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Research Article

An Improved Hardy-Rellich Inequality with Optimal Constant

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We show that a Hardy-Rellich inequality with optimal constants on a bounded domain can be refined by adding remainder terms. The procedure is based on decomposition into spherical harmonics.

1. Introduction

Hardy inequality in $\mathbb{R}^N$ reads, for all $u \in C_0^\infty(\mathbb{R}^N)$ and $N \geq 3$,

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx, \tag{1.1}$$

and $(N-2)^2/4$ is the best constant in (1.1) and is never achieved. A similar inequality with the same best constant holds if $\mathbb{R}^N$ is replaced by an arbitrary domain $\Omega \subset \mathbb{R}^N$ and Ω contains the origin. Moreover, Brezis and Vázquez [1] have improved it by establishing that for $u \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx + \Lambda(-\Delta, 2) \left(\frac{\omega_N}{|\Omega|} \right)^{2/N} \int_\Omega u^2 dx, \tag{1.2}$$

where ω_N and $|\Omega|$ denote the volume of the unit ball B_1 and Ω, respectively, and $\Lambda(-\Delta, 2)$ is the first eigenvalue of the Dirichlet Laplacian of the unit disc in $\mathbb{R}^2$. In case Ω is a ball centered at zero, the constant $\Lambda(-\Delta, 2)$ in (1.2) is sharp.

Similar improved inequalities have been recently proved if instead of (1.1) one considers the corresponding L^p Hardy inequalities. In all these cases a correction term is added on the right-hand side (see, e.g., [2–4]).

On the other hand, the classical Rellich inequality states that, for $N \geq 5$,

$$\int_{\mathbb{R}^N} |\Delta u|^2 dx \geq \left(\frac{N(N-4)}{4}\right)^2 \int_{\mathbb{R}^N} \frac{u^2}{|x|^4} dx, \quad u \in C_0^\infty\left(\mathbb{R}^N\right), \tag{1.3}$$

and $(N(N-4)/4)^2$ is the best constant in (1.3) and is never achieved (see [5]). And, more recently, Tertikas and Zographopoulos [6] obtained a stronger version of Rellich's inequality. That is, for all $u \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |\Delta u|^2 dx \geq \frac{N^2}{4} \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^2} dx, \quad N \geq 5. \tag{1.4}$$

Both inequalities are valid when $\mathbb{R}^N$ is replaced by a bounded domain $\Omega \subset \mathbb{R}^N$ containing the origin and the corresponding constants are known to be optimal. Recently, Gazzola et al. [4] have improved (1.3) by establishing that for $\Omega \subset B_R(0)$ and $u \in C_0^\infty(\Omega)$,

$$\begin{aligned}\int_{\Omega} |\Delta u|^2 dx \geq \left(\frac{N(N-4)}{4}\right)^2 \int_{\Omega} \frac{u^2}{|x|^4} dx + \frac{N(N-4)}{2} \Lambda(-\Delta, 2) R^{-2} \int_{\Omega} \frac{u^2}{|x|^2} dx \\ + \Lambda\left((-\Delta)^2, 4\right) R^{-4} \int_{\Omega} u^2 dx,\end{aligned} \tag{1.5}$$

where

$$\Lambda\left((-\Delta)^2, 4\right) = \inf_{u \in W^{2,2}(B_1^{(4)}) \setminus \{0\}} \frac{\int_{B_1^{(4)}} (\Delta u)^2 dx}{\int_{B_1^{(4)}} u^2 dx}, \tag{1.6}$$

and $B_1^{(4)}$ is the unit ball in $\mathbb{R}^4$. Our main concern in this note is to improve (1.4). In fact we have the following theorem.

Theorem 1.1. *There holds, for $N \geq 5$ and $u \in C_0^\infty(\Omega)$,*

$$\int_{\Omega} |\Delta u|^2 dx \geq \frac{N^2}{4} \int_{\Omega} \frac{|\nabla u|^2}{|x|^2} dx + \Lambda(-\Delta, 2) \left(\frac{\omega_N}{|\Omega|}\right)^{2/N} \int_{\Omega} |\nabla u|^2 dx. \tag{1.7}$$

Inequality (1.7) *is optimal in case Ω is a ball centered at zero.*

Combining Theorem 1.1 with (1.2), we have the following.

