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Research Article

Existence and Stability of Solutions for Nonautonomous Stochastic Functional Evolution Equations

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We establish the results on existence and exponent stability of solutions for a semilinear nonautonomous neutral stochastic evolution equation with finite delay; the linear part of this equation is dependent on time and generates a linear evolution system. The obtained results are applied to some neutral stochastic partial differential equations. These kinds of equations arise in systems related to couple oscillators in a noisy environment or in viscoelastic materials under random or stochastic influences.

1. Introduction

In this paper we study the existence and asymptotic behavior of mild solutions for the following neutral non-autonomous stochastic evolution equation with finite delay:

$$\begin{gathered} d(Z(t) - k(t, Z_t)) = [-A(t)(Z(t) - k(t, Z_t)) + F(t, Z_t)]dt + G(t, Z_t)dW(t), \quad 0 \le t \le T, \\ Z_0 = \phi \in L^p(\Omega, C_\alpha), \end{gathered} \tag{1.1}$$

where $A(t)$ generates a linear evolution system, or say linear evolution operator $\{U(t,s) : 0 \le s \le t \le T\}$ on a separable Hilbert space H with the inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. k; F and G are given functions to be specified later.

In recent years, existence, uniqueness, stability, invariant measures, and other quantitative and qualitative properties of solutions to stochastic partial differential equations have been extensively investigated by many authors. One of the important techniques to discuss these topics is the semigroup approach; see, for example, Da Prato and Zabczyk [1], Dawson [2], Ichikawa [3], and Kotelenez [4]. In paper [5] Taniguchi et al. have investigated the existence and asymptotic behavior of solutions for the following stochastic functional differential equation:

$$dZ(t) = \left[-AZ(t) + f(t, Z_t)\right]dt + g(t, Z_t)dW(t), \quad 0 \le t \le T,$$
$$Z_0 = \phi \in L^p(\Omega, C_\alpha), \tag{1.2}$$

by using analytic semigroups approach and fractional power operator arguments. In this work as well as other related literatures like [6–9], the linear part of the discussed equation is an operator independent of time t and generates a strongly continuous (one-parameter) semigroup or analytic semigroup so that the semigroup approach can be employed. We would also like to mention that some similar topics to the above for stochastic ordinary functional differential equations with finite delays have already been investigated successfully by various authors (cf. [6, 10–13] and references in [14] among others). Related work on functional stochastic evolution equations of McKean-Vlasov type and of second-order are discussed in [15, 16].

However, it occurs very often that the linear part of (1.2) is dependent on time t. Indeed, a lot of stochastic partial functional differential equations can be rewritten to semilinear non-autonomous equations having the form of (1.2) with $A = A(t)$. There exists much work on existence, asymptotic behavior, and controllability for deterministic non-autonomous partial (functional) differential equations with finite or infinite delays; see, for example, [17–20]. But little is known to us for non-autonomous stochastic differential equations in abstract space, especially for the case that $A(t)$ is a family of unbounded operators.

Our purpose in the present paper is to obtain results concerning existence, uniqueness, and stability of the solutions of the non-autonomous stochastic differential equations (1.1). A motivation example for this class of equations is the following non-autonomous boundary problem:

$$d(Z(t,x) - bZ(t-r,x)) = \left[a(t,x)\frac{\partial^2}{\partial^2 x}(Z(t,x) - bZ(t-r,x)) + f(t, Z(t + r_1(t), x))\right]dt$$
$$+ g(t, Z(t + r_1(t), x))d\beta(t),$$
$$Z(t,0) = Z(t,\pi) = 0, \quad t \ge 0,$$
$$Z(\theta, x) = \phi(x), \quad \theta \in [-r, 0],\ 0 \le x \le \pi. \tag{1.3}$$

As stated in paper [21], these problems arise in systems related to couple oscillators in a noisy environment or in viscoelastic materials under random or stochastic influences (see also [22] for the discussion for the corresponding determined systems). Therefore, it is meaningful to deal with (1.1) to acquire some results applicable to problem (1.3). In paper [21], Caraballo et al. have, under coercivity condition in an integral form, investigated the second moment (almost sure) exponential stability and ultimate boundedness of solutions to the following non-autonomous semilinear stochastic delay equation:

$$d(X(t) - k(t, X(t-r))) = [A(t)X(t) + F(t, X_t)]dt + G(t, X_t)dW(t), \quad t \geq 0,$$
$$X(t) = \phi(t) \quad t \in [-r, 0] \tag{1.4}$$

on a Hilbert space H, where $A(\cdot) \in L^\infty(0, T; \mathcal{L}(V, V^*))$ with $V \subset H \subset H^* \subset V^*$.

As we know, non-autonomous evolution equations are much more complicate than autonomous ones to be dealt with. Our approach here is inspired by the work in paper [5, 18, 19]. That is, we assume that $\{A(t) : t \geq 0\}$ is a family of unbounded linear operators on H with (common) dense domain such that it generates a linear evolution system. Thus we will apply the theory of linear evolution system and fractional power operators methods to discuss existence, uniqueness, and pth ($p > 2$) moment exponential stability of mild solutions to the stochastic partial functional differential equation (1.1). Clearly our work can be regarded as extension and development of that in [5, 21] and other related papers mentioned above.

We will firstly in Section 2 introduce some notations, concepts, and basic results about linear evolution system and stochastic process. The existence and uniqueness of mild solutions are discussed in Section 3 by using Banach fixed point theorem. In Section 4, we investigate the exponential stability for the mild solutions obtained in Section 3, and the conditions for stability are somewhat weaker than in [5]. Finally, in Section 5 we apply the obtained results to (1.3) to illustrate the applications.

2. Preliminaries

In this section we collect some notions, conceptions, and lemmas on stochastic process and linear evolution system which will be used throughout the whole paper.

Let $(\Omega, \mathcal{F}, P)$ be a probability space on which an increasing and right continuous family $\{\mathcal{F}_t\}_{t\in[0,+\infty)}$ of complete sub-σ-algebras of $\mathcal{F}$ is defined. H and K are two separable Hilbert space. Suppose that $W(t)$ is a given K-valued Wiener Process with a finite trace nuclear covariance operator $Q > 0$. Let $\beta_n(t)$, $n = 1, 2, \ldots$, be a sequence of real-valued one-dimensional standard Brownian motions mutually independent over $(\Omega, \mathcal{F}, P)$. Set

$$W(t) = \sum_{n=1}^{+\infty} \sqrt{\lambda_n} \beta_n(t) e_n, \quad t \geq 0, \tag{2.1}$$

where $\lambda_n \geq 0$ $(n = 1, 2, \ldots)$ are nonnegative real numbers, and $\{e_n\}$ $(n = 1, 2, \ldots)$ is a complete orthonormal basis in K. Let $Q \in \mathcal{L}(K, K)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $\mathrm{tr}Q = \sum_{n=1}^{+\infty} \lambda_n < \infty$. Then the above K-valued stochastic process $W(t)$ is called a Q-Wiener process.

Definition 2.1. Let $\sigma \in \mathcal{L}(K, H)$ and define

$$\|\sigma\|_{\mathcal{L}_2^0}^2 := \operatorname{tr}(\sigma Q \sigma^*) = \sum_{n=1}^{+\infty} \left\| \sqrt{\lambda_n} \sigma e_n \right\|^2. \tag{2.2}$$

If $\|\sigma\|_{\mathcal{L}_2^0} < \infty$, then σ is called a Q-Hilbert-Schmidt operator, and let $\mathcal{L}_2^0(K, H)$ denote, the space of all Q-Hilbert-Schmidt operators $\sigma : K \to H$.

In the next section the following lemma (see [1, Lemma 7.2]) plays an important role.

Lemma 2.2. *For any $p \geq 2$ and for arbitrary $\mathcal{L}_2^0$-valued predictable process $\Phi(t)$, $t \in [0, T]$, one has*

$$E\left(\sup_{s \in [0,t]} \left\| \int_0^s \Phi(s) dW(s) \right\|^p\right) \leq C_p E\left(\int_0^t \|\Phi(s)\|_{\mathcal{L}_2^0}^2 ds\right)^{p/2} \tag{2.3}$$

for some constant $C_p > 0$.

Now we turn to state some notations and basic facts from the theory of linear evolution system.

Throughout this paper, $\{A(t) : 0 \leq t \leq T\}$ is a family of linear operators defined on Hilbert space H, and for this family we always impose on the following restrictions.

(B_1) The domain $D(A)$ of $\{A(t) : 0 \leq t \leq T\}$ is dense in H and independent of t; $A(t)$ is closed linear operator.

(B_2) For each $t \in [0, T]$, the resolvent $R(\lambda, A(t))$ exists for all λ with $\mathrm{Re}\lambda \leq 0$, and there exists $C > 0$ so that $\|R(\lambda, A(t))\| \leq C/(|\lambda| + 1)$.

(B_3) There exists $0 < \delta \leq 1$ and $C > 0$ such that $\|(A(t) - A(s))\ A^{-1}(\tau)\| \leq C|t - s|^\delta$ for all $t, s, \tau \in [0, T]$.

Under these assumptions, the family $\{A(t) : 0 \leq t \leq T\}$ generates a unique linear evolution system, or called linear evolution operators $\{U(t, s), 0 \leq s \leq t \leq T\}$, and there exists a family of bounded linear operators $\{R(t, \tau) | 0 \leq \tau \leq t \leq T\}$ with $\|R(t, \tau)\| \leq K|t - \tau|^{\delta - 1}$ such that $U(t, s)$ has the representation

$$U(t, s) = e^{-(t-s)A(t)} + \int_s^t e^{-(t-\tau)A(\tau)} R(\tau, s) d\tau, \tag{2.4}$$

where $\exp(-\tau A(t))$ denotes the analytic semigroup having infinitesimal generator $-A(t)$ (note that Assumption (B_2) guarantees that $-A(t)$ generates an analytic semigroup on H).

