Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 702680, 7 pages
doi:10.1155/2009/702680

Research Article

A New Estimate on the Rate of Convergence of Durrmeyer-Bézier Operators

Pinghua Wang[1] and Yali Zhou[2]

[1] *Department of Mathematics, Quanzhou Normal University, Fujian 362000, China*
[2] *Liming University, Quanzhou, Fujian 362000, China*

Correspondence should be addressed to Pinghua Wang, xxc570@163.com

Received 20 February 2009; Accepted 13 April 2009

Recommended by Vijay Gupta

We obtain an estimate on the rate of convergence of Durrmeyer-Bézier operaters for functions of bounded variation by means of some probabilistic methods and inequality techniques. Our estimate improves the result of Zeng and Chen (2000).

1. Introdution

In 2000, Zeng and Chen [1] introduced the Durrmeyer-Bézier operators $D_{n,\alpha}$ which are defined as follows:

$$D_{n,\alpha}(f,x) = (n+1)\sum_{k=0}^{n} Q_{nk}^{(\alpha)}(x)\int_0^1 f(t)p_{nk}(t)dt, \tag{1.1}$$

where f is defined on $[0,1]$, $\alpha \geq 1$, $Q_{nk}^{(\alpha)}(x) = J_{nk}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$, $J_{nk}(x) = \sum_{j=k}^{n} p_{nj}(x)$, $k = 0,1,2,\ldots,n$ are Bézier basis functions, and $p_{nk}(x) = (n!/k!(n-k)!)(x^k\,(1-x)^{n-k})$, $k = 0,1,2,\ldots,n$ are Bernstein basis functions.

When $\alpha = 1$, $D_{n,1}(f)$ is just the well-known Durrmeyer operator

$$D_{n,1}(f,x) = (n+1)\sum_{k=0}^{n} p_{nk}(x)\int_0^1 f(t)p_{nk}(t)dt. \tag{1.2}$$

Concerning the approximation properties of operators $D_{n,1}(f)$ and some results on approximation of functions of bounded variation by positive linear operators, one can refer to [2–7]. Authors of [1] studied the rate of convergence of the operators $D_{n,\alpha}(f)$ for functions of bounded variation and presented the following important result.

Theorem A. *Let f be a function of bounded variation on $[0,1]$, ($f \in \mathrm{BV}[0,1]$), $\alpha \geq 1$, then for every $x \in (0,1)$ and $n \geq 1/x(1-x)$one has*

$$\left|D_{n,\alpha}(f,x) - \left[\frac{1}{\alpha+1}f(x+) + \frac{\alpha}{\alpha+1}f(x-)\right]\right| \leq \frac{8\alpha}{nx(1-x)}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x) + \frac{2\alpha}{\sqrt{nx(1-x)}}|f(x+) - f(x-)|, \tag{1.3}$$

where $\bigvee_a^b(g_x)$ is the total variation of g_x on $[a,b]$ and

$$g_x(t) = \begin{cases} f(t) - f(x+), & x < t \leq 1, \\ 0, & t = x, \\ f(t) - f(x-), & 0 \leq t < x. \end{cases} \tag{1.4}$$

Since the Durrmeyer-Bézier operators $D_{n,\alpha}$ are an important approximation operator of new type, the purpose of this paper is to continue studying the approximation properties of the operators $D_{n,\alpha}$ for functions of bounded variation, and give a better estimate than that of Theorem A by means of some probabilistic methods and inequality techniques. The result of this paper is as follows.

Theorem 1.1. *Let f be a function of bounded variation on $[0,1]$, ($f \in \mathrm{BV}[0,1]$), $\alpha \geq 1$, then for every $x \in (0,1)$ and $n > 1$ one has*

$$\left|D_{n,\alpha}(f,x) - \left[\frac{1}{\alpha+1}f(x+) + \frac{\alpha}{\alpha+1}f(x-)\right]\right| \leq \frac{4\alpha+1}{nx(1-x)}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x) + \frac{\alpha}{\sqrt{(n+1)x(1-x)}}|f(x+) - f(x-)|, \tag{1.5}$$

where $g_x(t)$ is defined in (1.4).

