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Research Article

A Coefficient Related to Some Geometric Properties of a Banach Space

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We introduce a new coefficient as a generalization of the modulus of smoothness and Pythagorean modulus of Banach space X. Some basic properties of this new coefficient are investigated. Moreover, some sufficient conditions which imply normal structure are presented.

1. Introduction

We will assume throughout this paper that X and X^* stand for a Banach space and its dual space, respectively. By $S(X)$ and $B(X)$ we denote the unit sphere and the unit ball of a Banach space X, respectively. The nontrival Banach space will mean later on that X is a real space and $\dim X \geq 2$. Let us recall some definitions of modulus in Banach space. The modulus of smoothness (see [5]) of X is the function $\rho_X(t)$ defined by

$$\rho_X(t) = \sup\left\{ \frac{\|x+ty\| + \|x-ty\|}{2} - 1 : x, y \in S(X) \right\}. \tag{1.1}$$

X is called uniformly smooth if $\lim_{t\to 0}(\rho_X(t))/t = 0$. X is called q-uniformly smooth $(1 < q \leq 2)$ if there exists a constant $K > 0$ such that $\rho_X(t) \leq Kt^q$ for all $t > 0$. Pythagorean modulus is introduced by Gao [6] is given by

$$E(t, X) = \sup\left\{ \|x+ty\|^2 + \|x-ty\|^2 : x, y \in S(X) \right\}, \quad \forall t > 0. \tag{1.2}$$

For $t > 0$, the parameterized James constant $J(t, X)$ is defined by

$$J(t, X) = \sup\{\min\{\|x + ty\|, \|x - ty\|\} : x, y \in S(X)\}. \tag{1.3}$$

Some basic properties concerning this constant were studied in [1].

A Banach space X is called uniformly nonsquare (see [7]) if there exists $\delta > 0$, such that $\|x+y\|/2 \leq 1-\delta$ or $\|x-y\|/2 \leq 1-\delta$ wherever $x, y \in S(X)$. The number $r(A) = \inf\{\sup\{\|x-y\| : y \in A\} : x \in A\}$ is called Chebyshev radius of A. The number $\operatorname{diam} A = \sup\{\|x-y\| : x, y \in A\}$ is called diameter of A. A Banach space X is said to have the normal structure provided $r(A) < \operatorname{diam} A$ for every bounded closed convex subset A of X with $\operatorname{diam} A > 0$.

Recall the ultraproduct of Banach spaces. Let $\mathcal{U}$ be a free ultrafilter on the set of natural numbers, the closed linear subspace of $l_\infty(X)$, $N_{\mathcal{U}} = \{\{x_i\} \in l_\infty(I, X_i) : \lim_{\mathcal{U}}\|x_i\| = 0\}$. The ultraproduct of $\{X_i\}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm. we write $\widetilde{X}$ to denote the ultraproduct. For more details see [8].

In this paper, we consider the coefficient $J_{X,p}(t)$ as a generalization of the modulus of smoothness and Pythagorean modulus of Banach space X. Some basic properties of this new coefficient are investigated, which generalized some known results. Meanwhile some sufficient conditions which imply the normal structure are obtained.

2. Some Properties on Coefficient $J_{X,p}(t)$

Definition 2.1. Let $x \in S(X), y \in S(X)$, for any $t > 0$, $1 \leq p < \infty$ we set

$$J_{X,p}(t) = \sup\left\{\left(\frac{\|x + ty\|^p + \|x - ty\|^p}{2}\right)^{1/p}\right\}. \tag{2.1}$$

It is easily seen that $J_{X,p}(t) \geq \rho_X(t) + 1$, the case of $p = 1, 2$, $J_{X,1}(t) = \rho_X(t) + 1$, $2J_{X,2}^2(t) = E(t, X)$, respectively.

The proof of the following proposition is trivial, so it is omitted.