For the linear evolution system $\{U(t,s), 0 \leq s \leq t \leq T\}$, the following properties are well known:

(a) $U(t,s) \in \mathcal{L}(H)$, the space of bounded linear transformations on H, whenever $0 \leq s \leq t \leq T$ and maps H into $D(A)$ as $t > s$. For each $x \in H$, the mapping $(t,s) \to U(t,s)x$ is continuous jointly in s and t;

(b) $U(t,s)U(s,\tau) = U(t,\tau)$ for $0 \leq \tau \leq s \leq t \leq T$;

(c) $U(t,t) = I$;

(d) $(\partial/\partial t)U(t,s) = -A(t)U(t,s)$, for $s < t$.

We also have the following inequalities:

$$\left\| e^{-tA(s)} \right\| \leq K, \quad t \geq 0,\ s \in [0,T],$$
$$\left\| A(s)e^{-tA(s)} \right\| \leq \frac{K}{t}, \quad t,s \in [0,T], \tag{2.5}$$
$$\|A(t)U(t,s)\| \leq \frac{K}{|t-s|}, \quad 0 \leq s \leq t \leq T.$$

Furthermore, Assumptions (B_1)–(B_3) imply that for each $t \in [0,T]$ the integral

$$A^{-\alpha}(t) = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} s^{\alpha-1} e^{-sA(t)} ds \tag{2.6}$$

exists for each $\alpha \in (0,1]$. The operator defined by (2.6) is a bounded linear operator and yields $A^{-\alpha}(t)A^{-\beta}(t) = A^{-(\alpha+\beta)}(t)$. Thus, we can define the fractional power as

$$A^{\alpha}(t) = \left[A^{-\alpha}(t)\right]^{-1}, \tag{2.7}$$

which is a closed linear operator with $D(A^{\alpha}(t))$ dense in H and $D(A^{\alpha}(t)) \subset D(A^{\beta}(t))$ for $\alpha \geq \beta$. $D(A^{\alpha}(t))$ becomes a Banach space endowed with the norm $\|x\|_{\alpha,t} = \|A^{\alpha}(t)x\|$, which is denoted by $H_{\alpha}(t)$.

The following estimates and Lemma 2.3 are from ([23, Part II]):

$$\left\| A^{\alpha}(t)A^{-\beta}(\tau) \right\| \leq K(\alpha,\beta) \tag{2.8}$$

for $t, \tau \in [0, T]$ and $0 \le \alpha < \beta$, and

$$\left\| A^{\beta}(t) e^{-sA(t)} \right\| \le \frac{K(\beta)}{s^{\beta}} e^{-ws}, \quad t > 0,\ \beta \le 0,\ w > 0, \tag{2.9}$$

$$\left\| A^{\beta}(t) U(t,s) \right\| \le \frac{K(\beta)}{|t-s|^{\beta}}, \quad 0 < \beta < t + 1 \tag{2.10}$$

for some $t > 0$, where $K(\alpha, \beta)$ and $K(\beta)$ indicate their dependence on the constants α, β.

Lemma 2.3. *Assume that* (B_1)–(B_3) *hold. If* $0 \le \gamma \le 1$, $0 \le \beta \le \alpha < 1 + \delta$, $0 < \alpha - \gamma \le 1$, *then, for any* $0 \le \tau < t + \Delta t \le t_0$, $0 \le \zeta \le T$,

$$\left\| A^{\gamma}(\zeta)(U(t+\Delta t, \tau) - U(t, \tau)) A^{-\beta}(\tau) \right\| \le C(\beta, \gamma, \alpha)(\Delta t)^{\alpha-\gamma} |t-\tau|^{\beta-\alpha}. \tag{2.11}$$

For more details about the theory of linear evolution system, operator semigroups, and fraction powers of operators, we can refer to [23–25].

In the sequel, we denote for brevity that $H_{\alpha} = D(A^{\alpha}(t_0))$ for some $t_0 > 0$, and $C_{\alpha} = C([-r, 0], H_{\alpha})$, the space of all continuous functions from $[-r, 0]$ into H_{α}. Suppose that $Z(t) : \Omega \to H_{\alpha}$, $t \ge -r$, is a continuous $\mathcal{F}_t$-adapted, H_{α}-valued stochastic process, we can associate with another process $Z_t : \Omega \to C_{\alpha}$, $t \ge 0$, by setting $Z_t(\theta)(\omega) = Z(t+\theta)(\omega)$, $\theta \in [-r, 0]$. Then we say that the process Z_t is generated by the process $Z(t)$. Let $MC_{\alpha}(p)$, $p > 2$, denote the space of all $\mathcal{F}_t$-measurable functions which belong to $L^p(\Omega, C_{\alpha})$; that is, $MC_{\alpha}(p)$, $p > 2$, is the space of all $\mathcal{F}_t$-measurable C_{α}-valued functions $\psi : \Omega \to C_{\alpha}$ with the norm $E\|\psi\|_{C_{\alpha}}^p = E(\sup_{\theta \in [-r,0]} \|A^{\alpha}(t_0)\psi(\theta)\|^p) < \infty$.

Now we end this section by stating the following result which is fundamental to the work of this note and can be proved by the similar method as that of [1, Proposition 4.15].

Lemma 2.4. *Let* $\Phi(t) : \Omega \to \mathcal{L}_2^0(K, H)$, $t \ge 0$ *be a predictable,* $\mathcal{F}_t$*-adapted process. If* $\Phi(t)k \in H_{\alpha}$, $t \ge 0$, *for arbitrary* $k \in K$, *and* $\int_0^t E\|\Phi(s)\|_{\mathcal{L}_2^0}^2 ds < +\infty$, $\int_0^t E\|A^{\alpha}(t_0)\Phi(s)\|_{\mathcal{L}_2^0}^2 ds < +\infty$, *then there holds*

$$A^{\alpha}(t_0) \int_0^t \Phi(s) dW(s) = \int_0^t A^{\alpha}(t_0) \Phi(s) dW(s). \tag{2.12}$$

3. Existence and Uniqueness

In this section we study the existence and uniqueness of mild solutions for (1.1). For this equation we assume that the following conditions hold (let $p > 2$).

(H_1) The function $k : [0, T] \times C_{\alpha} \to H_{\alpha}$ satisfies the following Lipschitz conditions: that is, there is a constant $L_0 > 0$ such that, for any ϕ_1, ϕ_2 and $\phi \in C_{\alpha}$, $t \in [0, T]$,

$$\begin{gathered} \|k(t, \phi_2) - k(t, \phi_1)\|_{\alpha}^p \le L_0 \|\phi_2 - \phi_1\|_{C_{\alpha}}^p, \\ \|k(t, \phi)\|_{\alpha}^p \le L_0 \|\phi\|_{C_{\alpha}}^p, \end{gathered} \tag{3.1}$$

and $3^{p-1} L_0 < 1$.

(H_2) The function $F : [0,T] \times C_\alpha \to H$ satisfies the following Lipschitz conditions: that is, there is a constant $L_1 > 0$ such that, for any $\phi_1, \phi_2 \in C_\alpha$ and $t \in [0,T]$,

$$\|F(t,\phi_2) - F(t,\phi_1)\|^p \le L_1 \|\phi_2 - \phi_1\|_{C_\alpha}^p. \tag{3.2}$$

(H_3) For function $G : [0,T] \times C_\alpha \to \mathcal{L}_2^0(K,H)$, there exists a constant $L_2 > 0$ such that

$$\|G(t,\phi_2) - G(t,\phi_1)\|_{\mathcal{L}_2^0}^p \le L_2 \|\phi_2 - \phi_1\|_{C_\alpha}^p \tag{3.3}$$

for any $\phi_1, \phi_2 \in C_\alpha$ and $t \in [0,T]$.

Under (H_2) and (H_3), we may suppose that there exists a constant $L_3 > 0$ such that

$$\|F(t,\phi)\|^p + \|G(t,\phi)\|_{\mathcal{L}_2^0}^p \le L_3 \left(\|\phi\|_{C_\alpha}^p + 1 \right) \tag{3.4}$$

for any $\phi \in C_\alpha$ and $t \in [0,T]$.

Similar to the deterministic situation we give the following definition of mild solutions for (1.1).

Definition 3.1. A continuous process $Z : [-r,T] \times \Omega \to H$ is said to be a mild solution of (1.1) if

(i) $Z(t)$ is measurable and $\mathcal{F}_t$-adapted for each $t \in [0,T]$;

(ii) $\int_0^T \|Z_s\|^p ds < \infty$, a.s.;

(iii) $Z(t)$ verifies the stochastic integral equation

$$\begin{aligned} Z(t) &= U(t,0)\left[\phi(0) - k(0,\phi)\right] + k(t,Z_t) \\ &\quad + \int_0^t U(t,s)F(s,Z_s)ds + \int_0^t U(t,s)G(s,Z_s)dW(s) \end{aligned} \tag{3.5}$$

on interval $[0,T]$, and $Z(\theta) = \phi(\theta)$ for $\theta \in [-r,0]$.

Next we prove the existence and uniqueness of mild solutions for (1.1).

Theorem 3.2. *Let $0 < \alpha < min\{(p-2)/2p, \delta\}$ and $p > 2/\delta$. Suppose that the assumptions (H_1)–(H_3) hold. Then there exists a unique (local) continuous mild solution to* (1.1) *for any initial value $\phi \in MC_\alpha(p)$.*

Proof. Denote by $\mathfrak{D}$ the Banach space of all the continuous processes $Z(t)$ which belong to the space $C([-r,T],L^p(\Omega,H))$ with $\|Z\|_{\mathfrak{D}}<\infty$, where

$$\|Z\|_{\mathfrak{D}} := \sup_{t\in[0,T]}\left(E\|Z_t\|_C^p\right)^{1/p},$$
$$E\|Z_t\|_C^p := E\left(\sup_{\theta\in[-r,0]}\|Z_t(\theta)\|^p\right). \tag{3.6}$$

Define the operator Ψ on $\mathfrak{D}$:

$$(\Psi Z)(t) = A^{\alpha}(t_0)U(t,0)\left[\phi(0)-k(0,\phi)\right] + A^{\alpha}(t_0)k\left(t,A^{-\alpha}(t_0)Z_t\right)$$
$$+\int_0^t A^{\alpha}(t_0)U(t,s)F\left(s,A^{-\alpha}(t_0)Z_s\right)ds+\int_0^t A^{\alpha}(t_0)U(t,s)G\left(s,A^{-\alpha}(t_0)Z_s\right)dW(s). \tag{3.7}$$

Then it is clear that to prove the existence of mild solutions to (1.1) is equivalent to find a fixed point for the operator Ψ. Next we will show by using Banach fixed point theorem that Ψ has a unique fixed point. We divide the subsequent proof into three steps.