Corollary 1.2. *There holds, for $N \geq 5$ and $u \in C_0^\infty(\Omega)$,*

$$\int_\Omega |\Delta u|^2 dx \geq \frac{N^2}{4}\int_\Omega \frac{|\nabla u|^2}{|x|^2}dx + \frac{(N-2)^2}{4}\Lambda(-\Delta,2)\left(\frac{\omega_N}{|\Omega|}\right)^{2/N}\int_\Omega \frac{u^2}{|x|^2}dx + \Lambda(-\Delta,2)^2\left(\frac{\omega_N}{|\Omega|}\right)^{4/N}\int_\Omega u^2 dx. \tag{1.8}$$

Next we consider analogous inequality (1.5). The main result is the following theorem.

Theorem 1.3. *Let $N \geq 8$ and let $\Omega \subset \mathbb{R}^N$ be such that $\Omega \subset B_R(0)$. Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega |\Delta u|^2 dx \geq \frac{N^2}{4}\int_\Omega \frac{|\nabla u|^2}{|x|^2}dx + \frac{N(N-8)}{4}\Lambda(-\Delta,2)R^{-2}\int_\Omega \frac{u^2}{|x|^2}dx + \Lambda\left((-\Delta)^2,4\right)R^{-4}\int_\Omega u^2 dx. \tag{1.9}$$

Remark 1.4. Since

$$\int_\Omega \frac{|\nabla u|^2}{|x|^2}dx \geq \frac{(N-4)^2}{4}\int_\Omega \frac{u^2}{|x|^4}dx + \Lambda(-\Delta,2)\left(\frac{\omega_N}{|\Omega|}\right)^{2/N}\int_\Omega \frac{u^2}{|x|^2}dx, \quad N \geq 5, \tag{1.10}$$

inequality (1.5) is implied by (1.9) in case of $N \geq 8$.

2. The Proofs

To prove the main results, we first need the following preliminary result.

Lemma 2.1. *Let $N \geq 5$ and $u \in C_0^\infty(\mathbb{R}^N)$. Set $r = |x|$. If $u(x)$ is a radial function, that is, $u(x) = u(r)$, then*

$$\int_{\mathbb{R}^N} |\Delta u|^2 dx - \frac{N^2}{4}\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^2}dx = \int_{\mathbb{R}^N} |\nabla u_r|^2 dx - \frac{(N-2)^2}{4}\int_{\mathbb{R}^N} \frac{u_r^2}{|x|^2}dx. \tag{2.1}$$

Proof. Observe that if $u(x) = u(r)$, then

$$|\nabla u| = |u_r|, \qquad \Delta u = \frac{d^2u}{dr^2} + \frac{N-1}{r}\cdot\frac{du}{dr}. \tag{2.2}$$

Therefore, we have

$$\int_{\mathbb{R}^N} |\Delta u|^2 dx = \int_{\mathbb{R}^N} \left| u_{rr} + \frac{N-1}{r} u_r \right|^2 dx$$
$$= \int_{\mathbb{R}^N} u_{rr}^2 dx + (N-1)^2 \int_{\mathbb{R}^N} \frac{u_r^2}{r^2} dx + 2(N-1) \int_{\mathbb{R}^N} \frac{u_{rr} u_r}{r} dx \tag{2.3}$$
$$= \int_{\mathbb{R}^N} u_{rr}^2 dx + (N-1)^2 \int_{\mathbb{R}^N} \frac{u_r^2}{r^2} dx + (N-1) \int_{\mathbb{R}^N} \frac{1}{r} \cdot \frac{d(u_r^2)}{dr} dx.$$

Though integration by parts, when $n \geq 3$,

$$\int_{\mathbb{R}^N} \frac{1}{r} \cdot \frac{d(u_r^2)}{dr} dx = \int_{S^{N-1}} d\sigma \int_0^\infty r^{N-2} \cdot \frac{d(u_r^2)}{dr} dr = -(N-2) \int_{\mathbb{R}^N} \frac{u_r^2}{r^2} dx, \tag{2.4}$$

and hence

$$\int_{\mathbb{R}^N} |\Delta u|^2 dx - \frac{N^2}{4} \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^2} dx = \int_{\mathbb{R}^N} u_{rr}^2 dx - \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{u_r^2}{r^2} dx$$
$$= \int_{\mathbb{R}^N} |\nabla u_r|^2 dx - \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{u_r^2}{|x|^2} dx. \tag{2.5}$$

□

By Lemma 2.1 and inequality (1.2), we have, when restricted to radial functions,

$$\int_\Omega |\Delta u|^2 dx - \frac{N^2}{4} \int_\Omega \frac{|\nabla u|^2}{|x|^2} dx \geq \Lambda(-\Delta, 2) \left(\frac{\omega_N}{|\Omega|} \right)^{2/N} \int_\Omega |\nabla u|^2 dx. \tag{2.6}$$

Our next step is to prove the following. If $u(x)$ is not a radial function, inequality (2.6) also holds.