Proposition 2.2. *Let X be a nontrival Banach space and $t > 0$. Then one has*

$$J_{X,p}(t) = \sup\{J_{Y,p}(t) : Y \in \mathcal{D}(X)\}, \tag{2.2}$$

where $\mathcal{D}(X) = \{Y : Y$ is a two-dimensional subspace of $X\}$.

Proposition 2.3. *Let X be a nontrival Banach space and $t > 0$. Then*

(1) $J_{X,p}(t)$ *is a nondecreasing function;*

(2) $J_{X,p}(t)$ *is a convex function;*

(3) $J_{X,p}(t)$ *is a continuous function;*

(4) $(J_{X,p}(t) - 1)/t$ *is a nondecreasing function.*

Proof. (1) Note that $f(t) = \|x + ty\|^p + \|x - ty\|^p$ is a convex and even function. Let $0 < t_1 \le t_2$, $x, y \in S(X)$. Then we have

$$\begin{aligned}
\|x + t_1 y\|^p + \|x - t_1 y\|^p &= f(t_1) = f\left(\frac{t_2 + t_1}{2t_2} t_2 + \frac{t_2 - t_1}{2t_2}(-t_2)\right) \\
&\le f(t_2) = \|x + t_2 y\|^p + \|x - t_2 y\|^p \\
&\le 2 J^p_{X,p}(t_2),
\end{aligned} \tag{2.3}$$

which implies that $2J^p_{X,p}(t_1) \le 2J^p_{X,p}(t_2)$, that is, the inequality $J_{X,p}(t_1) \le J_{X,p}(t_2)$ holds.

(2) Let $x, y \in S(X), t_1, t_2 > 0$, $\lambda \in (0,1)$ and $r(s) = \operatorname{sgn}(\sin 2\pi s)$. Then we have

$$\begin{aligned}
&\left(\int_0^1 \|x + r(s)(\lambda t_1 + (1-\lambda)t_2)y\|^p dt\right)^{1/p} \\
&\le \left(\int_0^1 (\lambda \|x + r(s)t_1 y\| + (1-\lambda)\|x + r(s)t_2 y\|)^p dt\right)^{1/p} \\
&\le \lambda \left(\int_0^1 \|x + r(s)t_1 y\|^p dt\right)^{1/p} + (1-\lambda)\left(\int_0^1 \|x + r(s)t_2 y\|^p dt\right)^{1/p} \\
&\le \lambda J_{X,p}(t) + (1-\lambda) J_{X,p}(t).
\end{aligned} \tag{2.4}$$

Since x, y are arbitrary, we have

$$J_{X,p}(\lambda t_1 + (1-\lambda)t_2) \le \lambda J_{X,p}(t_1) + (1-\lambda) J_{X,p}(t_2). \tag{2.5}$$

(2) The continuity of $J_{X,p}(t)$ follows from the case of (2).

(3) Let $0 < t_1 \le t_2$, then $t_1 = \lambda t_2 (0 < \lambda \le 1)$. Thus

$$\frac{J_{X,p}(t_1) - 1}{t_1} \le \frac{J_{X,p}((1-\lambda)0 + \lambda t_2) - 1}{\lambda t_2} \le \frac{J_{X,p}(t_2) - 1}{t_2}. \tag{2.6}$$

□

Proposition 2.4. *Let X be a nontrival Banach space and $t > 0$. Then*

$$\begin{aligned}
J_{X,p}(t) &= \sup\left\{\left(\frac{\|x + ty\|^p + \|x - ty\|^p}{2}\right)^{1/p} : x \in S(X),\ y \in B(X)\right\} \\
&= \sup\left\{\left(\frac{\|x + ty\|^p + \|x - ty\|^p}{2}\right)^{1/p} : x,\ y \in B(X)\right\}.
\end{aligned} \tag{2.7}$$

Proof. From Proposition 2.3(1), we have

$$\sup_{x\in S(X)}\sup_{y\in B(X)}\left\{\left(\frac{\|x+ty\|^p+\|x-ty\|^p}{2}\right)^{1/p}\right\}\le J_{X,p}(t\|y\|)\le J_{X,p}(t). \tag{2.8}$$

Since the opposite inequality holds obviously, we get the first equality.