Step 1. For arbitrary $Z\in\mathfrak{D}$, $(\Psi Z)(t)$ is continuous on the interval $[0,T]$ in the L^p-sense.

Let $0<t<T$ and $|h|$ be sufficiently small. Then for any fixed $Z\in\mathfrak{D}$, we have that

$$\begin{aligned}
E\|(\Psi Z)(t+h)-(\Psi Z)(t)\|^p \le{}& 4^{p-1}E\left\|A^{\alpha}(t_0)(U(t+h,0)-U(t,0))\left[\phi(0)-k\left(0,\phi\right)\right]\right\|^p\\
&+4^{p-1}E\|A^{\alpha}(t_0)[k(t+h,A^{\alpha}(t_0)Z_{t+h})-k(t,A^{\alpha}(t_0)Z_t)]\|^p\\
&+4^{p-1}E\left\|\int_0^{t+h}A^{\alpha}(t_0)U(t+h,s)F\left(s,A^{-\alpha}(t_0)Z_s\right)ds\right.\\
&\left.-\int_0^t A^{-\alpha}(t_0)U(t,s)F(s,A^{-\alpha}(t_0)Z_s)ds\right\|^p\\
&+4^{p-1}E\left\|\int_0^{t+h}A^{\alpha}(t_0)U(t+h,s)G\left(s,A^{-\alpha}(t_0)Z_s\right)dW(s)\right.\\
&\left.-\int_0^t A^{\alpha}(t_0)U(t,s)G(s,A^{-\alpha}(t_0)Z_s)dW(s)\right\|^p\\
:={}& I_1+I_2+I_3+I_4.
\end{aligned} \tag{3.8}$$

Thus, by Lemma 2.3 we get

$$\begin{aligned}
I_1 &= 4^{p-1}E\|A^{\alpha}(t_0)(U(t+h,0)-U(t,0))A^{-\beta}(t_0)A^{\beta}(t_0)\left[\phi(0)-k\left(0,\phi\right)\right]\|^p\\
&\le 4^{p-1}C^p|h|^{p(\beta-\alpha)}E\|A^{\beta}(t_0)[\phi(0)-k(0,\phi)]\|^p,
\end{aligned} \tag{3.9}$$

where $\beta > 0$ satisfies that $\alpha < \beta < 1$ and $C > 0$ are constant. From Condition (H_2) it follows that

$$
\begin{aligned}
I_3 &\le 8^{p-1} E\left\| \int_0^t A^\alpha(t_0)[U(t+h,s) - U(t,s)] F(s, A^{-\alpha}(t_0) Z_s) ds \right\|^p \\
&\quad + 8^{p-1} E\left\| \int_t^{t+h} A^\alpha(t_0) U(t+h,s) F(s, A^{-\alpha}(t_0) Z_s) ds \right\|^p \\
&= I_{31} + I_{32}.
\end{aligned} \tag{3.10}
$$

And by (2.8)–(2.10) one has that

$$
\begin{aligned}
I_{32} &\le 8^{p-1} E\left(\int_t^{t+h} \|A^\alpha(t_0) A^{-\beta}(t+h)\| \|A^\beta(t+h) U(t+h,s)\| \|F(s, A^{-\alpha}(t_0) Z_s)\| ds \right)^p \\
&\le 8^{p-1} (K(\alpha,\beta) K(\beta))^p E\left(\int_t^{t+h} \frac{1}{(t+h-s)^\beta} \|F(s, A^{-\alpha}(t_0) Z_s)\| ds \right)^p \\
&\le 8^{p-1} (K(\alpha,\beta) K(\beta))^p \left(\int_t^{t+h} \frac{1}{(t+h-s)^{q\beta}} ds \right)^{p/q} E \int_t^{t+h} \|F(s, A^{-\alpha}(t_0) Z_s)\|^p ds \\
&\le 8^{p-1} (K(\alpha,\beta) K(\beta))^p \left(\frac{1}{1-q\beta} \right)^{p-1} L_3 h^{(1-\beta)p} \left(1 + \|Z\|_{\mathfrak{D}}^p \right),
\end{aligned} \tag{3.11}
$$

where $q > 0$ solves $1/p + 1/q = 1$, and

$$
\begin{aligned}
I_{31} &= 8^{p-1} E\Bigg\| \int_0^t A^\alpha(t_0) \Bigg[e^{-(t+h-s)A(t+h)} + \int_s^{t+h} e^{-(t+h-\tau)A(\tau)} R(\tau,s) d\tau \\
&\qquad\qquad - e^{-(t-s)A(t)} - \int_s^t e^{-(t-\tau)A(\tau)} R(\tau,s) d\tau \Bigg] F(s, A^{-\alpha}(t_0) Z_s) ds \Bigg\|^p \\
&\le 8^{p-1} \Bigg(\int_0^t \Bigg\| A^\alpha(t_0) \Bigg[e^{-(t+h-s)A(t+h)} + \int_s^{t+h} e^{-(t+h-\tau)A(\tau)} R(\tau,s) d\tau \\
&\qquad\qquad - e^{-(t-s)A(t)} - \int_s^t e^{-(t-\tau)A(\tau)} R(\tau,s) d\tau \Bigg] \Bigg\|^q ds \Bigg)^{p/q} \\
&\quad \times E\left(\int_0^t \|F(s, A^{-\alpha}(t_0) Z_s)\|^p ds \right) \\
&:= 8^{p-1} I_{31}'^{p/q} E\left(\int_0^t \|F(s, A^{-\alpha}(t_0) Z_s)\|^p ds \right) \\
&\le 8^{p-1} L_3 I_{31}'^{p/q} T \left(1 + \|Z\|_{\mathfrak{D}}^p \right),
\end{aligned} \tag{3.12}
$$

while

$$
\begin{aligned}
I'_{31} &= \int_0^t \left\| A^{\alpha}(t_0)\left[e^{-(t+h-s)A(t+h)} + \int_s^{t+h} e^{-(t+h-\tau)A(\tau)} R(\tau,s)\,d\tau \right.\right. \\
&\qquad\qquad \left.\left. - e^{-(t-s)A(t)} - \int_s^t e^{-(t-\tau)A(\tau)} R(\tau,s)\,d\tau \right] \right\|^q ds \\
&\le 2^{q-1}\int_0^t \left\| A^{\alpha}(t_0)\left[e^{-(t+h-s)A(t+h)} - e^{-(t-s)A(t)} \right] \right\|^q ds \\
&\quad + 2^{q-1}\int_0^t \left\| A^{\alpha}(t_0)\int_s^t \left[e^{-(t+h-\tau)A(\tau)} - e^{-(t-\tau)A(\tau)} \right] R(\tau,s)\,d\tau \right. \\
&\qquad\qquad \left. + A^{\alpha}(t_0)\left[\int_t^{t+h} e^{-(t+h-\tau)A(\tau)} R(\tau,s)\,d\tau \right] \right\|^q ds \\
&\le 2^{q-1}\int_0^{t-\eta} \left\| A^{\alpha}(t_0)\left[e^{-(t+h-s)A(t+h)} - e^{-(t-s)A(t)} \right] \right\|^q ds \\
&\quad + 2^{q-1}\int_{t-\eta}^t \left\| A^{\alpha}(t_0)\left[e^{-(t+h-s)A(t+h)} - e^{-(t-s)A(t)} \right] \right\|^q ds \\
&\quad + 8^{q-1}\int_0^t \left(\int_s^{t-\eta} \left\| A^{\alpha}(t_0)\left[e^{-(t+h-\tau)A(\tau)} - e^{-(t-\tau)A(\tau)} \right] R(\tau,s) \right\| d\tau \right)^q ds \\
&\quad + 8^{q-1}\int_0^t \left(\int_{t-\eta}^t \left\| A^{\alpha}(t_0)\left[e^{-(t+h-\tau)A(\tau)} - e^{-(t-\tau)A(\tau)} \right] R(\tau,s) \right\| d\tau \right)^q ds \\
&\quad + 4^{q-1}\int_0^t \left\| \int_t^{t+h} A^{\alpha}(t_0) e^{-(t+h-\tau)A(\tau)} R(\tau,s)\,d\tau \right\|^q ds,
\end{aligned}
\tag{3.13}
$$

where $\eta > 0$ is very small. Since $A(t)e^{-\tau A(s)}$ is uniformly continuous in (t,τ,s) for $0 \le t \le T$, $m \le \tau \le T$ and $0 \le s \le T$, where m is any positive number (see [23, 25]), we deduce that

$$
\begin{aligned}
I'_{31} &\le 2^{q-1}(K(\alpha,\beta)K(\beta))^q \epsilon^q (T-\eta) \\
&\quad + \frac{4^{q-1}}{(1-q\beta)^q}(K(\alpha,\beta)K(\beta))^q \left[\left((h+\eta)^{1-q\beta} - h^{1-q\beta} \right) + \eta^{1-q\beta} \right] \\
&\quad + \frac{8^{q-1}}{(1+\delta q)^q}\left(K(\alpha,\beta)K(\beta)\left(\frac{C}{\delta}\right) \right)^q \left[(T+\eta)^{1+\delta q} + \eta^{1+\delta q} \right] \epsilon^q \\
&\quad + 8^{q-1}\left(\frac{K(\alpha,\beta)K(\beta)C}{(1-\beta)(1+(\delta-1)q)} \right)^q \left[(h+\eta)^{1-\beta} + h^{1-\beta} \right]^q \left[T^{1+(\delta-1)q} + \eta^{1+(\delta-1)q} \right] \\
&\quad + 4^{q-1}\left(\frac{K(\alpha,\beta)K(\beta)C}{(1-\beta)(1+(\delta-1)q)} \right)^q h^{(1-\beta)q} T^{1+(\delta-1)q}.
\end{aligned}
\tag{3.14}
$$

