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Research Article

Approximately n-Jordan Homomorphisms on Banach Algebras

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Let $n \in \mathbb{N}$, and let A, B be two rings. An additive map $h : A \to B$ is called n-Jordan homomorphism if $h(a^n) = (h(a))^n$ for all $a \in A$. In this paper, we establish the Hyers-Ulam-Rassias stability of n-Jordan homomorphisms on Banach algebras. Also we show that (a) to each approximate 3-Jordan homomorphism h from a Banach algebra into a semisimple commutative Banach algebra there corresponds a unique 3-ring homomorphism near to f, (b) to each approximate n-Jordan homomorphism h between two commutative Banach algebras there corresponds a unique n-ring homomorphism near to f for all $n \in \{3, 4, 5\}$.

1. Introduction and Preliminaries

Let A, B be two rings (algebras). An additive map $h : A \to B$ is called n-Jordan homomorphism (n-ring homomorphism) if $h(a^n) = (h(a))^n$ for all $a \in A$, ($h(\prod_{i=1}^n a_i) = \prod_{i=1}^n h(a_i)$, for all $a_1, a_2, \ldots, a_n \in A$). If $h : A \to B$ is a linear n-ring homomorphism, we say that h is n-homomorphism. The concept of n-homomorphisms was studied for complex algebras by Hejazian et al. [1] (see also [2, 3]). A 2-Jordan homomorphism is a Jordan homomorphism, in the usual sense, between rings. Every Jordan homomorphism is an n-Jordan homomorphism, for all $n \geq 2$, (e.g., [4, Lemma 6.3.2]), but the converse is false, in general. For instance, let A be an algebra over $\mathbb{C}$ and let $h : A \to A$ be a nonzero Jordan homomorphism on A. Then, $-h$ is a 3-Jordan homomorphism. It is easy to check that $-h$ is not 2-Jordan homomorphism or 4-Jordan homomorphism. The concept of n-Jordan homomorphisms was studied by the first author [5]. A classical question in the theory of functional equations is that "when is it true that a mapping which approximately satisfies a functional equation $\mathcal{E}$ must be somehow close to an exact solution of $\mathcal{E}$?" Such a problem was formulated by Ulam [6] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [7]. It gave rise to the *stability theory* for functional equations. Subsequently, various approaches to the problem have been introduced by several authors. For the history and various aspects of this theory we refer the reader to monographs [8–12]. Applying a theorem of Hyers [7], Rassias [13], and Gajda [14], Bourgin [15] proved the stability problem of ring homomorphisms between unital Banach algebras. Badora [16] proved the Hyers-Ulam-Rassias stability of ring homomorphisms, which generalizes the result of Bourgin. Recently, Miura et al. [17] proved the Hyers-Ulam-Rassias stability of Jordan homomorphisms. The stability problem of n-homomorphisms between Banach algebras, has been proved by the first author [18]. In this paper, we consider the stability, in the sense of Hyers-Ulam-Rassias, of n-Jordan homomorphisms on Banach algebras.

2. Main Result

By a following similar way as in [17], we obtain the next theorem.

Theorem 2.1. *Let A be a normed algebra, let B be a Banach algebra, let δ and ε be nonnegative real numbers, and let p,q be a real numbers such that $p,q<1$ or $p,q>1$, and that $q>0$. Assume that $f:A\to B$ satisfies the system of functional inequalities*

$$\|f(a+b)-f(a)-f(b)\|\le\varepsilon(\|a\|^p+\|b\|^p), \tag{2.1}$$

$$\|f(a^n)-f(a)^n\|\le\delta\|a\|^{nq} \tag{2.2}$$

for all $a,b\in A$. Then, there exists a unique n-Jordan homomorphism $h:A\to B$ such that

$$\|f(a)-h(a)\|\le\frac{2\varepsilon}{|2-2^p|}\|a\|^p \tag{2.3}$$

for all $a\in A$.

