda_0^n &= \sum_{i=0}^{r} \left(e_i \lambda_0^n + f_i \lambda_0^{(n-h_i)p} \right), \quad n \in \mathbb{Z}^0. \end{aligned} \tag{2.9}$$

This is equivalent to the existence of a solution $\lambda_0 \in (0,1)$ of equation $g(\lambda) = 0$, where g is the polynomial defined by (2.6).

Now, $g(0) = \lim_{\lambda\to 0^+} g(\lambda) = -\infty < 0$ in view of $\sum_{i=0}^{r} d_i e > 0$. On the other hand, $g(1) = a - b - (c+d)(e+f) > 0$ in view of (2.3). As a consequence, there exists $\lambda_0 \in (0,1)$ such that $g(\lambda_0) = 0$. Hence, (λ_0, α) is a solution of (2.9) with $\alpha = e + \sum_{i=0}^{r} f_i \lambda_0^{-n+(n-h_i)p} > 0$.

For this value of λ_0, the pair $\{K\lambda_0^n, K\alpha\lambda_0^n\}$ is a solution of (2.8) for every $K \geq 0$. Thus, choosing $K = \max_{0\leq i\leq r}\{u_{-h_i}, \alpha^{-1} v_{-h_i}\}$, we have that $u_n \leq K\lambda_0^n = x_n$, and $v_n \leq K\alpha\lambda_0^n = y_n$ for all $n = -h_r, \ldots, 0$.

Hence, using the first part of the proof, we can conclude that $u_n \leq x_n$, and $v_n \leq y_n$ for all $n \in \mathbb{Z}_0$. □

By the similar argument used in Theorem 2.1, we obtain the following result.

Theorem 2.2. *Let* $a, b, c, d, e, f \in \mathbb{R}_0^+$, $h_i \in \mathbb{Z}^0$, $i = 0, \ldots, r$, $0 = h_0 < h_1 < \cdots < h_r$; $r \geq 1$, *and*

$$b + c(e + f) + d(e + f)^{r+1} < a \leq 1 \tag{2.10}$$

with $ce > 0$. *Also, let* $\{u_n, v_n\}_{n \in \mathbb{Z}^{-h_r}}$ *be a sequence of nonnegative real numbers satisfying the system of inequalities*

$$\begin{gathered}
\Delta u_n \leq -a u_n + \prod_{i=0}^{r} b u_{n-h_i} + c v_n + \prod_{i=0}^{r} d v_{n-h_i}, \quad n \in \mathbb{Z}^0, \\
v_n \leq e u_n + \prod_{i=0}^{r} f u_{n-h_i}, \quad n \in \mathbb{Z}^0.
\end{gathered} \tag{2.11}$$

Then there exist constants $K_1 \geq 0$, $K_2 \geq 0$, *and* $\lambda_0 \in (0,1)$ *such that*

$$u_n \leq K_1 \lambda_0^n, \quad v_n \leq K_2 \lambda_0^n, \quad n \in \mathbb{Z}^0, \tag{2.12}$$

where $K_1 = \max_{0 \leq i \leq r}\{u_{-h_i}, \rho^{-1} v_{-h_i}\}$, *and* $K_2 = \rho K_1$ *with* $\rho = e + f \prod_{i=0}^{r} \lambda_0^{-h_i}$. *Moreover,* λ_0 *can be chosen as the smallest root in the interval* $(0,1)$ *of equation* $F(\lambda) = 0$, *where*

$$F(\lambda) = \lambda - (1 - a + ce) - \left[b + cf + d\left(e + f\lambda^{rn-h}\right)^{r+1}\right]\lambda^{rn-h} \tag{2.13}$$

with $n \in \mathbb{Z}^0$, $h = \sum_{i=0}^{r} h_i$.

Proof. Let $\{x_n, y_n\}_{n \in \mathbb{Z}^{-h_r}}$ be a sequence of nonnegative real numbers satisfying the system of inequalities

$$\begin{gathered}
\Delta x_n = -a x_n + b \prod_{i=0}^{r} x_{n-h_i} + c y_n + d \prod_{i=0}^{r} y_{n-h_i}, \quad n \in \mathbb{Z}^0, \\
y_n = e x_n + f \prod_{i=0}^{r} x_{n-h_i}, \quad n \in \mathbb{Z}^0.
\end{gathered} \tag{2.14}$$

Since $(1 - a) \geq 0$, it is easy to prove by induction that if $u_n \leq x_n$ and $v_n \leq y_n$ for $n = -h_r, \ldots, 0$, then $u_n \leq x_n$ and $v_n \leq y_n$ for all $n \in \mathbb{Z}^0$.