In a similar way, we have that

$$
\begin{aligned}
I_4 &\le 6^{p-1} E\left\| \int_0^t A^{-\alpha}(t_0)[U(t+h,s) - U(t-s)] G\left(s, A^{-\alpha}(t_0) Z_s\right) dW(s) \right\|^p \\
&\quad + 6^{p-1} E\left\| \int_t^{t+h} A^{-\alpha}(t_0) U(t+h,s) G\left(s, A^{-\alpha}(t_0) Z_s\right) dW(s) \right\|^p \\
&:= I_{41} + I_{42}.
\end{aligned}
\tag{3.15}
$$

By virtue of Condition (H_3) and by using Lemma 2.2, we infer that

$$
\begin{aligned}
I_{42} &\le 8^{p-1} E\left(\int_t^{t+h} \left\| A^{\alpha}(t_0) A^{-\beta}(t+h) \right\| \left\| A^{\beta}(t+h) U(t+h,s) \right\| \left\| G\left(s, A^{-\alpha}(t_0) Z_s\right) \right\|_{\mathcal{L}_2}^2 ds \right)^{p/2} \\
&\le 8^{p-1} (K(\alpha,\beta) K(\beta))^p E\left(\int_t^{t+h} \frac{1}{(t+h-s)^{2\beta}} \left\| G(s, A^{-\alpha}(t_0) Z_s) \right\|_{\mathcal{L}_2^0}^2 ds \right)^{p/2} \\
&\le 8^{p-1} (K(\alpha,\beta) K(\beta))^p \left(\int_t^{t+h} \frac{1}{(t+h-s)^{2p\beta/(p-2)}} ds \right)^{(p-2)/2} E\int_t^{t+h} \left\| \left\| G(s, A^{-\alpha}(t_0) Z_s) \right\|_{\mathcal{L}_2^0}^2 \right\|^p ds \\
&\le 8^{p-1} (K(\alpha,\beta) K(\beta))^p \left(\frac{p-2}{p-2-2p\beta} \right)^{(p-2)/2} L_3 h^{(p-2)/(p-2-2p\beta)+1} \left(1 + \|Z\|_{\mathfrak{D}}^p \right), \\
I_{41} &= 8^{p-1} E\Bigg\| \int_0^t A^{\alpha}(t_0) \Bigg[e^{-(t+h-s)A(t+h)} + \int_s^{t+h} e^{-(t+h-\tau)A(\tau)} R(\tau,s) d\tau \\
&\qquad - e^{-(t-s)A(t)} - \int_s^t e^{-(t-\tau)A(\tau)} R(\tau,s) d\tau \Bigg] G(s, A^{-\alpha}(t_0) Z_s) dW(s) \Bigg\|^p \\
&\le 8^{p-1} C_p E\Bigg(\int_0^t \Bigg\| A^{\alpha}(t_0) \Bigg[e^{-(t+h-s)A(t+h)} + \int_s^{t+h} e^{-(t+h-\tau)A(\tau)} R(\tau,s) d\tau \\
&\qquad - e^{-(t-s)A(t)} - \int_s^t e^{-(t-\tau)A(\tau)} R(\tau,s) d\tau \Bigg] G(s, A^{-\alpha}(t_0) Z_s) \Bigg\|_{\mathcal{L}_2^0}^2 ds \Bigg)^{p/2}
\end{aligned}
$$

$$\leq 8^{p-1}C_p\left(\int_0^t\left\|A^\alpha(t_0)\left[e^{-(t+h-s)A(t+h)}+\int_s^{t+h}e^{-(t+h-\tau)A(\tau)}R(\tau,s)d\tau\right.\right.\right.$$

$$\left.\left.\left.-e^{-(t-s)A(t)}-\int_s^t e^{-(t-\tau)A(\tau)}R(\tau,s)d\tau\right]\right\|^{p/(p-2)}ds\right)^{(p-2)/2}$$

$$\times E\left(\int_0^t\|G(s,A^{-\alpha}(t_0)Z_s)\|^p ds\right)$$

$$:= 8^{p-1}C_p I'^{(p-2)/2}_{41}E\left(\int_0^t\|G(s,A^{-\alpha}(t_0)Z_s)\|^p ds\right)$$

$$\leq 8^{p-1}C_p I'^{(p-2)/2}_{41}L_3T\left(1+\|Z\|^p_{\mathfrak{D}}\right),\tag{3.16}$$

while

$$I'_{41}=\int_0^t\left\|A^\alpha(t_0)\left[e^{-(t+h-s)A(t+h)}+\int_s^{t+h}e^{-(t+h-\tau)A(\tau)}R(\tau,s)d\tau\right.\right.$$

$$\left.\left.-e^{-(t-s)A(t)}-\int_s^t e^{-(t-\tau)A(\tau)}R(\tau,s)d\tau\right]\right\|^{p/(p-2)}ds$$

$$\leq 2^{2/(p-2)}\int_0^t\left\|A^\alpha(t_0)\left[e^{-(t+h-s)A(t+h)}-e^{-(t-s)A(t)}\right]\right\|^{p/(p-2)}ds$$

$$+2^{2/(p-2)}\int_0^t\left\|A^\alpha(t_0)\int_s^t\left[e^{-(t+h-\tau)A(\tau)}-e^{-(t-\tau)A(\tau)}\right]R(\tau,s)d\tau\right.$$

$$\left.+A^\alpha(t_0)\left[\int_t^{t+h}e^{-(t+h-\tau)A(\tau)}R(\tau,s)d\tau\right\|^{p/(p-2)}ds$$

$$\leq 2^{2/(p-2)}\int_0^{t-\eta}\left\|A^\alpha(t_0)\left[e^{-(t+h-s)A(t+h)}-e^{-(t-s)A(t)}\right]\right\|^{p/(p-2)}ds$$

$$+2^{2/(p-2)}\int_{t-\eta}^t\left\|A^\alpha(t_0)\left[e^{-(t+h-s)A(t+h)}-e^{-(t-s)A(t)}\right]\right\|^{p/(p-2)}ds$$

$$+8^{2/(p-2)}\int_0^t\left(\int_s^{t-\eta}\left\|A^\alpha(t_0)\left[e^{-(t+h-\tau)A(\tau)}-e^{-(t-\tau)A(\tau)}\right]R(\tau,s)\right\|d\tau\right)^{p/(p-2)}ds$$

$$
+ 8^{2/(p-2)} \int_0^t \left\| A^{\alpha}(t_0) \int_{t-\eta}^{t} \left[e^{-(t+h-\tau)A(\tau)} - e^{-(t-\tau)A(\tau)} \right] R(\tau, s) \right\| d\tau \Big)^{p/(p-2)} ds
$$

$$
+ 4^{2/(p-2)} \int_0^t \left\| \int_t^{t+h} A^{\alpha}(t_0) e^{-(t+h-\tau)A(\tau)} R(\tau, s) d\tau \right\|^{p/(p-2)} ds. \tag{3.17}
$$

Again by the uniform continuity of $A(t)^{-\tau A(s)}$ and (2.9) and (2.10), we can compute that

$$
\begin{aligned}
I'_{41} \leq{}& 2^{q_1-1} (\alpha, \beta)^{q_1} K(\beta)^{q_1} \epsilon^{q_1} (T - \eta) \\
&+ \frac{4^{q_1-1}}{(1 - q_1\beta)^{q_1}} (K(\alpha, \beta) K(\beta))^{q_1} \left[\left((h + \eta)^{1-q_1\beta} - h^{1-q_1\beta} \right) + \eta^{1-q_1\beta} \right] \\
&+ \frac{8^{q_1-1}}{(1 + \delta q_1)^{q_1}} \left(K(\alpha, \beta) K(\beta) \left(\frac{C}{\delta} \right) \right)^{q_1} \left[(T + \eta)^{1+\delta q_1} + \eta^{1+\delta q_1} \right] \epsilon^{q_1} \\
&+ 8^{q_1-1} \left(\frac{K(\alpha, \beta) K(\beta) C}{(1 - \beta)(1 + (\delta - 1) q_1)} \right)^{q_1} \left[(h + \eta)^{1-\beta} + h^{1-\beta} \right]^{q_1} \left[T^{1+(\delta-1)q_1} + \eta^{1+(\delta-1)q_1} \right] \\
&+ 4^{q_1-1} \left(\frac{K(\alpha, \beta) K(\beta) C}{(1 - \beta)(1 + (\delta - 1) q_1)} \right)^{q_1} h^{(1-\beta)q_1} T^{1+(\delta-1)q_1},
\end{aligned} \tag{3.18}
$$

where $q_1 = p/(p - 2)$.

The above arguments show that I_1, $I_3 = I_{31} + I_{32}$ and $I_4 = I_{41} + I_{42}$ are all tend to 0 as $|h| \to 0$ and $\eta \to 0$, and I_2 also clearly tends to 0 from Condition (H_1). Therefore, $(\Psi Z)(t)$ is continuous on the interval $[0, T]$ in the L^p-sense.

Step 2. We prove that $\Psi(\mathfrak{D}) \subset \mathfrak{D}$.