It is obvious that the estimate (1.5) is better than the estimate (1.3). More important, the estimate (1.5) is true for all $n > 1$. This is an important improvement comparing with the fact that estimate (1.3) holds only for $n \geq 1/x(1-x)$.

2. Some Lemmas

In order to prove Theorem 1.1, we need the following preliminary results.

Lemma 2.1. *Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of independent and identically distributed random variables, ξ_1 is a random variable with two-point distribution $P(\xi_1 = i) = x^i(1-x)^{1-i}$ ($i = 0, 1$, and $x \in [0,1]$ is* *a parameter). Set* $\eta_n = \sum_{k=1}^{n} \xi_k$, *with the mathematical expectation* $E(\eta_n) = \mu_n \in (-\infty, +\infty)$, *and with the variance* $D(\eta_n) = \sigma_n^2 > 0$. *Then for* $k = 1, 2, \ldots, n+1$,*one has*

$$\left|P(\eta_n \le k-1) - P(\eta_{n+1} \le k)\right| \le \frac{\sigma_{n+1}}{\mu_{n+1}}, \tag{2.1}$$

$$\left|P(\eta_n \le k) - P(\eta_{n+1} \le k)\right| \le \frac{\sigma_{n+1}}{(n+1-\mu_{n+1})}. \tag{2.2}$$

Proof. Since $\eta_n = \sum_{k=1}^{n} \xi_k$, from the distribution series of ξ_k, by convolution computation we get

$$P(\eta_n = j) = \frac{n!}{j!(n-j)!} x^j (1-x)^{n-j}, \quad 0 \le j \le n. \tag{2.3}$$

Furthermore by direct computations we have

$$\begin{gathered} \mu_{n+1} = (n+1)x, \\ P(\eta_n = j-1) = \frac{j}{(n+1)x} P(\eta_{n+1} = j), \quad 1 \le j \le n+1. \end{gathered} \tag{2.4}$$

Thus we deduce that

$$\begin{aligned} \left|P(\eta_n \le k-1) - P(\eta_{n+1} \le k)\right| &= \left|\sum_{j=1}^{k} P(\eta_n = j-1) - \sum_{j=1}^{k} P(\eta_{n+1} = j) - P(\eta_{n+1} = 0)\right| \\ &= \left|\sum_{j=0}^{k} \left(\frac{j}{(n+1)x} - 1\right) P(\eta_{n+1} = j)\right| \\ &\le \frac{1}{(n+1)x} \sum_{j=0}^{k} \left|j - (n+1)x\right| P(\eta_{n+1} = j) \\ &\le \frac{1}{(n+1)x} \sum_{j=0}^{n+1} \left|j - (n+1)x\right| P(\eta_{n+1} = j) \\ &\le \frac{1}{\mu_{n+1}} E\left|\eta_{n+1} - \mu_{n+1}\right|. \end{aligned} \tag{2.5}$$

By Schwarz's inequality, it follows that

$$\frac{1}{\mu_{n+1}} E\left|\eta_{n+1} - \mu_{n+1}\right| \le \frac{\sqrt{E(\eta_{n+1} - \mu_{n+1})^2}}{\mu_{n+1}} = \frac{\sigma_{n+1}}{\mu_{n+1}}. \tag{2.6}$$

The inequality (2.1) is proved.

Similarly, by using the identities

$$n+1-\mu_{n+1} = (n+1)(1-x),$$
$$P(\eta_n = j) = \frac{(n+1)-j}{(n+1)(1-x)} P(\eta_{n+1} = j), \quad 1 \le j \le n+1, \tag{2.7}$$

we get the inequality (2.2). Lemma 2.1 is proved. □

Lemma 2.2. *Let* $\alpha \ge 1$, $k = 0, 1, 2, \ldots, n$, $p_{nk}(x) = (n!/k!(n-k)!)x^k\ (1-x)^{n-k}$ *be Bernstein basis functions, and let* $J_{nk}(x) = \sum_{j=k}^{n} p_{nj}(x)$ *be Bézier basis functions, then one has*

$$\left| J_{nk}^{\alpha}(x) - J_{n+1,k+1}^{\alpha}(x) \right| \le \frac{\alpha}{\sqrt{(n+1)x(1-x)}},$$
$$\left| J_{nk}^{\alpha}(x) - J_{n+1,k}^{\alpha}(x) \right| \le \frac{\alpha}{\sqrt{(n+1)x(1-x)}}. \tag{2.8}$$