Let $u \in C_0^\infty(\Omega)$. If we extend u as zero outside Ω, we may consider $u \in C_0^\infty(\mathbb{R}^N)$. Decomposing u into spherical harmonics we get

$$u = \sum_{k=0}^{\infty} u_k := \sum_{k=0}^{\infty} f_k(r) \phi_k(\sigma), \tag{2.7}$$

where $\phi_k(\sigma)$ are the orthonormal eigenfunctions of the Laplace-Beltrami operator with responding eigenvalues

$$c_k = k(N+k-2), \quad k \geq 0. \tag{2.8}$$

The functions $f_k(r)$ belong to $C_0^\infty(\Omega)$, satisfying $f_k(r) = O(r^k)$ and $f_k'(r) = O(r^{k-1})$ as $r \to 0$. In particular, $\phi_0(\sigma) = 1$ and $u_0(r) = (1/|\partial B_r|)\int_{\partial B_r} u\, d\sigma$, for any $r > 0$. Then, for any $k \in \mathbb{N}$, we have

$$\Delta u_k = \left(\Delta f_k(r) - \frac{c_k}{r^2} f_k(r)\right)\phi_k(\sigma). \tag{2.9}$$

So

$$\begin{aligned}\int_{\mathbb{R}^N} |\Delta u_k|^2 dx &= \int_{\mathbb{R}^N} \left(\Delta f_k(r) - \frac{c_k}{r^2} f_k(r)\right)^2 dx,\\ \int_{\mathbb{R}^N} |\nabla u_k|^2 dx &= \int_{\mathbb{R}^N} \left(\left|\nabla f_k(r)\right|^2 + \frac{c_k}{r^2} f_k^2(r)\right) dx.\end{aligned} \tag{2.10}$$

In addition,

$$\begin{aligned}\int_{\mathbb{R}^N} |\Delta u|^2 dx &= \sum_{k=0}^{\infty} \int_{\mathbb{R}^N} |\Delta u_k|^2 dx = \sum_{k=0}^{\infty} \int_{\mathbb{R}^N} \left(\Delta f_k(r) - \frac{c_k}{r^2} f_k(r)\right)^2 dx,\\ \int_{\mathbb{R}^N} |\nabla u|^2 dx &= \sum_{k=0}^{\infty} \int_{\mathbb{R}^N} |\nabla u_k|^2 dx = \sum_{k=0}^{\infty} \int_{\mathbb{R}^N} \left(\left|\nabla f_k(r)\right|^2 + \frac{c_k}{r^2} f_k^2(r)\right) dx.\end{aligned} \tag{2.11}$$

Using equality (2.10), we have that (see, e.g., [6, page 452])

$$\begin{aligned}\int_{\mathbb{R}^N} |\Delta u_k|^2 dx &= \int_{\mathbb{R}^N} (f_k'')^2 dx + (N - 1 + 2c_k) \int_{\mathbb{R}^N} r^{-2} (f_k')^2 dx\\ &\quad + c_k [c_k + 2(N-4)] \int_{\mathbb{R}^N} r^{-4} f_k^2 dx,\\ \int_{\mathbb{R}^N} \frac{|\nabla u_k|^2}{|x|^2} dx &= \int_{\mathbb{R}^N} \frac{\left|\nabla f_k(r)\right|^2}{r^2} dx + c_k \int_{\mathbb{R}^N} \frac{f_k^2(r)}{r^4} dx.\end{aligned} \tag{2.12}$$

Therefore, we have that, by (2.12),

$$\begin{aligned}&\int_{\mathbb{R}^N} |\Delta u_k|^2 dx - \frac{N^2}{4} \int_{\mathbb{R}^N} \frac{|\nabla u_k|^2}{|x|^2} dx\\ &\quad = \int_{\mathbb{R}^N} (f_k'')^2 dx - \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{(f_k')^2}{r^2} dx\\ &\qquad + c_k \left[2 \int_{\mathbb{R}^N} \frac{(f_k')^2}{r^2} dx + \left(c_k - \frac{N^2 - 8N + 32}{4}\right) \int_{\mathbb{R}^N} \frac{(f_k)^2}{r^4} dx \right].\end{aligned} \tag{2.13}$$