To this end, let $Z \in \mathfrak{D}$. Then we have that

$$
\begin{aligned}
E\|(\Psi Z)_t\|_C^p \leq{}& 4^{p-1} E \sup_{-r\leq\theta\leq 0} \| A^{\alpha}(t_0) U(t + \theta, 0) [\phi(0) - k(0, \phi)] \|^p \\
&+ 4^{p-1} E \sup_{-r\leq\theta\leq 0} \| A^{\alpha}(t_0) k(t + \theta, A^{-\alpha}(t_0) Z_{t+\theta}) \|^p \\
&+ 4^{p-1} E \sup_{-r\leq\theta\leq 0} \left\| \int_0^{t+\theta} A^{\alpha}(t_0) U(t + \theta, s) F(s, A^{-\alpha}(t_0) Z_s) ds \right\|^p \\
&+ 4^{p-1} E \sup_{-r\leq\theta\leq 0} \left\| \int_0^{t+\theta} A^{\alpha}(t_0) U(t + \theta, s) G(s, A^{-\alpha}(t_0) Z_s) dW(s) \right\|^p \\
:={}& I_5 + I_6 + I_7 + I_8.
\end{aligned} \tag{3.19}
$$

Again by Lemma 2.3, we get that

$$\begin{aligned} I_5 &\le 4^{p-1} E \sup_{-r\le\theta\le 0} \Big\| A^{\alpha}(t_0)(U(t+\theta,0)-U(0,0))A^{-\beta}(t_0)A^{\beta}(t_0)[\phi(0)-k(0,\phi)] \\ &\qquad -A^{\alpha}(t_0)[\phi(0)-k(0,\phi)]\Big\|^p \\ &\le 8^{p-1}\left(E \sup_{-r\le\theta\le 0} \Big\| A^{\alpha}(t_0)(U(t+\theta,0)-U(0,0))A^{-\beta}(t_0)A^{\beta}(t_0)[\phi(0)-k(0,\phi)]\Big\|^p \right. \\ &\qquad \left. +E\| A^{\alpha}(t_0)[\phi(0)-k(0,\phi)]\|^p \right) \\ &\le 8^{p-1}\left(C^p T^{p(\beta-\alpha)} E\|[\phi(0)-k(0,\phi)]\|_{\alpha}^p + E\|[\phi(0)-k(0,\phi)]\|_{\alpha}^p \right). \end{aligned} \tag{3.20}$$

By Condition (H_1) one easily has

$$I_6 \le 4^{p-1} L_0 \|Z\|_{\mathfrak{D}}^p. \tag{3.21}$$

And (2.9) and (2.10) imply that

$$\begin{aligned} I_7 &\le 4^{p-1} E \sup_{-r\le\theta\le 0} \left\| \int_0^{t+\theta} A^{\alpha}(t_0)U(t+\theta,s)F(s,A^{-\alpha}(t_0)Z_s)ds \right\|^p \\ &\le 4^{p-1} E \sup_{-r\le\theta\le 0} \left(\int_0^{t+\theta} \left\| A^{\alpha}(t_0)A^{-\beta}(t+\theta)\right\| \left\| A^{\beta}(t+\theta)U(t+\theta,s)\right\| \left\| F(s,A^{-\alpha}(t_0)Z_s)\right\| ds \right)^p \\ &\le 4^{p-1}(K(\alpha,\beta)K(\beta))^p E \sup_{-r\le\theta\le 0} \left(\int_0^{t+\theta} \frac{1}{(t+\theta-s)^{\beta}} \|F(s,A^{-\alpha}(t_0)Z_s)\| ds \right)^p \\ &\le 4^{p-1}(K(\alpha,\beta)K(\beta))^p E \sup_{-r\le\theta\le 0} \left(\int_0^{t+\theta} \frac{1}{(t+\theta-s)^{q\beta}} ds \right)^{p/q} \left(\int_0^{t+\theta} \|F(s,A^{-\alpha}(t_0)Z_s)\|^p ds \right) \\ &\le 4^{p-1} L_3 (K(\alpha,\beta)K(\beta))^p \left(\frac{p-1}{p-p\beta-1} \right)^{p-1} T^{p-p\beta}\left(1+\|Z\|_{\mathfrak{D}}^p\right). \end{aligned} \tag{3.22}$$

From the inequality

$$\int_{\sigma}^{t} (t-s)^{\alpha-1}(s-\sigma)^{-\alpha} ds = \frac{\pi}{\sin \pi\alpha}, \quad \text{for } \sigma \le s \le t,\ \alpha \in (0,1), \tag{3.23}$$

established in [26], it follows that

$$I_8 = 3^{p-1}\left|\frac{\sin\pi\gamma}{\pi}\right|^p E\sup_{-r\le\theta\le 0}\left\|\int_0^{t+\theta} A^{\alpha}(t_0)U(t+\theta,s)(t+\theta-s)^{\gamma-1}R(s)ds\right\|^p, \tag{3.24}$$

where

$$R(s) = \int_0^s (s-\sigma)^{-\gamma}U(s,\sigma)G(\sigma, A^{-\alpha}(t_0)Z_{\sigma})dW(\sigma), \tag{3.25}$$

and $0 < \gamma < (p-2)/2p$. Thus,

$$\begin{aligned} I_8 &\le 3^{p-1}\left(K(\alpha,\beta)K(\beta)\left|\frac{\sin\pi\gamma}{\pi}\right|\right)^p E\sup_{-r\le\theta\le 0}\left\|\int_0^{t+\theta}(t+\theta-s)^{\gamma-\beta-1}R(s)ds\right\|^p \\ &\le 3^{p-1}L_3C_p\left(K(\alpha,\beta)K(\beta)M_0\right)^p\left(\frac{p-1}{p(\beta-\gamma)-1}\right)^{p-1}T^{p(\beta-\gamma)-1} \\ &\quad\cdot\frac{1}{1-2p\gamma/(p-2)}T^{4-2p\gamma/(p-2)}\left(1+\|Z\|_{\mathfrak{D}}^p\right), \end{aligned} \tag{3.26}$$

where $M_0 = \sup_{0\le\sigma\le s\le T}\|U(s,\sigma)\|$. Hence $\|\Psi Z\|_{\mathfrak{D}}^p < \infty$ and so $\Psi(\mathfrak{D}) \subset \mathfrak{D}$.

Step 3. It remains to verify that Ψ is a contraction on $\mathfrak{D}$.

Suppose that Z_1, $Z_2 \in \mathfrak{D}$, then for any fixed $t \in [0,T]$,

$$\begin{aligned} &E\|(\Psi Z_2)_t - (\Psi Z_1)_t\|_C^p \\ &\quad\le 3^{p-1}E\sup_{-r\le\theta\le 0}\left\|A^{\alpha}(t_0)k(t+\theta, A^{-\alpha}(t_0)Z_{2,t+\theta}) - A^{\alpha}(t_0)k\left(t+\theta, A^{-\alpha}(t_0)Z_{1,t+\theta}\right)\right\|^p \\ &\qquad+3^{p-1}E\sup_{-r\le\theta\le 0}\left\|\int_0^{t+\theta}A^{\alpha}(t_0)U(t+\theta,s)\left[F\left(s,A^{-\alpha}(t_0)Z_2(s)\right)-F\left(s,A^{-\alpha}(t_0)Z_1(s)\right)\right]ds\right\|^p \\ &\qquad+3^{p-1}E\sup_{-r\le\theta\le 0}\left\|\int_0^{t+\theta}A^{\alpha}(t_0)U(t+\theta,s)\left[G\left(s,A^{-\alpha}(t_0)Z_2(s)\right)-G\left(s,A^{-\alpha}(t_0)Z_1(s)\right)\right]dW(s)\right\|^p \\ &\quad:= I_9 + I_{10} + I_{11}. \end{aligned} \tag{3.27}$$

Clearly,

$$
\begin{aligned}
I_9 &\leq 3^{p-1} L_0 \|Z_2 - Z_1\|_{\mathfrak{D}}^p, \\
I_{10} &\leq \left(K(\alpha,\beta)K(\beta)\right)^p E \sup_{-r\leq\theta\leq 0} \left(\int_0^{t+\theta} \frac{1}{(t+\theta-s)^{q\beta}} ds \right)^{p/q} \\
&\quad \times \left(\int_0^{t+\theta} \left\| F\left(s, A^{-\alpha}(t_0) Z_2(s)\right) - F\left(s, A^{-\alpha}(t_0) Z_1(s)\right) \right\|^p ds \right) \\
&\leq L_1 \left(K(\alpha,\beta)K(\beta)\right)^p \left(\frac{p-1}{p-1+p\beta} \right)^{p-1} T^{p+p\beta} \|Z_2 - Z_1\|_{\mathfrak{D}}^p.
\end{aligned}
\tag{3.28}
$$

Next, let

$$
R_1(u) = \int_0^s (s-\sigma)^{-\gamma} U(s,\sigma) \left[G\left(s, A^{-\alpha}(t_0) Z_2(s)\right) - G\left(s, A^{-\alpha}(t_0) Z_1(s)\right) \right] dW(\sigma), \tag{3.29}
$$

then there holds that

$$
\begin{aligned}
I_{11} &\leq 3^{p-1} \left(K(\alpha,\beta)K(\beta) \left| \frac{\sin \pi\gamma}{\pi} \right| \right)^p E \sup_{-r\leq\theta\leq 0} \left\| \int_0^{t+\theta} (t+\theta-s)^{\gamma-\beta-1} R_1(s) ds \right\|^p \\
&\leq 3^{p-1} L_2 C_p \left(K(\alpha,\beta)K(\beta)M_0\right)^p \left(\frac{p-1}{p(\gamma-\beta)-1} \right)^{p-1} T^{p(\gamma-\beta)-1} \cdot \frac{1}{1-2p\gamma/(p-2)} T^{4-2p\gamma/(p-2)} \\
&\quad \cdot \|Z_2 - Z_1\|_{\mathfrak{D}}^p.
\end{aligned}
\tag{3.30}
$$

Therefore,

$$
\|\Psi(Z_2) - \Psi(Z_1)\|_{\mathfrak{D}}^p \leq \Theta(T) \|Z_2 - Z_1\|_{\mathfrak{D}}^p \tag{3.31}
$$

with

$$
\begin{aligned}
\Theta(T) = \frac{1}{1-3^{p-1}L_0} \Bigg[& \left(K(\alpha,\beta)K(\beta)\right)^p \left(\frac{p-1}{p-1+p\beta} \right)^{p-1} T^{p+p\beta} \\
& + 3^{p-1} L_2 C_p \left(K(\alpha,\beta)K(\beta)M_0 \left| \frac{\sin \pi\gamma}{\pi} \right| \right)^p \left(\frac{p-1}{p(\gamma-\beta)-1} \right)^{p-1} T^{p(\gamma-\beta)-1} \\
& \cdot \frac{1}{1-(2p\gamma/(p-2))} T^{4-2p\gamma/(p-2)} \Bigg].
\end{aligned}
\tag{3.32}
$$

Then we can take a suitable $0 < T_1 < T$ sufficient small such that $\Theta(T_1) < 1$, and hence Ψ is a contraction on $\mathfrak{D}_{T_1}$ ($\mathfrak{D}_{T_1}$ denotes $\mathfrak{D}$ with T substituted by T_1). Thus, by the well-known

Banach fixed point theorem we obtain a unique fixed point $Z^* \in \mathfrak{D}_{T_1}$ for operator Ψ, and hence $Z(t) = A^{-\alpha}(t_0)Z^*(t)$ is a mild solution of (1.1). This procedure can be repeated to extend the solution to the entire interval $[0,T]$ in finitely many similar steps, thereby completing the proof for the existence and uniqueness of mild solutions on the whole interval $[0,T]$. □

For the globe existence of mild solutions for (1.1), it is easy to prove the following result.