Let t be fixed. And we set $h(\lambda)=\|\lambda x+ty\|^p+\|\lambda x-ty\|^p$. Then $h(\lambda)$ is a convex and even function, therefore $h(\lambda)\ge h(1)$ for all $\lambda\ge 1$. For $x,y\in B(X)$ we have

$$\left\|\frac{x}{\|x\|}+ty\right\|^p+\left\|\frac{x}{\|x\|}-ty\right\|^p\ge\|x+ty\|^p+\|x-ty\|^p. \tag{2.9}$$

Therefore

$$\sup_{x\in S(X)}\sup_{y\in B(X)}\left(\|x+ty\|^p+\|x-ty\|^p\right)\ge\sup_{x\in B(X)}\sup_{y\in B(X)}\left(\|x+ty\|^p+\|x-ty\|^p\right). \tag{2.10}$$

Since the opposite inequality holds obviously, then we obtain the second equality. □

Theorem 2.5. *For any nontrival Banach space X, let $1\le p<\infty$, $t>0$. Then the following conditions are equivalent:*

(1) $J_{X,p}(t)<1+t$;

(2) $J(t,X)<1+t$.

Proof. (1)⇒(2) . It is well known that $J_{X,p}(t)\le 1+t$ for all p. Suppose that $J(t,X)=1+t$. From the definition of $J(t,X)$, for any $\epsilon>0$ there are $x,y\in S(X)$ such that

$$\min\{\|x+ty\|,\|x-ty\|\}\ge(1+t-\epsilon). \tag{2.11}$$

Then we have

$$\left(\frac{\|x+ty\|^p+\|x-ty\|^p}{2}\right)^{1/p}\ge(1+t-\epsilon). \tag{2.12}$$

Since ϵ are arbitrary this implies that $J_{X,p}(t)\ge 1+t$—a contradiction

(2)⇒(1) . Similarly suppose that $J_{X,p}(t)=1+t$, for any $\epsilon>0$ there are $x,y\in S(X)$ such that

$$\left(\|x+ty\|^p+\|x-ty\|^p\right)\ge 2(1+t-\epsilon)^p, \tag{2.13}$$

and $\|x+ty\|^p+\|x-ty\|^p\le 2(1+t)^p$. Since ϵ are arbitrary, we have

$$\|x+ty\|=\|x-ty\|=1+t. \tag{2.14}$$

From the equivalent definition of $J(t, X)$, we get $J(t, X) \geq 1 + t$. This is a contradiction and thus we complete the proof. □

Corollary 2.6. *Let $1 \leq p < \infty$, $t > 0$. Then the following conditions are equivalent:*

(1) *X is uniformly nonsquare;*

(2) *$J_{X,p}(t) < 1 + t$, for some $t > 0$;*

(3) *$J_{X,p}(t) < 1 + t$, for all $t > 0$.*

Proof. This follows from Theorem 2.5 and the conclusion of $J(t, X)$ in [1]. □

Theorem 2.7. *A Banach space X is uniformly smooth if and only if*

$$\lim_{t \to 0} \left(\frac{J_{X,p}(t) - 1}{t} \right) = 0. \tag{2.15}$$

Proof. The sufficiency is trivial since $(\rho_X(t) + 1) \leq J_{X,p}(t)$ holds for any $t > 0$ and $1 \leq p < \infty$. To see the necessity, we suppose that $\lim_{t \to 0}(J_{X,p}(t) - 1/t) > 0$. Proposition 2.3(4) implies that there exist a $c \in (0, 1)$ such that $J_{X,p}(t) - 1/t \geq c$ for any $t > 0$. In particular, let $0 < t < 1$ and choose x, y with $\|x\| = 1, \|y\| = t$ such that

$$\|x + y\|^p + \|x - y\|^p \geq 2(1 + ct)^p. \tag{2.16}$$

Without loss of generality, we assume that $\min\{\|x + y\|, \|x - y\|\} = \|x - y\| = h$ then $h \in [1 - t, 1 + ct]$. From the above inequality we get that