Lemma 2.2. *There holds, for $N \geq 4$ and $k \geq 1$,*

$$2\int_\Omega \frac{(f_k')^2}{r^2}dx + \left(c_k - \frac{N^2-8N+32}{4}\right)\int_\Omega \frac{(f_k)^2}{r^4}dx \geq 2\Lambda(-\Delta,2)\left(\frac{\omega_N}{|\Omega|}\right)^{2/N}\int_\Omega \frac{(f_k)^2}{r^2}dx. \tag{2.14}$$

Proof. Set $g_k = f_k/r$. Then g_k satisfies $g_k(r) = O(r^{k-1})$ and $g_k'(r) = O(r^{k-2})$ as $r \to 0$. Moreover, since $f_k(r)$ belong to $C_0^\infty(\Omega)$, we have that

$$\begin{aligned}\int_\Omega (g_k')^2 dx &= \int_\Omega \frac{(f_k')^2}{r^2}dx - 2\int_\Omega \frac{f_k' f_k}{r^3}dx + \int_\Omega \frac{f_k^2}{r^4}dx\\ &= \int_\Omega \frac{(f_k')^2}{r^2}dx + (N-3)\int_\Omega \frac{f_k^2}{r^4}dx\\ &= \int_\Omega \frac{(f_k')^2}{r^2}dx + (N-3)\int_\Omega \frac{g_k^2}{r^2}dx.\end{aligned} \tag{2.15}$$

Here we use the fact when $N \geq 4$ and $k \geq 1$,

$$2\int_\Omega \frac{f_k' f_k}{r^3}dx = \int_{S^{N-1}} d\sigma \int_0^\infty r^{N-4}\cdot\frac{d(f_k^2)}{dr}dr = -(N-4)\int_\Omega \frac{f_k^2}{r^4}dx. \tag{2.16}$$

Using inequalities (1.2) and (2.15), we have that, for $N \geq 4$ and $k \geq 1$,

$$\begin{aligned}&2\int_\Omega \frac{(f_k')^2}{r^2}dx + \left(c_k - \frac{N^2-8N+32}{4}\right)\int_\Omega \frac{(f_k)^2}{r^4}dx\\ &\quad= 2\int_\Omega (g_k')^2 dx + \left(c_k - \frac{N^2+8}{4}\right)\int_\Omega \frac{g_k^2}{r^2}dx\\ &\quad\geq \frac{(N-2)^2}{2}\int_\Omega \frac{g_k^2}{r^2}dx + 2\Lambda(-\Delta,2)\left(\frac{\omega_N}{|\Omega|}\right)^{2/N}\int_\Omega g_k^2 dx + \left(c_k - \frac{N^2+8}{4}\right)\int_\Omega \frac{g_k^2}{r^2}dx\\ &\quad= \frac{N^2-8N+4c_k}{4}\int_\Omega \frac{g_k^2}{r^2}dx + 2\Lambda(-\Delta,2)\left(\frac{\omega_N}{|\Omega|}\right)^{2/N}\int_\Omega g_k^2 dx\\ &\quad\geq \frac{N^2-8N+4c_1}{4}\int_\Omega \frac{g_k^2}{r^2}dx + 2\Lambda(-\Delta,2)\left(\frac{\omega_N}{|\Omega|}\right)^{2/N}\int_\Omega g_k^2 dx\end{aligned}$$

$$= \frac{N^2-4N-4}{4}\int_\Omega \frac{g_k^2}{r^2}dx + 2\Lambda(-\Delta,2)\left(\frac{\omega_N}{|\Omega|}\right)^{2/N}\int_\Omega g_k^2 dx$$
$$\geq 2\Lambda(-\Delta,2)\left(\frac{\omega_N}{|\Omega|}\right)^{2/N}\int_\Omega g_k^2 dx$$
$$= 2\Lambda(-\Delta,2)\left(\frac{\omega_N}{|\Omega|}\right)^{2/N}\int_\Omega \frac{(f_k)^2}{r^2}dx. \tag{2.17}$$

□

An immediate consequence of the inequalities (2.13) and Lemma 2.2 is the following result. For $k \geq 1$,

$$\int_{\mathbb{R}^N}|\Delta u_k|^2 dx - \frac{N^2}{4}\int_{\mathbb{R}^N}\frac{|\nabla u_k|^2}{|x|^2}dx$$
$$\geq \int_{\mathbb{R}^N}(f_k'')^2 dx - \frac{(N-2)^2}{4}\int_{\mathbb{R}^N}\frac{(f_k')^2}{r^2}dx + 2c_k\Lambda(-\Delta,2)\left(\frac{\omega_N}{|\Omega|}\right)^{2/N}\int_\Omega\frac{(f_k)^2}{r^2}dx. \tag{2.18}$$