Next we prove that, under the assumptions of the theorem, there exists a solution $\{x_n, y_n\}_{n \in \mathbb{Z}^{-h_r}}$ to system (2.14) in the form $x_n = \lambda^n$, $y_n = \rho\lambda^n$ with $\rho > 0$, $\lambda \in (0,1)$. Indeed, such $\{x_n, y_n\}_{n \in \mathbb{Z}^{-h_r}}$ is a solution of (2.14) if and only if

$$\begin{gathered}
\lambda^{n+1} = (1 - a)\lambda^n + b \prod_{i=0}^{r} \lambda^{n-h_i} + c\rho\lambda^n + d \prod_{i=0}^{r} \rho\lambda^{n-h_i}, \quad n \in \mathbb{Z}^0, \\
\rho\lambda^n = e\lambda^n + f \prod_{i=0}^{r} \lambda^{n-h_i}, \quad n \in \mathbb{Z}^0.
\end{gathered} \tag{2.15}$$

This is equivalent to the existence of a solution $\lambda \in (0,1)$ of equation $F(\lambda) = 0$, where F is the polynomial defined in (2.13).

Now, in view of $ce > 0$, we have $F(0) = -1 + a - ac < 0$ in case $rn > h$, $F(0) = -1 + a - ac - [b + cf + d(e+f)^{r+1}] < 0$ in case $rn = h$, and $F(0) = \lim_{\lambda \to 0^+} F(\lambda) = -\infty < 0$ in case $rn < h$.

On the other hand, $F(1) = a - b - c(e+f) - d(e+f)^{r+1} > 0$ in view of (2.10). As a consequence, there exists $\lambda_0 \in (0,1)$ such that $F(\lambda_0) = 0$. Hence, (λ_0, ρ) is a solution of (2.15) with $\rho = e + f\prod_{i=0}^{r} \lambda_0^{-h_i} > 0$.

For this value of λ_0, the pair $\{K\lambda_0^n, K\rho\lambda_0^n\}$ is a solution of (2.14) for every $K \geq 0$. Thus, choosing $K = \max_{0 \leq i \leq r}\{u_{-h_i}, \rho^{-1}v_{-h_i}\}$, we have $u_n \leq K\lambda_0^n$, and $v_n \leq K\rho\lambda_0^n$ for all $n = -h_r, \ldots, 0$. These imply $u_n \leq x_n$, and $v_n \leq y_n$ for all $n = -h_r, \ldots, 0$. Hence, using the first part of the proof, we can conclude that $u_n \leq x_n$, and $v_n \leq y_n$ for all $n \in \mathbb{Z}_0$. □

Remark 2.3. In [10], a discrete Halanay-type inequality was given as in Theorem A, where the inequalities were replaced by

$$\begin{gathered} \Delta u_n \leq -au_n + b\tilde{u}_n + cv_n + d\hat{v}_n, \quad n \geq 0, \\ v_n \leq eu_n + f\tilde{u}_n, \quad n \geq 0, \end{gathered} \tag{2.16}$$

where $\Delta u_n = u_{n+1} - u_n$, $\tilde{u}_n = \max\{u_n, \ldots, u_{n-r}\}$, $\hat{v}_n = \max\{v_{n-1}, \ldots, v_{n-r}\}$, and $r \geq 1$ is a natural number. Note that if a sequence $\{u_n\}_{n \in \mathbb{Z}^{-r}}$ of positive real numbers satisfies (2.16), then it also satisfies (2.4). On the other hand, let $p = r = 1$, $h_i = i$; $a = \sum_{i=0}^{1} a_i = 1$, $b = b_0 = b_1 = 1/7$, $c = c_0 = c_1 = 0$, $d = d_0 = d_1 = 1/7$, $e = e_0 = e_1 = 1/7$, and $f = f_0 = f_1 = 1/7$. Then we might easily show that the sequence $\{1/2^n\}_{n \in \mathbb{Z}^{-1}}$ satisfies (2.4) but not (2.16). Indeed,

$$\begin{aligned} \Delta u_n &= \frac{1}{2^{n+1}} - \frac{1}{2^n} \\ &= -\frac{1}{2^{n+1}}, \\ &< -\frac{1}{2^n} + \frac{1}{7}\left(\frac{1}{2^n} + \frac{1}{2^{n-1}}\right) + \frac{1}{7}\left(\frac{1}{2^n}\frac{5}{7} + \frac{1}{2^{n-1}}\frac{5}{7}\right) \\ &= -\frac{13}{49}\frac{1}{2^n}, \end{aligned} \tag{2.17}$$