$$\|x + y\| + \|x - y\| \geq h + (2(1 + ct)^p - h^p)^{1/p} =: f(h). \tag{2.17}$$

Note that $f(h)$ attain its minimum at $h = 1 - t$; in the view of the definition $\rho_X(t)$ implies that

$$\frac{\rho_X(t)}{t} \geq \frac{f(1 - t) - 2}{2t} = \frac{1 - t + (2(1 + ct)^p - (1 - t)^p)^{1/p} - 2}{2t}. \tag{2.18}$$

Letting $t \to 0$, and using L'Hôpital's rule, we get

$$\lim_{t \to 0} \frac{\rho_X(t)}{t} \geq c > 0. \tag{2.19}$$

This is a contradiction, and thus we complete the proof. □

Theorem 2.8 ([2])**.** *Let $1 \leq p < \infty$ and $1 < q \leq 2$. Then X is q-uniformly smooth if and only if there exists $K \geq 1$ such that*

$$\frac{\|x + y\|^p + \|x - y\|^p}{2} \leq \|x\|^q + \|Ky\|^q, \quad \forall x, y \in X. \tag{2.20}$$

Theorem 2.9. *Let $1 \le p < \infty$ and $1 < q \le 2$. The following conditions are equivalent:*

(1) *X is q-uniformly smooth;*

(2) *there is $K \ge 1$ such that*

$$J_{X,p}(t) \le (1 + Kt^q)^{1/q}, \quad \forall t > 0. \tag{2.21}$$

Proof. This follows from Theorem 2.8 and the definition of $J_{X,p}(t)$. □

Theorem 2.10. *Let X be the space l_r or $L_r[0, 1]$ with $\dim X \ge 2$.*

(1) *Let $1 < r \le 2$ and $1/r + 1/r' = 1$. Then for all $t > 0$*

if $1 < p < r'$ then $J_{X,p}(t) = (1 + t^r)^{1/r}$.
If $r' \le p < \infty$ then $J_{X,p}(t) \le (1 + Kt^r)^{1/r}$, for some $K \ge 1$.

(1) *Let $2 \le r < \infty$,$1 \le p < \infty$ and $h = \max\{r, p\}$. Then*

$$J_{X,p}(t) = \left(\frac{(1+t)^h + |1-t|^h}{2}\right)^{1/h}, \quad \forall t > 0. \tag{2.22}$$

Proof. Note that when $1 < r \le 2$, l_r, $L_r[0,1]$ are r-uniformly smooth and l_r, $L_r[0,1]$ satisfying Clarkson's inequality

$$\left(\frac{\|x+y\|^{r'} + \|x-y\|^{r'}}{2}\right)^{1/r'} \le \left(\|x\|^r + \|y\|^r\right)^{1/r}. \tag{2.23}$$

In the case of $1 < p < r'$, we get that $K = 1$ in Theorem 2.8 from [2, Remark 1]; therefore

$$J_{X,p}(t) \le (1 + t^r)^{1/r}, \quad \forall t \ge 0. \tag{2.24}$$

On the other hand, we take $x = (1, 0, \ldots), y = (0, 1, 0, \ldots)$. Then $\|x\| = \|y\| = 1$, and

$$\left(\frac{\|x+ty\|_r^p + \|x-ty\|_r^p}{2}\right)^{1/p} = (1 + t^r)^{1/r}. \tag{2.25}$$

Hence $J_{X,p}(t) = (1 + t^r)^{1/r}$ when $1 < p < r'$.