Proof. Put $s:=-\operatorname{sgn}(p-1)$, and $h(a):=\lim_m(1/2^{sm})f(2^{sm}a)$ for all $a\in A$. It follows from [13, 14] that h is additive map satisfies (2.3). We will show that h is n-Jordan homomorphism. Since $\lim_m 2^{smn(q-1)}=0$, it follows from (2.2) that

$$\begin{aligned}
&\lim_m\frac{1}{2^{smn}}\{\|f((2^{sm}a)(2^{sm}a)\cdots(2^{sm}a))-(f(2^{sm}a))^n\|\}\\
&\quad\le\lim_m\frac{1}{2^{smn}}\delta\|2^{sm}a\|^{nq}\\
&\quad=\lim_m(2^{smn(q-1)})\delta\|a\|^{nq}=0.
\end{aligned}\tag{2.4}$$

Hence, we have

$$\begin{aligned} h(a^n) &= \lim_m \frac{1}{2^{smn}} f\left(2^{smn}(a^n)\right) \\ &= \lim_m \frac{1}{2^{smn}} f\left((2^{sm}a)(2^{sm}a)\cdots(2^{sm}a)\right) \\ &= \lim_m \frac{1}{2^{smn}} \left\{ f\left((2^{sm}a)(2^{sm}a)\cdots(2^{sm}a)\right) - \left(f(2^{sm}a)\right)^n + \left(f(2^{sm}a)\right)^n \right\} \\ &= (h(a))^n \end{aligned} \tag{2.5}$$

for all $a \in A$. In other words, h is n-Jordan homomorphism. The uniqueness property of h follows from [13, 14]. □

Theorem 2.2. *Let A be a normed algebra, let B be a Banach algebra, let δ and ε be nonnegative real numbers, and let p, q be real numbers such that $p < 1$ and $q < 0$. If $f : A \to B$ is a mapping, with $f(0) = 0$, such that the inequalities (2.1) and (2.2) are valid. Then, there exists a unique n-Jordan homomorphism $h : A \to B$ such that*

$$\left\| f(a) - h(a) \right\| \leq \frac{2\varepsilon}{|2 - 2^p|} \|a\|^p \tag{2.6}$$

for all $a \in A$.

Proof. Assume that $\|0\|^p = \infty$. It follows from [13] that there exists an additive map $h : A \to B$ satisfies (2.6). It suffices to show that $h(a^n) = h(a)^n$ for all $a \in A$. Since h is additive, we get $h(0) = 0$, and so the case $a = 0$ is omitted. Let $a \in A - \{0\}$ be arbitrarily. If $a^n \neq 0$, then the proof of Theorem 2.1 works well, and $h(a^n) = h(a)^n$. Thus we need to consider only the case $a^n = 0$. Since $f(0) = 0$, it follows from (2.2), that

$$\left\| \frac{1}{2^{mn}} \left(f(2^m a)\right)^n \right\| \leq \frac{1}{2^{mn}} \delta \|2^m a\|^{nq} = 2^{mn(q-1)} \delta \|a\|^{nq}. \tag{2.7}$$

Hence, we have

$$\lim_m \frac{1}{2^{mn}} \left(f(2^m a)\right)^n = 0. \tag{2.8}$$

On the other hand, we have

$$h(a) = \lim_m \frac{1}{2^m} \left(f(2^m a)\right). \tag{2.9}$$

It follows from (2.8) and (2.9) that

$$h(a)^n = \lim_m \left\{ \frac{1}{2^{mn}} \left(f(2^m a) \right)^n \right\} = 0, \tag{2.10}$$

which proves $h(a^n) = 0 = h(a)^n$, whenever $a^n = 0$. This completes the proof. □

By [17, Theorem 1.1] and [5, Theorem 2.5], we have the following theorem.

Theorem 2.3. *Let $n \in \{2,3\}$ be fixed. Suppose A is a Banach algebra, which needs not to be commutative, and suppose B is a semisimple commutative Banach algebra. Then, each n-Jordan homomorphism $h : A \to B$ is a n-ring homomorphism.*

Let $n \in \{2,3\}$ be fixed. As a direct corollary, we show that to each approximate n-Jordan homomorphism f from a Banach algebra into a semisimple commutative Banach algebra there corresponds a unique n-ring homomorphism near to f.

Corollary 2.4. *Let $n \in \{2,3\}$ be fixed. Suppose A is a Banach algebra, which needs not to be commutative, and suppose B is a semisimple commutative Banach algebra. Let δ and ε be nonnegative real numbers and let p, q be a real numbers such that $(p-1)(q-1) > 0$, $q \geq 0$ or $(p-1)(q-1) > 0$, $q < 0$ and $f(0) = 0$. Assume that $f : A \to B$ satisfies the system of functional inequalities*

$$\begin{gathered} \left\| f(a+b) - f(a) - f(b) \right\| \leq \varepsilon \left(\|a\|^p + \|b\|^p \right), \\ \left\| f(a^n) - f(a)^n \right\| \leq \delta \|a\|^{nq} \end{gathered} \tag{2.11}$$

for all $a, b \in A$. Then, there exists a unique n-ring homomorphism $h : A \to B$ such that

$$\left\| f(a) - h(a) \right\| \leq \frac{2\varepsilon}{|2 - 2^p|} \|a\|^p \tag{2.12}$$

for all $a \in A$.