with $\sum_{i=0}^{1} b_i + (\sum_{i=0}^{1} c_i + \sum_{i=0}^{1} d_i)(\sum_{i=0}^{1} e_i + \sum_{i=0}^{1} f_i) < \sum_{i=0}^{1} a_i = 1$. On the other hand,

$$\begin{aligned} \Delta u_n &= -\frac{1}{2^{n+1}} \\ &> -\frac{1}{2^n} + \frac{1}{7}\max\left\{\frac{1}{2^n}, \frac{1}{2^{n-1}}\right\} + \frac{1}{7}\left(\frac{1}{7}\frac{1}{2^{n-1}} + \frac{1}{7}\max\left\{\frac{1}{2^{n-1}}\right\}\right) \\ &= -\frac{31}{49}\frac{1}{2^n}. \end{aligned} \tag{2.18}$$

Therefore, in the case of positive sequences, the discrete inequality (2.4) is less conservative than the discrete Halanay-type inequality given by (2.16).

3. Global Stability of Difference Equations

In order to show the applicability of the previous result, in this section we consider the generalized difference equation

$$\Delta x_n = -ax_n - bf(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}), \tag{3.1}$$

where $n, h_i \in \mathbb{Z}^+,\ i = 1, \dots, r,$ and $b > 0$.

Although, for every initial string $\{x_{-h_r}, x_{-h_r+1}, \dots, x_0\}$, the solution $\{x_n\}$ of (3.1) can be explicitly calculated by a recurrence formula similar to (2.2), it is in general difficult to investigate the asymptotic behavior of the solutions using that formula. The next result gives an asymptotic estimate by a simple use of the discrete Halanay inequality.

Theorem 3.1. *For all* $(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) \in \mathbb{Z}^0 \times \mathbb{R}^{r+1}$, *assume that* f *satisfies the following inequalities:*

$$\left|f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r})\right| \le \sum_{j=0}^{r} \beta_j \left|x_{n-h_j}\right|^p, \tag{3.2}$$

$$\left|f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) - x_n\right| \le \sum_{j=0}^{r} \gamma_j \left|\Delta x_{n-h_j}\right|, \tag{3.3}$$

where $\beta_j, \gamma_j, p \in \mathbb{R}_0^+,\ \sum_{i=0}^{r} \gamma_i |a| > 0,\ h_j \in \mathbb{Z}^0\ (j = 0, \dots, r-1)$, *and* $h_r \in \mathbb{Z}^+$ *with* $0 = h_0 < h_1 < \cdots < h_r$. *If either*

(a) $0 \le a \le 1 - b,\ 0 < b\gamma < 1$, *and* $0 < \beta \le 1$, *or*

(b) $a < 0$ *and* $0 < b\gamma < (a+b)(-a+b\beta)^{-1}$

hold, then there exists a constant $\lambda_0 \in (0,1)$ *for every solution* $\{x_n\}$ *of* (3.1) *such that*

$$|x_n| \le \left(\max_{-h_r \le i \le 0}\left\{|x_i|, \alpha_1^{-1}|\Delta x_i|\right\}\right)\lambda_0^n, \quad n \in \mathbb{Z}^0, \tag{3.4}$$

where $\alpha_1 = |a| + b\sum_{i=0}^{r} \beta_i \lambda_0^{-n+(n-h_i)p}$, $\beta = \sum_{i=0}^{r} \beta_i$, $\gamma = \sum_{i=0}^{r} \gamma_i$, *and* λ_0 *can be chosen as the smallest root in the interval* $(0,1)$ *of equation* $g_1(\lambda) = 0$, *where*

$$g_1(\lambda) = \lambda - (1 - a - b) - \sum_{i=0}^{r} b|a|\gamma_i \lambda^{-h_i} - \sum_{i=0}^{r} b\gamma_i \left(\sum_{j=0}^{r} b\beta_j \lambda^{(n-h_j-h_i)p-n}\right) \tag{3.5}$$

with $n \in \mathbb{Z}^0$.