In the case of $L_r[0,1]$ we take $x(s)$, $y(s)$ such that

$$\int_0^b |x(s)|^r ds = 1, \qquad \int_b^1 |y(s)|^r ds = 1. \tag{2.26}$$

Set

$$x_1(s)=\begin{cases}x(s), & \text{if } 0\le s<b,\\ 0, & \text{if } b\le s\le 1,\end{cases}$$
$$y_1(s)=\begin{cases}0, & \text{if } 0\le s<b,\\ y(s), & \text{if } b\le s\le 1.\end{cases} \tag{2.27}$$

Then $\|x_1(s)\|=1$, $\|y_1(s)\|=1$ and

$$\left(\frac{\|x_1(s)+ty_1(s)\|_r^p+\|x_1(s)-ty_1(s)\|_r^p}{2}\right)^{1/p}=(1+t^r)^{1/r}. \tag{2.28}$$

Hence $J_{X,p}(t)=(1+t^r)^{1/r}$ when $1<p<r'$. If $r'\le p<\infty$, then $J_{X,p}(t)\le(1+Kt^r)^{1/r}$, where $K\ge 1$ from Theorem 2.8. (2) Note that when $2\le r<\infty$, l_r, $L_r[0,1]$ satisfying Hanner's inequality

$$\|x+y\|^r+\|x-y\|^r\le\big|\|x\|+\|y\|\big|^r+\big|\|x\|-\|y\|\big|^r. \tag{2.29}$$

From [3] we know that the inequality

$$\|x+y\|^r+\|x-y\|^r\le\big|\|x\|+\|\gamma y\|\big|^r+\big|\|x\|-\|\gamma y\|\big|^r \tag{2.30}$$

holds if and only if the inequality

$$\left(\frac{\|x+y\|^s+\|x-y\|^s}{2}\right)^{1/s}\le\left(\frac{\big|\|x\|+\|\gamma y\|\big|^\alpha+\big|\|x\|-\|\gamma y\|\big|^\alpha}{2}\right)^{1/a} \tag{2.31}$$

holds with some $\gamma>0$, where $1<r,s,a<\infty$. First let $s=a=p$. We get

$$J_{X,p}(t)\le\left(\frac{(1+t)^p+|1-t|^p}{2}\right)^{1/p}. \tag{2.32}$$

Similarly, let $s=p$ and $a=r$. We also get

$$J_{X,p}(t)\le\left(\frac{(1+t)^r+|1-t|^r}{2}\right)^{1/r}. \tag{2.33}$$

On the other hand, we take $x_1 = y_1 = (1,0,\ldots), x_2 = ((1/2)^{1/r},(1/2)^{1/r},\ldots)$ and $y_2 = ((1/2)^{1/r},(1/2)^{1/r},\ldots)$. Then $\|x_i\| = \|y_j\| = 1$, $i,j = 1,2$, and

$$\begin{aligned}\left(\frac{\|x_1+ty_1\|_r^p+\|x_1-ty_1\|_r^p}{2}\right)^{1/p} &= \left(\frac{(1+t)^p+|1-t|^p}{2}\right)^{1/p},\\ \left(\frac{\|x_2+ty_2\|_r^p+\|x_2-ty_2\|_r^p}{2}\right)^{1/p} &= \left(\frac{(1+t)^r+|1-t|^r}{2}\right)^{1/r}.\end{aligned} \tag{2.34}$$

Therefore we get the conclusion (2).

In the case of $L_r[0,1]$, we take $x(s) \in S(L_r[0,1])$. Then $\int_0^1|x(s)|^r ds = 1$. Take $b \in [0,1]$ such that $\int_0^b|x(s)|^r ds = 1/2$. Then $\int_b^1|x(s)|^r ds = 1/2$. Let

$$y(s) = \begin{cases} x(s), & \text{if } 0 \le s < b,\\ -x(s), & \text{if } b \le s \le 1,\end{cases} \tag{2.35}$$

and set $x_1(s) = y_1(s) = x(s)$, $x_2(s) = x(s)$, and $y_2(s) = y(s)$. Then $x_i(s) \in S(L_r[0,1])$, $y_i(s) \in S(L_r[0,1])$, $i = 1,2$, and

$$\begin{aligned}\left(\frac{\|x_1(s)+ty_1(s)\|_r^p+\|x_1(s)-ty_1(s)\|_r^p}{2}\right)^{1/p} &= \left(\frac{(1+t)^p+|1-t|^p}{2}\right)^{1/p},\\ \left(\frac{\|x_2(s)+ty_2(s)\|_r^p+\|x_2(s)-ty_2(s)\|_r^p}{2}\right)^{1/p} &= \left(\frac{(1+t)^r+|1-t|^r}{2}\right)^{1/r}.\end{aligned} \tag{2.36}$$