Proof. Note that $0 \le J_{nk}(x),\ J_{n+1,k+1}(x) \le 1$, $\mu_{n+1} = (n+1)x$, $\sigma_{n+1}^2 = (n+1)x(1-x)$, and $\alpha \ge 1$. Thus

$$\begin{aligned}
\left| J_{nk}^{\alpha}(x) - J_{n+1,k+1}^{\alpha}(x) \right| &\le \alpha \left| J_{nk}(x) - J_{n+1,k+1}(x) \right| \\
&= \alpha \left| \sum_{j=k}^{n} p_{nj} - \sum_{j=k+1}^{n+1} p_{n+1,j} \right| \\
&= \alpha \left| \left(1 - \sum_{j=k}^{n} p_{nj} \right) - \left(1 - \sum_{j=k+1}^{n+1} p_{n+1,j} \right) \right| \\
&= \alpha \left| P(\eta_n \le k-1) - P(\eta_{n+1} \le k) \right|.
\end{aligned} \tag{2.9}$$

Now by inequality (2.1) of Lemma 2.1 we obtain

$$\left| J_{nk}^{\alpha}(x) - J_{n+1,k+1}^{\alpha}(x) \right| \le \alpha \frac{1-x}{\sqrt{(n+1)x(1-x)}} \le \frac{\alpha}{\sqrt{(n+1)x(1-x)}}. \tag{2.10}$$

Similarly, by using inequality (2.2), we obtain

$$\left| J_{nk}^{\alpha}(x) - J_{n+1,k}^{\alpha}(x) \right| \le \alpha \frac{x}{\sqrt{(n+1)x(1-x)}} \le \frac{\alpha}{\sqrt{(n+1)x(1-x)}}. \tag{2.11}$$

Thus Lemma 2.2 is proved. □

3. Proof of Theorem 1.1

Let f satisfy the conditions of Theorem 1.1, then f can be decomposed as

$$\begin{aligned} f(t) = {} & \frac{1}{\alpha+1} f(x+) + \frac{\alpha}{\alpha+1} f(x-) + g_x(t) \\ & + \frac{f(x+) - f(x-)}{2}\left(\operatorname{sgn}(t-x) + \frac{\alpha-1}{\alpha+1}\right) \\ & + \delta_x(t)\left(f(x) - \frac{1}{2}f(x+) - \frac{1}{2}f(x-)\right), \end{aligned} \tag{3.1}$$

where

$$\operatorname{sgn}(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0, \\ -1, & t < 0, \end{cases} \qquad \delta_x(t) = \begin{cases} 0, & t \neq x, \\ 1, & t = x. \end{cases} \tag{3.2}$$

Obviously $D_{n,\alpha}(\delta_x, x) = 0$, thus from (3.1) we get

$$\begin{aligned} & \left| D_{n,\alpha}(f,x) - \left(\frac{1}{\alpha+1} f(x+) + \frac{\alpha}{\alpha+1} f(x-) \right) \right| \\ & \quad \leq \left| D_{n,\alpha}(g_{x,},x) \right| + \left| \frac{f(x+) - f(x-)}{2} \left(D_{n,\alpha}(\operatorname{sgn}(t-x),x) + \frac{\alpha-1}{\alpha+1} \right) \right|. \end{aligned} \tag{3.3}$$

We first estimate $|D_{n,\alpha}(\operatorname{sgn}(t-x),x) + (\alpha-1)/(\alpha+1)|$, from [1, page 11] we have the following equation:

$$D_{n,\alpha}(\operatorname{sgn}(t-x),x) + \frac{\alpha-1}{\alpha+1} = 2\sum_{k=0}^{n+1} p_{n+1,k}(x) J_{nk}^{\alpha}(x) - 2\sum_{k=0}^{n+1} p_{n+1,k}(x) \gamma_{nk}^{\alpha}(x), \tag{3.4}$$

where $J_{n+1,k+1}^{\alpha}(x) < \gamma_{nk}^{\alpha}(x) < J_{n+1,k}^{\alpha}(x)$.