As a consequence, the trivial solution of (3.1) *is globally asymptotically stable.*

Proof. Let $\{x_n\}$ be a solution of (3.1). Equation (3.1) can be written in the form

$$\Delta x_n = -(a+b)x_n - b\left[f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) - x_n\right]. \tag{3.6}$$

Hence, we know that

$$x_n = \left[1 - (a+b)\right]^n x_0 + \sum_{i=0}^{n-1} \left[1 - (a+b)\right]^{n-i-1} (-b) \left[f\left(i, x_i, x_{i-h_1}, \ldots, x_{i-h_r}\right) - x_i\right], \tag{3.7}$$

where $n \in \mathbb{Z}^0$. Thus, using inequality (3.3), we obtain

$$\left|x_n\right| \le \left[1 - (a+b)\right]^n \left|x_0\right| + \sum_{i=0}^{n-1} \sum_{j=0}^{r} \left[1 - (a+b)\right]^{n-i-1} b\gamma_j \left|\Delta x_{i-h_j}\right|. \tag{3.8}$$

Denote $u_n = |x_n|$ for $n = -h_r, \ldots, 0$, and

$$u_n = \left[1 - (a+b)\right]^n \left|x_0\right| + \sum_{i=0}^{n-1} \sum_{j=0}^{r} \left[1 - (a+b)\right]^{n-i-1} b\gamma_j \left|\Delta x_{i-h_j}\right| \tag{3.9}$$

for $n \in \mathbb{Z}^+$. Then we have $|x_n| \le u_n$ and, from inequality (3.9), we obtain

$$\Delta u_n = -(a+b) u_n + \sum_{j=0}^{r} b\gamma_j \left|\Delta x_{n-h_j}\right| \tag{3.10}$$

for $n \in \mathbb{Z}^+$. On the other hand, using hypothesis (3.2) in (3.1), we have

$$\begin{aligned} \left|\Delta x_n\right| &\le |-a| \left|x_n\right| + b \sum_{j=0}^{r} \beta_j \left|x_{n-h_j}\right|^p \\ &\le |a| u_n + b \sum_{j=0}^{r} \beta_j u_{n-h_j}^p. \end{aligned} \tag{3.11}$$

Denote $v_n = |\Delta x_n|$. We can apply Theorem 2.1 to the system of inequalities (3.10) and (3.11) with $\sum_{i=0}^{r} a_i = a + b$, $b_i = 0$, $c_i = 0$, $d_i = b\gamma_j$, $\sum_{i=0}^{r} e_i = |a|$, and $f_i = b\beta_j$. Consequently, Theorem 2.1 ensures the validity of the following inequality:

$$\left|x_n\right| \le \left(\max_{-h_r \le i \le 0} \left\{\left|x_i\right|, \alpha_1^{-1} \left|\Delta x_i\right|\right\}\right) \lambda_0^n, \quad n \in \mathbb{Z}^0, \tag{3.12}$$

where λ_0 and α_1 are chosen as in Theorem 3.1. This completes the proof of the theorem. □

Next, we obtain new conditions for the asymptotic stability of (3.1) using inequality (3.13) instead of (3.3).

Corollary 3.2. *For all* $(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) \in \mathbb{Z}^0 \times \mathbb{R}^{r+1}$, *assume that* f *satisfies inequality* (3.2) *and the following condition:*

$$\left|f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) - x_n\right| \le \sum_{j=0}^{r} \left(\gamma_j \left|x_{n-h_j}\right|^p + \delta_j \left|\Delta x_n\right| + \eta_j \left|\Delta x_{n-h_j}\right|\right), \tag{3.13}$$

where $\gamma_j, \delta_j, \eta_j, p \in \mathbb{R}_0^+$, $\sum_{j=0}^{r} \eta_j |a| > 0$, $h_j \in \mathbb{Z}^0$ $(j = 0, \dots, r-1)$, *and* $h_r \in \mathbb{Z}^+$ *with* $0 = h_0 < h_1 < \cdots < h_r$. *If*

$$b\gamma + b(\delta + \eta)(|a| + b\beta) < a + b \le 1 \tag{3.14}$$

holds, where $\beta = \sum_{i=0}^{r} \beta_i$, $\gamma = \sum_{i=0}^{r} \gamma_i$, $\delta = \sum_{i=0}^{r} \delta_i$, *and* $\eta = \sum_{i=0}^{r} \eta_i$, *then there exists a constant* $\lambda_0 \in (0,1)$ *for every solution* $\{x_n\}$ *of* (3.1) *such that*