Using inequalities (2.18) and Lemma 2.1, we have that, since $f_k(r) \in C_0^\infty(\Omega)$, for $k \geq 1$,

$$\int_{\mathbb{R}^N}|\Delta u_k|^2 dx - \frac{N^2}{4}\int_{\mathbb{R}^N}\frac{|\nabla u_k|^2}{|x|^2}dx$$
$$\geq \Lambda(-\Delta,2)\left(\frac{\omega_N}{|\Omega|}\right)^{2/N}\int_{\mathbb{R}^N}(f_k')dx + 2c_k\Lambda(-\Delta,2)\left(\frac{\omega_N}{|\Omega|}\right)^{2/N}\int_\Omega\frac{(f_k)^2}{r^2}dx$$
$$\geq \Lambda(-\Delta,2)\left(\frac{\omega_N}{|\Omega|}\right)^{2/N}\left(\int_{\mathbb{R}^N}(f_k')dx + c_k\int_\Omega\frac{(f_k)^2}{r^2}dx\right)$$
$$= \Lambda(-\Delta,2)\left(\frac{\omega_N}{|\Omega|}\right)^{2/N}\int_{\mathbb{R}^N}|\nabla u_k|^2 dx. \tag{2.19}$$

Inequality (2.19) implies that, if $u(x)$ is not a radial function, then

$$\int_\Omega|\Delta u|^2 dx - \frac{N^2}{4}\int_\Omega\frac{|\nabla u|^2}{|x|^2}dx \geq \Lambda(-\Delta,2)\left(\frac{\omega_N}{|\Omega|}\right)^{2/N}\int_\Omega|\nabla u|^2\ dx. \tag{2.20}$$

Proof of Theorem 1.1. Using inequality (2.6) and (2.20), we have that, for $N \geq 5$ and $u \in C_0^\infty(\Omega)$,

$$\int_\Omega|\Delta u|^2 dx \geq \frac{N^2}{4}\int_\Omega\frac{|\nabla u|^2}{|x|^2}dx + \Lambda(-\Delta,2)\left(\frac{\omega_N}{|\Omega|}\right)^{2/N}\int_\Omega|\nabla u|^2 dx. \tag{2.21}$$

In case Ω is a ball centered at zero, a simple scaling allows to consider the case $\Omega = B_1$. Set

$$H = \inf_{u\in C_0^\infty(B_1)\setminus\{0\}} \frac{\int_{B_1}|\Delta u|^2 dx - (N^2/4)\int_{B_1}\left(|\nabla u|^2/|x|^2\right)dx}{\int_{B_1}|\nabla u|^2 dx}. \tag{2.22}$$

Using Lemma 2.1 and inequality (1.2), we have that $H \leq H_{\text{radial}} = \Lambda(-\Delta, 2)$. On the other hand, we have, by inequality (2.21), $H \geq \Lambda(-\Delta, 2)$. Thus $H = \Lambda(-\Delta, 2)$. The proof is complete. □

Proof of Theorem 1.3. A scaling argument shows that we may assume $R = 1$ and $\Omega = B_1 = B$. □

Step 1. Assume u is radial, $r = |x|$ and $v(r) = |x|^{(N-4)/2}u(r)$, then (see [6, Lemma 2.3])

$$\int_B |\Delta u|^2 dx - \frac{N^2}{4}\int_B \frac{|\nabla u|^2}{|x|^2}dx = \int_B \frac{|\Delta v|^2}{|x|^{N-4}}dx + \left(\frac{N(N-8)}{4} - N(N-4)\right)\int_B \frac{v_r^2}{|x|^{N-2}}dx, \tag{2.23}$$

and (see [6, (6.4)])

$$\int_B \frac{|\Delta v|^2}{|x|^{N-4}}dx = \int_B \frac{v_{rr}^2}{|x|^{N-4}}dx + (N-1)(N-3)\int_B \frac{v_r^2}{|x|^{N-2}}dx. \tag{2.24}$$