Thus by Lemma 2.2, we get $|J_{nk}^{\alpha}(x) - \gamma_{nk}^{\alpha}(x)| \leq \alpha/\sqrt{(n+1)x(1-x)}$. Note that $\sum_{k=0}^{n+1} p_{n+1,k}(x) = 1$, we have

$$\left| D_{n,\alpha}(\operatorname{sgn}(t-x),x) + \frac{\alpha-1}{\alpha+1} \right| = \left| 2\sum_{k=0}^{n+1} p_{n+1,k}(x) \left(J_{nk}^{\alpha}(x) - \gamma_{nk}^{\alpha}(x) \right) \right| \leq \frac{2\alpha}{\sqrt{(n+1)x(1-x)}}. \tag{3.5}$$

Next we estimate $|D_{n,\alpha}(g_x,x)|$. From (15) of [1], it follows the inequality

$$\left| D_{n,\alpha}(g_x,x) \right| \leq 4\alpha \frac{nx(1-x)+1}{n^2x^2(1-x)^2} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}} (g_x). \tag{3.6}$$

That is,

$$n^2x^2(1-x)^2\left|D_{n,\alpha}(g_x,x)\right| \le 4\alpha(nx(1-x)+1)\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x). \tag{3.7}$$

On the other hand, note that $g_x(x)=0$, we have

$$\begin{aligned}\left|D_{n,\alpha}(g_x,x)\right| &\le D_{n,\alpha}(\left|g_x(t)-g_x(x)\right|,x)\\ &\le\bigvee_{0}^{1}(g_x)D_{n,\alpha}(1,x)\\ &=\bigvee_{0}^{1}(g_x)\le\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x).\end{aligned} \tag{3.8}$$

From (3.7) and (3.8) we obtain

$$\left|D_{n,\alpha}(g_x,x)\right| \le \frac{4\alpha nx(1-x)+4\alpha+4\alpha}{n^2x^2(1-x)^2+4\alpha}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x). \tag{3.9}$$

Using inequality

$$n^2x^2(1-x)^2+16\alpha^2+4\alpha > 8\alpha nx(1-x), \tag{3.10}$$

we get

$$\frac{4\alpha nx(1-x)+4\alpha+4\alpha}{n^2x^2(1-x)^2+4\alpha} < \frac{4\alpha+1}{nx(1-x)}, \quad \forall n>1. \tag{3.11}$$

Thus from (3.9) we obtain

$$\left|D_{n,\alpha}(g_x,x)\right| \le \frac{4\alpha+1}{nx(1-x)}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x). \tag{3.12}$$

Theorem 1.1 now follows by collecting the estimations (3.3), (3.5), and (3.12).

Acknowledgment

The present work is supported by Project 2007J0188 of Fujian Provincial Science Foundation of China.

References

[1] X.-M. Zeng and W. Chen, "On the rate of convergence of the generalized Durrmeyer type operators for functions of bounded variation," *Journal of Approximation Theory*, vol. 102, no. 1, pp. 1–12, 2000.
[2] R. Bojanić and F. H. Chêng, "Rate of convergence of Bernstein polynomials for functions with derivatives of bounded variation," *Journal of Mathematical Analysis and Applications*, vol. 141, no. 1, pp. 136–151, 1989.
[3] M. M. Derriennic, "Sur l'approximation de fonctions intégrables sur [0, 1] par des polynômes de Bernstein modifies," *Journal of Approximation Theory*, vol. 31, no. 4, pp. 325–343, 1981.
[4] S. S. Guo, "On the rate of convergence of the Durrmeyer operator for functions of bounded variation," *Journal of Approximation Theory*, vol. 51, no. 2, pp. 183–192, 1987.
[5] V. Gupta and R. P. Pant, "Rate of convergence for the modified Szász-Mirakyan operators on functions of bounded variation," *Journal of Mathematical Analysis and Applications*, vol. 233, no. 2, pp. 476–483, 1999.
[6] A. N. Shiryayev, *Probability*, vol. 95 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1984.
[7] X.-M. Zeng and V. Gupta, "Rate of convergence of Baskakov-Bézier type operators for locally bounded functions," *Computers & Mathematics with Applications*, vol. 44, no. 10-11, pp. 1445–1453, 2002.