□

Theorem 2.11. *The following statements are equivalent:*

(1) X *is isometric to a Hilbert space;*

(2) $J_{X,p}(t) = (1+t^2)^{1/2}$ *for all* $t > 0$ *and* $1 \le p \le 2$.

Proof. (1)⇒(2). A Banach space X is isometric to a Hilbert space l_2; then $J_{X,p}(t) = (1+t^2)^{1/2}$ for all $t \ge 0$ from Theorem 2.10 when $1 \le p \le 2$.

(2)⇒(1). In the case of $p = 1$, $J_{X,1}(t) = \rho_X(t) \le \sqrt{1+t^2} - 1$; therefore X is isometric to a Hilbert space (see [4]). □

Remark 2.12. The above theorem is not true for the case of $p > 2$. In fact if $p > 2$, let X be a Hilbert space, then $J_{X,p}(t) = ((1+t)^p + |1-t|^p/2)^{1/p}$ for all $t > 0$ from Theorem 2.10.

3. Banach-Mazur Distance and Constant's Stability

Let X and Y be isomorphic Banach space. The Banach-Mazur distance between X and Y, denoted by $d(X,Y)$, is defined to be the infimum of $\|T\|\,\|T^{-1}\|$ taken over all isomorphisms T from X and Y.

Theorem 3.1. *If X and Y be isomorphic Banach space, then for* $t > 0$, $1 \le p < \infty$

$$\frac{J_{X,p}(t)}{d(X,Y)} \le J_{Y,p}(t) \le J_{X,p}(t) d(X,Y). \tag{3.1}$$

Proof. Let $x, y \in S(X)$. For each $\epsilon > 0$ there exist an isomorphism T from X and Y such that $\|T\| \, \|T^{-1}\| \le (1+\epsilon) d(X,Y)$. Set $x' = Tx/\|T\|, y' = Ty/\|T^{-1}\|$. Then $x', y' \in B(Y)$. By Proposition 2.4, we obtain that

$$\begin{aligned}\left(\frac{\|x+ty\|^p + \|x-ty\|^p}{2}\right)^{1/p} &= \|T\| \left(\frac{\|T^{-1}(x'+ty')\|^p + \|T^{-1}(x'-ty')\|^p}{2}\right)^{1/p} \\ &\le (1+\epsilon) d(X,Y) \left(\frac{\|x'+ty'\|^p + \|x'-ty'\|^p}{2}\right)^{1/p} \\ &\le (1+\epsilon) d(X,Y) J_{Y,p}(t).\end{aligned} \tag{3.2}$$

Since $\epsilon > 0$ are arbitrary, it follows that

$$J_{X,p}(t) \le d(X,Y) J_{Y,p}(t). \tag{3.3}$$

The second inequality follows by simply interchanging X and Y. □

Corollary 3.2. *Let X be a Banach space and* $t > 0$, $X_1 = (X, \|\cdot\|_1)$ *where* $\|\cdot\|_1$ *is an equivalent norm on X satisfying , for* $a, b > 0$, *and* $x \in X$,

$$a\|x\| \le \|x\|_1 \le b\|x\|, \tag{3.4}$$

then $a/b J_{X,p}(t) \le J_{X_1,p}(t) \le b/a J_{X,p}(t)$.

Proof. It follows from Theorem 3.1 and the fact that $d(X, X_1) \le b/a$. □

A Banach space X is finitely representable in a Banach space Y if for every $\epsilon > 0$ and for every finite-dimensional subspace X_0 of X, there exist is a finite-dimensional subspace Y_0 of Y with $\dim(X_0)=\dim(Y_0)$ such that $d(X_0, Y_0) \le 1+\epsilon$.