$$\left|x_n\right| \le \left(\max_{-h_r \le i \le 0} \left\{\left|x_i\right|, \alpha_1^{-1} \left|\Delta x_i\right|\right\}\right) \lambda_0^n, \quad n \in \mathbb{Z}^0, \tag{3.15}$$

where $\alpha_1 = |a| + b\sum_{i=0}^{r} \beta_i \lambda_0^{-n+(n-h_i)p}$, *and* λ_0 *can be chosen as the smallest root in the interval* $(0,1)$ *of equation* $g_2(\lambda) = 0$, *where*

$$\begin{aligned} g_2(\lambda) = \lambda - \left(1 - a - b + b|a|\delta\right) - \sum_{i=0}^{r} b\left(\gamma_i + b\delta\beta_i\right)\lambda^{(n-h_i)p-n} \\ - \sum_{i=0}^{r} b|a|\eta_i \lambda^{-h_i} - \sum_{i=0}^{r} b\eta_i \left(\sum_{j=0}^{r} b\beta_j \lambda^{(n-h_j-h_i)p-n}\right) \end{aligned} \tag{3.16}$$

with $n \in \mathbb{Z}^0$.

As a consequence, the trivial solution of (3.1) *is globally asymptotically stable.*

Similarly, using Theorem 2.2 instead of Theorem 2.1, we obtain the following result.

Theorem 3.3. *For all* $(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) \in \mathbb{Z}^0 \times \mathbb{R}^{r+1}$, *assume that* f *satisfies the following inequalities:*

$$\begin{aligned} \left|f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r})\right| &\le \beta \prod_{j=0}^{r} \left|x_{n-h_j}\right|, \\ \left|f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) - x_n\right| &\le \gamma \prod_{j=0}^{r} \left|x_{n-h_j}\right| + \delta \left|\Delta x_n\right| + \eta \prod_{j=0}^{r} \left|\Delta x_{n-h_j}\right|, \end{aligned} \tag{3.17}$$

where $\beta, \gamma, \delta, \eta \in \mathbb{R}_0^+$, $h_j \in \mathbb{Z}^0$, $j = 0, \dots, r-1$, *and* $h_r \in \mathbb{Z}^+$ *with* $0 = h_0 < h_1 < \cdots < h_r$. *If* $|a|\delta > 0$ *and*

$$b\gamma + b\delta\left(|a| + b\beta\right) + b\eta\left(|a| + b\beta\right)^{r+1} < a + b \le 1, \tag{3.18}$$

then there exists a constant $\lambda_0 \in (0,1)$ *for every solution* $\{x_n\}$ *of* (3.1) *such that*

$$|x_n| \leq \left(\max_{-h_r \leq i \leq 0}\{|x_i|, \rho_1^{-1}|\Delta x_i|\}\right)\lambda_0^n, \quad n \in \mathbb{Z}^0, \tag{3.19}$$

where $\rho_1 = |a| + b\beta\prod_{i=0}^{r}\lambda_0^{h_i}$, *and* λ_0 *can be chosen as the smallest root in the interval* $(0,1)$ *of equation* $F_1(\lambda) = 0$, *where*

$$F_1(\lambda) = \lambda - (1 - (a+b) + |a|b\delta) - b\left[\gamma + b\beta\delta + \eta\left(|a| + b\beta\lambda^{rn-h}\right)^{r+1}\right]\lambda^{rn-h} \tag{3.20}$$

with $n \in \mathbb{Z}^0$, $h = \sum_{i=0}^{r} h_i$.

As a consequence, the trivial solution of (3.1) *is globally asymptotically stable.*

Remark 3.4. Equation (3.1) covers a variety of difference equations. For instance, we can consider the following difference equation:

$$\Delta x_n = -a x_n - b f(x_{n-k}), \quad b > 0. \tag{3.21}$$

Next, we study the asymptotic behavior of the solutions of (3.21). We can apply Theorem 3.1, Corollary 3.2, or Theorem 3.3 to obtain some relations between coefficients a and b that ensure the global asymptotic stability of the zero solution. Moreover, from Theorem 3.1 we know that if there exists $\beta, \gamma \in \mathbb{R}^+$ such that $|f(x)| \leq \beta|x|^p$, $|f(x) - x| \leq \gamma|\Delta x|$ for all x, and if either

(a) $0 < a \leq 1 - b$, $0 < b\gamma < 1$, and $0 < \beta \leq 1$, or

(b) $a < 0$ and $0 < b\gamma < (a+b)(-a+b\beta)^{-1}$

hold, then all solutions of (3.21) converge to zero.

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