Therefore

$$\int_B |\Delta u|^2 dx - \frac{N^2}{4}\int_B \frac{|\nabla u|^2}{|x|^2}dx = \int_B \frac{v_{rr}^2}{|x|^{N-4}}dx + 3\int_B \frac{v_r^2}{|x|^{N-2}}dx + \frac{N(N-8)}{4}\int_B \frac{v_r^2}{|x|^{N-2}}dx. \tag{2.25}$$

Since v is radial,

$$\begin{aligned} \int_B \frac{v_r^2}{|x|^{N-2}}dx &\geq \Lambda(-\Delta, 2)\int_B \frac{v^2}{|x|^{N-2}}dx; \\ \int_B \frac{v_{rr}^2}{|x|^{N-4}}dx + 3\int_B \frac{v_r^2}{|x|^{N-2}}dx &= \frac{\Sigma_N}{\Sigma_4}\int_{B^{(4)}} v_{rr}^2 dx + 3\frac{\Sigma_N}{\Sigma_4}\int_{B^{(4)}} \frac{v_r^2}{|x|^2}dx \\ &= \frac{\Sigma_N}{\Sigma_4}\int_{B^{(4)}} |\Delta_{\text{rad},4} v|^2 dx \\ &\geq \frac{\Sigma_N}{\Sigma_4}\Lambda\left((-\Delta)^2, 4\right)\int_{B^{(4)}} v^2 dx \\ &= \Lambda\left((-\Delta)^2, 4\right)\int_B \frac{v^2}{|x|^{N-4}}dx, \end{aligned} \tag{2.26}$$

where Σ_k denote the surface area of the unit sphere in $\mathbb{R}^k$, $B^{(4)}$ is the unit ball in $\mathbb{R}^4$, and

$$\Delta_{\text{rad},4} = \frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} \tag{2.27}$$

is the radial Laplacian in $\mathbb{R}^4$.

Therefore, for $N \geq 8$,

$$\begin{aligned}\int_B |\Delta u|^2 dx - \frac{N^2}{4}\int_B \frac{|\nabla u|^2}{|x|^2}dx \\ \geq \Lambda(-\Delta,2)\int_B \frac{v^2}{|x|^{N-2}}dx + \frac{N(N-8)}{4}\Lambda\Big((-\Delta)^2,4\Big)\int_B \frac{v^2}{|x|^{N-4}}dx \\ = \Lambda(-\Delta,2)\int_B \frac{u^2}{|x|^2}dx + \frac{N(N-8)}{4}\Lambda\Big((-\Delta)^2,4\Big)\int_B u^2 dx.\end{aligned} \tag{2.28}$$

Step 2. For $u \in C_0^\infty(B)$, set

$$u = \sum_{k=0}^{\infty} u_k := \sum_{k=0}^{\infty} f_k(r)\phi_k(\sigma). \tag{2.29}$$

We get, by (2.18),

$$\begin{aligned}\int_B |\Delta u_k|^2 dx - \frac{N^2}{4}\int_B \frac{|\nabla u_k|^2}{|x|^2}dx \geq \int_B (f_k'')^2 dx - \frac{(N-2)^2}{4}\int_B \frac{(f_k')^2}{r^2}dx \\ = \int_B |\Delta f_k|^2 dx - \frac{N^2}{4}\int_B \frac{|\nabla f_k|^2}{|x|^2}dx.\end{aligned} \tag{2.30}$$

In getting the last equality, we used Lemma 2.1.

Using inequality (1.9) for radial functions from step 1,

$$\begin{aligned}\int_B |\Delta u_k|^2 dx - \frac{N^2}{4}\int_B \frac{|\nabla u_k|^2}{|x|^2}dx \\ \geq \Lambda(-\Delta,2)\int_B \frac{f_k^2}{|x|^2}dx + \frac{N(N-8)}{4}\Lambda\Big((-\Delta)^2,4\Big)\int_B f_k^2 dx \\ = \Lambda(-\Delta,2)\int_B \frac{u_k^2}{|x|^2}dx + \frac{N(N-8)}{4}\Lambda\Big((-\Delta)^2,4\Big)\int_B u_k^2 dx,\end{aligned} \tag{2.31}$$

one obtains, by (2.11),

$$\int_B |\Delta u|^2 dx - \frac{N^2}{4}\int_B \frac{|\nabla u|^2}{|x|^2} dx \geq \Lambda(-\Delta, 2)\int_B \frac{u^2}{|x|^2} dx + \frac{N(N-8)}{4}\Lambda\left((-\Delta)^2, 4\right)\int_B u^2 dx \tag{2.32}$$

which demonstrates inequality (1.9).

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