Proof. It follows from Theorems 2.1, 2.2, and 2.3. □

Theorem 2.5. *Let $n \in \{3,4,5\}$ be fixed, A, B be two commutative algebras, and let $h : A \to B$ be a n-Jordan homomorphism. Then, h is n-ring homomorphism.*

Proof. For $n = 3,4$, (see [5, Theorem 2.2]). Now suppose $n = 5$. Then, h is additive and $h(a^5) = (h(a))^5$ for all $a \in A$. Replacing a by $a + b$ to get

$$\begin{aligned} & h\left(5a^4 b + 10a^3 b^2 + 10a^2 b^3 + 5ab^4\right) \\ & \quad = 5h(a^4)h(b) + 10h(a^3)h(b^2) + 10h(a^2)h(b^3) + 5h(a)h(b^4). \end{aligned} \tag{2.13}$$

Now, replacing a by $x+y$ in (2.13), we obtain that

$$\begin{aligned}
&h\{5(x^4+4x^3y+6x^2y^2+4xy^3+y^4)b+10(x^3+3x^2y+3xy^2+y^3)b^2\\
&\quad+10(x^2+2xy+y^2)b^3+5(x+y)b^4\}\\
&=5\left[h(x)^4+4h(x)^3h(y)+6h(x)^2h(y)^2+4h(x)h(y)^3+h(y)^4\right]h(b)\\
&\quad+10\left[h(x)^3+3h(x)^2h(y)+3h(x)h(y)^2+h(y)^3\right]h(b)^2\\
&\quad+10\left[h(x)^2+2h(x)h(y)+h(y)^2\right]h(b)^3+5\left[h(x)+h(y)\right]h(b)^4.
\end{aligned} \tag{2.14}$$

By (2.13) and (2.14), we get

$$\begin{aligned}
&h\{(20x^3yb+30x^2y^2b+20xy^3b+30x^2yb^2+30xy^2b^2+20xyb^3)\}\\
&=20h(x)^3h(y)h(b)+30h(x)^2h(y)^2h(b)+20h(x)h(y)^3h(b)\\
&\quad+30h(x)^2h(y)h(b)^2+30h(x)h(y)^2h(b)^2+20h(x)h(y)h(b)^3.
\end{aligned} \tag{2.15}$$

By (2.15) it follows that

$$\begin{aligned}
&h\{xyb\left[20(x^2+y^2+b^2)+30(xy+xb+yb)\right]\}\\
&=h(x)h(y)h(b)\left[20h(x)^2+30h(x)h(y)+20h(y)^2\right.\\
&\quad\left.+30h(x)h(b)+30h(y)h(b)+20h(b)^2\right].
\end{aligned} \tag{2.16}$$

Replacing b by $z+w$ in (2.16), we obtain

$$\begin{aligned}
&h\{xyz\left[20(w^2+2zw)+30(xy+xw+yw)\right]+xyw\left[20(z^2+2zw)+30(xy+xz+yz)\right]\}\\
&=h(x)h(y)h(z)\left[20(h(w)^2+2h(z)h(w))+30(h(x)h(y)+h(x)h(w)+h(y)h(w))\right]\\
&\quad+h(x)h(y)h(w)\left[20(h(z)^2+2h(z)h(w))+30(h(x)h(y)+h(x)h(z)+h(y)h(z))\right].
\end{aligned} \tag{2.17}$$

Replacing z by $t+s$ in (2.17), we get

$$\begin{aligned}
&h\{xy(t+s)\left[20(w^2+2(t+s)w)+30(xy+xw+yw)\right]\\
&\quad+xyw\left[20((t+s)^2+2(t+s)w)+30(xy+x(t+s)+y(t+s))\right]\}\\
&=h(x)h(y)h(t+s)\left[20(h(w)^2+2h(t+s)h(w))\right.\\
&\qquad\left.+30(h(x)h(y)+h(x)h(w)+h(y)h(w))\right]\\
&\quad+h(x)h(y)h(w)\left[20(h(t+s)^2+2h(t+s)h(w))\right.\\
&\qquad\left.+30(h(x)h(y)+h(x)h(t+s)+h(y)h(t+s))\right].
\end{aligned} \tag{2.18}$$