Theorem 3.3. *Suppose that the family $A(t)_{t\geq 0}$ satisfies (B_1)–(B_3) on interval $[0,+\infty)$ such that $U(t,s)$ is defined for all $0 \leq s \leq t < \infty$. Let the functions $k : [0,+\infty)\times C_\alpha \to H_\alpha$, $F : [0,+\infty)\times C_\alpha \to H$ and $G : [0,+\infty) \times C_\alpha \to \mathcal{L}_2^0(K,H)$ satisfy the Assumptions (H_1)–(H_3), respectively. Then there exists a unique, global, continuous solution $Z : [-r,+\infty) \times \Omega \to H$ to (1.1) for any initial value $\phi \in MC_\alpha(p)$.*

Proof. Since T is arbitrary in the proof of the previous theorem, this assertion follows immediately. □

4. Exponential Stability

Now, we consider the stability result of mild solutions to (1.1). For this purpose we need to assume further that the family $A(t)_{t\geq 0}$ verifies additionally the following.

(B_4) $\sup_{0<t,s<\infty}\|A(t)A^{-1}(s)\| < \infty$, and there exists a closed operator $A(\infty)$ with bounded inverse and domain $D(A)$ such that

$$\left\|(A(t) - A(\infty))A^{-1}(0)\right\| \longrightarrow 0 \tag{4.1}$$

as $t \to +\infty$. Then the following inequalities are true:

$$\begin{gathered}\|U(t,s)\| \leq Me^{-\omega(t-s)},\\ \|A^\alpha(t)U(t,s)\| \leq M_1 e^{-\omega(t-s)}(t-s)^{-\alpha},\\ \|A^\alpha(t_0)U(t,s)\| \leq M_1 e^{-\omega(t-s)}(t-s)^{-\alpha}\end{gathered} \tag{4.2}$$

for all $t \geq s \geq 0$ (see [23, 25]).

Theorem 4.1. *Let the functions $k : [0,+\infty) \times C_\alpha \to H_\alpha$, $F : [0,+\infty) \times C_\alpha \to H$, $G : [0,+\infty) \times C_\alpha \to \mathcal{L}_2^0(K,H)$ satisfy the Lipschitz conditions (H_1)–(H_3), respectively. Furthermore, assume that there exist nonnegative real numbers $N_i \geq 0$ and continuous functions $c_i : [0,+\infty) \to \mathbb{R}^+$ with $c_i(t) \leq P_i e^{-\omega t}$ $(i = 0,1,2)$, $P_i > 0$, such that*

$$\begin{gathered}E\|k(t,Z_t)\|_{C_\alpha}^p \leq N_0 E\|Z_t\|_{C_\alpha}^p + c_0(t), \quad t \geq 0,\\ E\|F(t,Z_t)\|^p \leq N_1 E\|Z_t\|_{C_\alpha}^p + c_1(t), \quad t \geq 0,\\ E\|G(t,Z_t)\|_{\mathcal{L}_2^0}^p \leq N_2 E\|Z_t\|_{C_\alpha}^p + c_1(t), \quad t \geq 0\end{gathered} \tag{4.3}$$

for any mild solution $Z(t)$ of (1.1). If the constants N_1 and N_2 are small enough such that $\omega - K_3/(1-4^{p-1}N_0) > 0$ with K_3 determined by (4.13) below, then the solution is (the pth moment) exponentially stable. In other words, there exist positive constants $\epsilon > 0$ and $K(p,\epsilon,\phi) > 0$ such that, for each $t \geq 0$,

$$E\|Z_t\|_{C_\alpha}^p \leq K(p,\epsilon,\phi)e^{-et}. \tag{4.4}$$

Proof. Let $Z(t)$ be a mild solution of (1.1), then

$$\begin{aligned}
E\|Z_t\|_{C_\alpha}^p &= E\left(\sup_{-r\leq\theta\leq 0}\|Z(t+\theta)\|_\alpha^p\right)\\
&\leq 4^{p-1}E\left(\sup_{-r\leq\theta\leq 0}\|U(t+\theta,0)[\phi(0)-k(0,\phi)]\|_\alpha^p\right)\\
&\quad + 4^{p-1}E\left(\sup_{-r\leq\theta\leq 0}\|k(t+\theta,Z_{t+\theta})\|_\alpha^p\right)\\
&\quad + 4^{p-1}E\left(\sup_{-r\leq\theta\leq 0}\left\|\int_0^{t+\theta}U(t+\theta,s)F(s,Z_s)ds\right\|_\alpha^p\right)\\
&\quad + 4^{p-1}E\left(\sup_{-r\leq\theta\leq 0}\left\|\int_0^{t+\theta}U(t+\theta,s)G(s,Z_s)dW(s)\right\|_\alpha^p\right)\\
&:= I_{12}+I_{13}+I_{14}+I_{15},\\
I_{12} &\leq 4^{p-1}E\left(\sup_{-r\leq\theta\leq 0}M_1^p e^{-p\omega(t+\theta)}(t+\theta)^{-p\alpha}\|[\phi(0)-k(0,\phi)]\|^p\right)\\
&\leq 4^{p-1}M_1^p e^{-p\omega(t-r)}(t-r)^{-p\alpha}E\|[\phi(0)-k(0,\phi)]\|^p\\
&\leq 4^{p-1}M_1^p e^{p\omega r}r^{-p\alpha}E\|[\phi(0)-k(0,\phi)]\|^p\cdot e^{-\omega t}\\
&:= K_0\cdot e^{-\omega t},\\
I_{13} &\leq 4^{p-1}\left(N_0 E\|Z_t\|_{C_\alpha}^p + c_0(t)\right),\\
I_{14} &\leq 4^{p-1}M_1^p E\left(\sup_{-r\leq\theta\leq 0}\left(\int_0^{t+\theta}(t+\theta-s)^{-\alpha}e^{-\omega(t+\theta-s)}\|F(s,Z_s)\|ds\right)^p\right)\\
&= 4^{p-1}M_1^p E\left(\sup_{-r\leq\theta\leq 0}\left(\int_0^{t+\theta}(t+\theta-s)^{-\alpha}e^{-((p-1)\omega/p)(t+\theta-s)}e^{-(\omega/p)(t+\theta-s)}\|F(s,Z_s)\|ds\right)^p\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq 4^{p-1} M_1^p E\left(\sup_{-r\leq\theta\leq 0}\left(\int_0^{t+\theta}(t+\theta-s)^{-(p/(p-1))\alpha}e^{-\omega(t+\theta-s)}ds\right)^{p-1}\right.\\
&\qquad\qquad \left.\times\left(\int_0^{t+\theta}e^{-\omega(t+\theta-s)}\|F(s,Z_s)\|^p ds\right)\right)\\
&\leq 4^{p-1}M_1^p e^{\omega r}\left(\int_0^{+\infty}(t+\theta-s)^{-(p/(p-1))\alpha}e^{-\omega(t+\theta-s)}ds\right)^{p-1}E\left(\int_0^t e^{-\omega(t-s)}\|F(s,Z_s)\|^p ds\right)\\
&\leq 4^{p-1}M_1^p e^{\omega r}\left(\Gamma\left(1-\frac{p}{p-1}\alpha\right)\omega^{(p/(p-1))\alpha-1}\right)^{p-1}\cdot\int_0^t e^{-\omega(t-s)}\left[N_1 E\|Z_s\|_{C_\alpha}^p + c_1(s)\right]ds\\
&:= K_1\cdot\int_0^t e^{-\omega(t-s)}\left[N_1 E\|Z_s\|_{C_\alpha}^p + c_1(s)\right]ds.
\end{aligned}
\tag{4.5}
$$

For I_{15} we have that

$$
I_{15} = 4^{p-1}E\left(\sup_{-r\leq\theta\leq 0}\left\|\int_0^{t+\theta}A^\alpha(t_0)U(t+\theta,s)(t+\theta-s)^{\gamma-1}R_2(s)ds\right\|^p\right),
\tag{4.6}
$$

where

$$
R_2(u) = \int_0^s (s-\sigma)^{-\gamma}U(s,\sigma)G(s,Z_s)dW(\sigma)
\tag{4.7}
$$

with γ satisfying $\alpha + 1/p < \gamma < 1/2$. Then,

$$
\begin{aligned}
I_{15} &\leq 4^{p-1}M_1^p\left|\frac{\sin\pi\gamma}{\pi}\right|^p E\left(\sup_{-r\leq\theta\leq 0}\left[\int_0^{t+\theta}(t+\theta-s)^{q(\gamma-\alpha-1)}e^{-\omega(t+\theta-s)}ds\right]^{p-1}\right.\\
&\qquad\qquad\left.\cdot\int_0^{t+\theta}e^{-\omega(t+\theta-s)}\|R_2(s)\|^p ds\right)\\
&\leq 4^{p-1}e^{\omega r}M_1^p\left[\Gamma\left(1-\frac{p}{p-1}(1+\alpha-\gamma)\right)\omega^{(p/(p-1))(1+\alpha-\gamma)-1}\right]^{p-1}\\
&\quad\cdot\int_0^t e^{-\omega(t-s)}E\|R_2(s)\|^p ds\Big).
\end{aligned}
\tag{4.8}
$$

Since, by Lemma 2.2,

$$\begin{aligned}
&\int_0^t e^{-\omega(t-s)} E\|R_2(s)\|^p ds \\
&\quad = \int_0^t e^{-\omega(t-s)} E\left\| \int_0^s (s-\sigma)^{-\gamma} U(s,\sigma) G(s,Z_s) dW(\sigma) \right\|^p ds \\
&\quad \le C_p M^p \int_0^t e^{-\omega(t-s)} E\left(\int_0^s (s-\sigma)^{-2\gamma} e^{-2\omega(s-\sigma)} \|G(s,Z_s)\|^2_{\mathcal{L}_2^0} d\sigma \right)^{p/2} ds \\
&\quad \le C_p M^p E \int_0^t \left(\int_0^s (s-\sigma)^{-2\gamma} e^{-2\omega(1/q)(s-\sigma)} e^{-2\omega(1/p)(s-\sigma)} \|G(s,Z_s)\|^2_{\mathcal{L}_2^0} d\sigma \right)^{p/2} ds,
\end{aligned} \tag{4.9}$$

and the Young inequality enables us to get immediately that

$$\begin{aligned}
&E\left(\int_0^t e^{-\omega(t-s)} \|R_2(s)\|^p ds \right) \\
&\quad \le C_p M^p \left(\int_0^t (s-\sigma)^{-2\gamma} e^{-2\omega(1/q)(s-\sigma)} ds \right)^{p/2} \left(\int_0^t e^{-\omega(t-\sigma)} E\|G(s,Z_s)\|^p_{\mathcal{L}_2^0} d\sigma \right), \\
&\quad \le C_p M^p \left(\Gamma(1-2\gamma) \left(\frac{2\omega}{q} \right)^{2\gamma-1} \right)^{p/2} \left(\int_0^t e^{-\omega(t-\sigma)} E\|G(s,Z_s)\|^p_{\mathcal{L}_2^0} d\sigma \right), \\
&\quad \le C_p M^P \left(\Gamma(1-2\gamma) \left(\frac{2\omega}{q} \right)^{2\gamma-1} \right)^{p/2} \left(\int_0^t e^{-\omega(t-\sigma)} \left[N_2 E\|Z_s\|^p_{C_\alpha} + c_2(s) \right] d\sigma \right).
\end{aligned} \tag{4.10}$$

Hence,

$$\begin{aligned}
I_{15} &\le 4^{p-1} e^{\omega r} M_1^p \left| \frac{\sin \pi\gamma}{\pi} \right|^p \left[\Gamma\left(1 - \frac{p}{p-1}(1+\alpha-\gamma) \right) \omega^{(p/(p-1))(1+\alpha-\gamma)-1} \right]^{p-1} \\
&\quad \cdot C_p M^P \left(\Gamma(1-2\gamma) \left(\frac{2\omega}{q} \right)^{2\gamma-1} \right)^{p/2} \left(\int_0^t e^{-\omega(t-s)} \left[N_2 E\|Z_s\|^p_{C_\alpha} + c_2(s) \right] ds \right) \\
&:= K_2 \cdot \int_0^t e^{-\omega(t-s)} \left[N_2 E\|Z_s\|^p_{C_\alpha} + c_2(s) \right] ds.
\end{aligned} \tag{4.11}$$

Therefore, combining the above estimates yields that

$$\begin{aligned}
\left(1 - 4^{p-1} N_0 \right) E\|Z_t\|^p_{C_\alpha} &\le K_0 e^{-\omega t} + 4^{p-1} c_0(t) + K_3 e^{-\omega t} \int_0^t e^{\omega s} E\|Z_s\|^p_{C_\alpha} ds \\
&\quad + e^{-\omega t} \int_0^t e^{\omega s} (K_1 c_1(s) + K_2 c_2(s)) ds,
\end{aligned} \tag{4.12}$$

where

$$K_3 = K_1 N_1 + K_2 N_2. \tag{4.13}$$

Now, Taking arbitrarily $\epsilon \in \mathbb{R}$ with $0 < \epsilon < \omega$ and $T > 0$ large enough, we obtain that

$$\begin{aligned}
&\left(1 - 4^{p-1} N_0\right) \int_0^T e^{\epsilon t} E\|Z_t\|_{C_\alpha}^p dt \\
&\quad \le K_0 \int_0^T e^{-(\omega-\epsilon)t} dt + \int_0^T 4^{p-1} c_0(t) e^{\epsilon t} dt + K_3 \int_0^T e^{\epsilon t - \omega t} \int_0^t e^{\omega s} E\|Z_s\|_{C_\alpha}^p ds\, dt \\
&\quad\quad + \int_0^T e^{-(\omega-\epsilon)t} \int_0^t e^{\omega s} (K_1 c_1(s) + K_2 c_2(s)) ds\, dt.
\end{aligned} \tag{4.14}$$

While,

$$\begin{aligned}
\int_0^T e^{\epsilon t - \omega t} \int_0^t e^{\omega s} E\|Z_s\|_{C_\alpha}^p ds\, dt &= \int_0^T \int_s^T e^{\omega s} E\|Z_s\|_{C_\alpha}^p e^{\epsilon t - \omega t} dt\, ds \\
&\le \frac{1}{\omega - \epsilon} \int_0^T e^{\epsilon s} E\|Z_s\|_{C_\alpha}^p ds.
\end{aligned} \tag{4.15}$$

Substituting (4.15) into (4.14) gives that

$$\begin{aligned}
&\left(1 - 4^{p-1} N_0\right) \int_0^T e^{\epsilon t} E\|Z_t\|_{C_\alpha}^p dt \\
&\quad \le K_0 \int_0^T e^{-(\omega-\epsilon)t} dt + \int_0^T 4^{p-1} c_0(t) e^{\epsilon t} dt + \frac{K_3}{\omega - \epsilon} \int_0^T e^{\epsilon s} E\|Z_s\|_{C_\alpha}^p ds \\
&\quad\quad + \int_0^T e^{-(\omega-\epsilon)t} \int_0^t e^{\omega s} (K_1 c_1(s) + K_2 c_2(s)) ds\, dt.
\end{aligned} \tag{4.16}$$

Since K_3 can be small enough by assumption, it is possible to choose a suitable $\epsilon \in \mathbb{R}^+$ with $0 < \epsilon < \omega - K_3/(1 - 4^{p-1} N_0)$ such that

$$1 - 4^{p-1} N_0 - \frac{K_3}{\omega - \epsilon} > 0. \tag{4.17}$$

Hence letting $T \to +\infty$ in (4.16) yields that

$$\int_0^{+\infty} e^{\epsilon t} E\|Z_t\|_{C_\alpha}^p dt \le \frac{1}{1-4^{p-1}N_0 - K_3/\omega - \epsilon}\left[K\int_0^{+\infty} e^{-(\omega-\epsilon)t}dt + \int_0^{+\infty} 4^{p-1}c_0(t)e^{\epsilon t}dt + \int_0^{+\infty} e^{-(\omega-\epsilon)t}\int_0^t e^{\omega s}(K_1c_1(s)+K_2c_2(s))ds\,dt\right] := K'(p,\epsilon,\phi) < \infty. \tag{4.18}$$

On the other hand, from the deduction of (4.12) it is not difficult to see that it also holds with ω substituted by ϵ, that is,

$$\left(1-4^{p-1}N_0\right)E\|Z_t\|_{C_\alpha}^p \le K_0e^{-\epsilon t} + 4^{p-1}c_0(t) + K_3e^{-\epsilon t}\int_0^t e^{\epsilon s}E\|Z_s\|_{C_\alpha}^p ds + e^{-\epsilon t}\int_0^t e^{\epsilon s}(K_1c_1(s)+K_2c_2(s))ds, \tag{4.19}$$

then it follows from (4.18) and (4.19) that (note the conditions for $c_i(\cdot)$)

$$E\|Z_t\|_{C_\alpha}^p \le \left[K_0 + 4^{p-1}c_0(t)e^{\epsilon t} + K_3K'(p,\epsilon,\phi) + \int_0^{+\infty} e^{\epsilon s}(K_1c_1(s)+K_2c_2(s))ds\right]\cdot e^{-\epsilon t} := K(p,\epsilon,\phi)e^{-\epsilon t}, \tag{4.20}$$

which is our desired inequality. Then the proof is completed. □

We also have the following result for almost surely exponential stability.

Theorem 4.2. *Suppose that all the conditions of Theorem 4.1 are satisfied. Then the solution is almost surely exponentially stable. Moreover, there exists a positive constant $\epsilon > 0$ such that*

$$\limsup_{t\to+\infty} \frac{\log\|Z(t)\|_\alpha}{t} \le -\frac{\varepsilon}{2p}, \quad a.s. \tag{4.21}$$

Proof. The proof is similar to that of [5, Theorem 3.3] and we omit it. □

5. Examples

Now we apply the results obtained above to consider the following non-autonomous stochastic functional differential equation with finite delay (i.e. , (1.3)).

Example 5.1. We have

$$
\begin{aligned}
& d(Z(t,x) - bZ(t-r,x)) \\
&\quad = \left[a(t,x)\frac{\partial^2}{\partial^2 x}(Z(t,x) - bZ(t-r,x)) + f(t, Z(t+r_1(t),x)) \right] dt \\
&\qquad + g(t, Z(t+r_2(t),x))d\beta(t), \\
&\qquad\qquad Z(t,0) = Z(t,\pi) = 0, \quad t \geq 0, \\
&\qquad\qquad Z(\theta,x) = \phi(x), \quad \theta \in [-r,0],\ 0 \leq x \leq \pi,
\end{aligned}
\tag{5.1}
$$

where $-r < r_i(t) < 0$, $a(t,x)$ is a continuous function and is uniformly Hölder continuous in t (with exponent $0 < \delta \leq 1$) and satisfies that $\sup_x |a(t,x) - a(+\infty,x)| \to 0$ as $t \to +\infty$. Let $H = L^2([0,\pi])$ and $K = \mathbb{R}$, $\beta(t)$ denote a one-dimensional standard Brownian motion.