Hence, we get

$$
\begin{aligned}
&h[40xywts - 30x^2y^2w - 30xys(xy + xw + yw)] \\
&\quad = 40h(x)h(y)h(w)h(t)h(s) - 30h(x)^2h(y)^2h(w) \\
&\qquad - 30h(x)h(y)h(s)[h(x)h(y) + h(x)h(w) + h(y)h(w)], \\
&h[40xywts - 30x^2y^2w - 30xyt(xy + xw + yw)] \\
&\quad = 40h(x)h(y)h(w)h(t)h(s) - 30h(x)^2h(y)^2h(w) \\
&\qquad - 30h(x)h(y)h(t)[h(x)h(y) + h(x)h(w) + h(y)h(w)].
\end{aligned}
\tag{2.19}
$$

By (2.19) it follows that

$$
\begin{aligned}
&h[xy(s - t)(xy + xw + yw)] \\
&\quad = h(x)h(y)(h(s) - h(t))[h(x)h(y) + h(x)h(w) + h(y)h(w)].
\end{aligned}
\tag{2.20}
$$

Replacing t by $-s$ in (2.20), we obtain

$$
h[2xys(xy + xw + yw)] = 2h(x)h(y)h(s)[h(x)h(y) + h(x)h(w) + h(y)h(w)]. \tag{2.21}
$$

Replacing y, w by x in (2.21), we get

$$
h(x^4s) = h(x)^4h(s). \tag{2.22}
$$

Replacing x by $x + y$ in above equality to get

$$
h((4x^3y + 6x^2y^2 + 4xy^3)s) = (4h(x)^3h(y) + 6h(x)^2h(y)^2 + 4h(x)h(y)^3)h(s). \tag{2.23}
$$

Replacing x by $x + z$ in (2.23), we obtain

$$
\begin{aligned}
&h\{[(4x^3y + 6x^2y^2 + 4xy^3) + (4z^3y + 6z^2y^2 + 4zy^3) + 12(x^2zy + xz^2y + xzy^2)]s\} \\
&\quad = \{(4h(x)^3h(y) + 6h(x)^2h(y)^2 + 4h(x)h(y)^3) \\
&\qquad + (4h(z)^3h(y) + 6h(z)^2h(y)^2 + 4h(z)h(y)^3) \\
&\qquad + 12(h(x)^2h(z)h(y) + h(x)h(z)^2h(y) + h(x)h(z)h(y)^2)\}h(s).
\end{aligned}
\tag{2.24}
$$

Combining (2.23) by (2.24), we get

$$
h\{(xyz)(x + y + z)s\} = [(h(x)h(y)h(z))(h(x) + h(y) + h(z))]h(s). \tag{2.25}
$$

Replacing z by $-x$ in (2.25) to obtain

$$h(x^2y^2s) = h(x)^2h(y)^2h(s), \tag{2.26}$$

replacing y by $y + w$ in (2.26), we get

$$h(x^2yws) = h(x)^2h(y)h(w)h(s). \tag{2.27}$$

Now, replace x by $x + t$ in (2.27), we obtain

$$h(xtyws) = h(x)h(t)h(y)h(w)h(s). \tag{2.28}$$

Hence, h is 5-ring homomorphism. □

Corollary 2.6. *Let $n \in \{3, 4, 5\}$ be fixed. Suppose A, B are commutative Banach algebras. Let δ and ε be nonnegative real numbers and let p, q be a real numbers such that $(p-1)(q-1) > 0$, $q \geq 0$ or $(p-1)(q-1) > 0$, $q < 0$, and $f(0) = 0$. Assume that $f : A \longrightarrow B$ satisfies the system of functional inequalities*

$$\begin{gathered}\left\| f(a+b) - f(a) - f(b) \right\| \leq \varepsilon\left(\|a\|^p + \|b\|^p\right), \\ \left\| f(a^n) - f(a)^n \right\| \leq \delta \|a\|^{nq}\end{gathered} \tag{2.29}$$

for all $a, b \in A$. Then, there exists a unique n-ring homomorphism $h : A \longrightarrow B$ such that

$$\left\| f(a) - h(a) \right\| \leq \frac{2\varepsilon}{|2 - 2^p|}\|a\|^p \tag{2.30}$$

for all $a \in A$.

Proof. It follows from Theorems 2.1, 2.2, and 2.5. □

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