We define the operators $A(t)$ by

$$
A(t)u = -a(t,x)u'' \tag{5.2}
$$

with the domain

$$
D(A) = \{u(\cdot) \in H : u, u' \text{ absolutely continuous}, u'' \in H, u(0) = u(\pi) = 0\}. \tag{5.3}
$$

Then $A(t)$ generates an evolution operator $U(t,s)$ satisfying assumptions (B_1)–(B_4) (see [23]). Set $H_\alpha = D(A^\alpha(t_0))$ for some $t_0 > 0$ and $C_\alpha = C([-r,0], H_\alpha)$.

In order to discuss the system (5.1), we also need the following assumptions on functions f and g.

(A_1) The functions $f : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$, $g : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ are continuous and global Lipschitz continuous in the second variable.

(A_2) There exist real numbers N_3, $N_4 > 0$ and continuous functions $c_3(\cdot)$, $c_4(\cdot)$: $[0,+\infty) \to \mathbb{R}_+$ such that

$$
\begin{aligned}
|F(t,y)|^p &\leq N_3 |y|^P + c_3(t), \quad \text{for } t \geq 0,\ y \in \mathbb{R}, \\
|G(t,y)|^p &\leq N_4 |y|^P + c_4(t), \quad \text{for } t \geq 0,\ y \in \mathbb{R},
\end{aligned}
\tag{5.4}
$$

where c_3, c_4 satisfy that $|c_i(t)| \leq P_i e^{-\omega t}$, for some $P_i > 0$ small enough $(i = 3, 4)$.

Now we can define $k : [0,+\infty)\times C([-r,0];H_\alpha) \to H_\alpha$, $F : [0,+\infty)\times C([-r,0];H_\alpha) \to H$ and $G : [0,+\infty)\times C([-r,0];H_\alpha) \to \mathcal{L}_2^0(\mathbb{R},H)$ as

$$\begin{aligned} k(t,\phi)(x) &:= b\phi(-r)(x), \\ F(t,\phi)(x) &:= f(t,\phi(r_1(t))(x)), \\ G(t,\phi)(x) &:= g(t,\phi(r_1(t))(x)). \end{aligned} \tag{5.5}$$

Then it is not difficult to verify that k, F, and G satisfy the conditions (H_1), (H_2), and (H_3), respectively, due to Assumptions (A_1), (A_2), since, by the embody property of H_α (also see [25, Corollary 2.6.11]), $\|x\| \le C'\|A^\alpha(t_0)x\|$ for some constant $C' > 0$. Hence we have, by Theorems 3.3 and 4.1, the following.

Theorem 5.2. *Let $0 < \alpha < min\{(p-2)/2p, \delta\}$ and $p > 2/\delta$. Suppose that all the above assumptions are satisfied. Then for the stochastic system* (5.1) *there exists a global mild solution $Z(t) = Z(t,0,\phi) \in H_\alpha$, and it is exponentially stable provided that N_3 and N_4 are small enough.*

We present another system for which the linear evolution is given explicitly, and so all the coefficients for the conditions of the obtained results can be estimated properly.

Example 5.3. We have

$$\begin{gathered} d\left[Z(t,x) + \int_{-r}^{t}\int_{0}^{\pi} b(s-t,y,x)Z(s,y)\,dy\,ds\right] \\ = \left[\left(\frac{\partial^2}{\partial x^2} + a(t)\right)\left[Z(t,x) + \int_{-r}^{t}\int_{0}^{\pi} b(s-t,y,x)Z(s,y)\,dy\,ds\right] + f(t,Z(t-r,x))\right]dt \\ + g(t,Z(t-r,x))\,d\beta(t), \quad t \ge 0, \\ Z(t,0) = Z(t,\pi) = 0, \\ Z(\theta,x) = \phi(\theta,x), -r \le \theta \le 0, \quad 0 \le x \le \pi, \end{gathered} \tag{5.6}$$

where $a(t)$ is a positive function and is Hölder continuous in t with parameter $0 < \delta < 1$. f, g and β are as in Example 5.1, $b \in C^1$ and $b(s,y,\pi) = b(s,y,0) = 0$.

Let $H = L^2([0,\pi])$, $K = \mathbb{R}$. $A(t)$ is defined by

$$A(t)f = -f'' - a(t)f \tag{5.7}$$

with the domain

$$D(A) = \{f(\cdot) \in H : f, f' \text{ absolutely continuous}, f'' \in H, f(0) = f(\pi) = 0\}. \tag{5.8}$$

Then it is not difficult to verify that $A(t)$ generates an evolution operator $U(t,s)$ satisfying assumptions (B_1)–(B_4) and

$$U(t,s) = T(t-s)e^{-\int_s^t a(\tau)d\tau}, \tag{5.9}$$

where $T(t)$ is the compact analytic semigroup generated by the operator $-A$ with $-Af = -f''$ for $f \in D(A)$. It is easy to compute that A has a discrete spectrum, and the eigenvalues are $n^2, n \in \mathbb{N}$, with the corresponding normalized eigenvectors $z_n(x) = \sqrt{(2/\pi)}\sin(nx)$. Thus for $f \in D(A)$, there holds

$$-A(t)f = \sum_{n=1}^{\infty}\left(-n^2 - a(t)\right)\langle f, z_n\rangle z_n, \tag{5.10}$$

and clearly the common domain coincides with that of the operator A. Furthermore, We may define $A^{\alpha}(t_0)$ $(t_0 \in [0,a])$ for self-adjoint operator $A(t_0)$ by the classical spectral theorem, and it is easy to deduce that

$$A^{\alpha}(t_0)f = \sum_{n=1}^{\infty}\left(n^2 + a(t_0)\right)^{\alpha}\langle f, z_n\rangle z_n \tag{5.11}$$

on the domain $D[A^{\alpha}] = \{f(\cdot) \in H, \sum_{n=1}^{\infty}(n^2 + a(t_0))^{\alpha}\langle f, z_n\rangle z_n \in H\}$. Particularly,

$$A^{1/2}(t_0)f = \sum_{n=1}^{\infty}\sqrt{n^2 + a(t_0)}\langle f, z_n\rangle z_n. \tag{5.12}$$

Therefore, we have that, for each $f \in H$,

$$\begin{aligned} U(t,s)f &= \sum_{n=1}^{\infty} e^{-n^2(t-s)-\int_s^t a(\tau)d\tau}\langle f, z_n\rangle z_n, \\ A^{\alpha}(t_0)A^{-\beta}(t_0)f &= \sum_{n=1}^{\infty}\left(n^2 + a(t_0)\right)^{\alpha-\beta}\langle f, z_n\rangle z_n, \\ A^{\alpha}(t_0)U(t,s)f &= \sum_{n=1}^{\infty}\left(n^2 + a(t_0)\right)^{\alpha} e^{-n^2(t-s)-\int_s^t a(\tau)d\tau}\langle f, z_n\rangle z_n. \end{aligned} \tag{5.13}$$

Then,

$$\left\|A^{\alpha}(t)A^{-\beta}(s)\right\| \leq (1 + \|a(\cdot)\|)^{\alpha}, \qquad \left\|A^{\beta}(t)U(t,s)A^{-\beta}(s)\right\| \leq (1 + \|a(\cdot)\|)^{\beta} \tag{5.14}$$

for $t, s \in [0, T]$, $0 < \alpha < \beta$. And

$$
\begin{aligned}
&\left\| A^{\beta}(t)U(t,s)f \right\|^2 \\
&\quad = \sum_{n=1}^{\infty} \left(n^2 + a(t)\right)^{2\beta} e^{-2n^2(t-s)-2\int_s^t a(\tau)d\tau} \left|\langle f, z_n\rangle\right|^2 \\
&\quad = (t-s)^{-2\beta} \sum_{n=1}^{\infty} \left[\left(n^2 + a(t)\right)(t-s)\right]^{2\beta} e^{-2(n^2+a(t))(t-s)+a(t)(t-s)-2\int_s^t a(\tau)d\tau} \left|\langle f, z_n\rangle\right|^2 \\
&\quad = (t-s)^{-2\beta} \sum_{n=1}^{\infty} e^{2\beta \log\left[\left(n^2+a(t)\right)(t-s)\right]-2\left(n^2+a(t)\right)(t-s)+a(t)(t-s)-2\int_s^t a(\tau)d\tau} \left|\langle f, z_n\rangle\right|^2 \\
&\quad \le (t-s)^{-2\beta} \sum_{n=1}^{\infty} \beta^{2\beta} e^{a(t)(t-s)-2\int_s^t a(\tau)d\tau} \left|\langle f, z_n\rangle\right|^2, \quad (\text{note } c \log x - x \le c \log c - c),
\end{aligned}
\tag{5.15}
$$

which shows that $\|A^{\beta}(t)U(t,s)\| \le C_{\beta}/(t-s)^{\beta}$ for $C_{\beta} = \beta^{\beta} \max\{e^{a(t)(t-s)-\int_s^t a(\tau)d\tau} : t, s \in [0, T]\} > 0$.

Now we define F, G as above and

$$
k(t, \varphi)(x) = \int_{-r}^{0} \int_{0}^{\pi} b(s, y, x)\varphi(s, y)\, dy\, ds \tag{5.16}
$$

for all $\varphi \in C_{\alpha} = C([-r, 0], H_{\alpha})$ and any $x \in [0, \pi]$. Thus (5.6) has the form (1.1). Thus we can easily obtain its existence and stability of mild solutions for (5.6) by Theorems 3.3 and 4.1 under some proper conditions.

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