Corollary 3.3. *Let X be a Banach space, X be finitely representable in Y and* $t > 0$. *Then*

(1) $J_{X,p}(t) \le J_{Y,p}(t)$,

(2) $J_{X,p}(t) = J_{X^{**},p}(t)$.

Proof. (1) For any $x, y \in S(X)$, let X_0 be a two-dimensional subspace that contains x and y. For any $\epsilon > 0$, since X is finitely representable in Y, there exist is a two-dimensional subspace Y_0 of Y such that $d(X_0, Y_0) \le 1+\epsilon$. Applying Theorem 3.1 to the pair of X_0 and Y_0 , we obtain $J_{X,p}(t) \le (1+\epsilon) J_{Y,p}(t)$. The proof is complete since $\epsilon > 0$ is arbitrary.

(2) For any Banach space X, by the principle of local reflexivity, X^{**} is always finitely representable in X. Then $J_{X,p}(t) \geq J_{X^{**},p}(t)$ by (1). On the other hand, X is isometric to a subspace of X^{**}; therefore $J_{X,p}(t) \leq J_{X^{**},p}(t)$. □

Next we illustrate the above results by the following examples, which can give rough estimates of the constant. For $\lambda > 0$, let Z_λ be $\mathbb{R}^2$ with the norm

$$|x|_\lambda = \left(\|x\|_2^2 + \lambda\|x\|_\infty^2\right)^{1/2}, \tag{3.5}$$

then we have

$$\sqrt{(\lambda+2)/2}\|x\|_2 \leq |x|_\lambda \leq \sqrt{\lambda+1}\|x\|_2, \quad \forall x \in Z_\lambda. \tag{3.6}$$

From Corollary 3.2 we get

$$J_{Z_\lambda,p}(t) \leq \sqrt{\frac{2(\lambda+1)}{\lambda+2}} J_{l_2,p}(t). \tag{3.7}$$

Similarly we get

$$\begin{aligned} J_{X_{\lambda,r},p}(t) &\leq \lambda J_{l_r,p}(t), & J_{Y_{\lambda,r},p}(t) &\leq \lambda J_{L_r,p}(t), \\ J_{l_{r,r'},p}(t) &\leq 2^{1/r'-1/r} J_{l_r,p}(t), & J_{b_{r,r'},p}(t) &\leq 2^{1/r'-1/r} J_{l_r,p}(t), \end{aligned} \tag{3.8}$$

where $X_{\lambda,r}(\lambda \geq 1)$ is the space $l_r(2 \leq r < \infty)$ with the norm

$$\|x\|_{\lambda,r} = \max\{\|x\|_r, \lambda\|x\|_\infty\}, \tag{3.9}$$

$Y_{\lambda,r}(\lambda \geq 1)$ is the space $L_r[0,1](1 \leq r \leq 2)$ with the norm

$$\|x\|_{\lambda,r} = \max\{\|x\|_r, \lambda\|x\|_1\}, \tag{3.10}$$

and $l_{r,r'}$ is the Day-James spaces, and $b_{r,r'}$ is the Bynum spaces, respectively. Unfortunately we cannot get the exact value of $J_{X,p}(t)$ in the above spaces. However we have the following result.

Let $X = R^2$ with the norm defined by

$$\|x\| = \begin{cases} \|x\|_\infty, & x_1x_2 \geq 0, \\ \|x\|_1, & x_1x_2 \leq 0. \end{cases} \tag{3.11}$$

Then we have

$$J_{X,p}(t) = \left[\frac{1+(1+t)^p}{2}\right]^{1/p}, \quad (0 < t \le 1),$$
$$J_{X,p}(t) = \left[\frac{t^p+(1+t)^p}{2}\right]^{1/p}, \quad (1 < t < \infty). \tag{3.12}$$

Proof. It is well known that $\rho_X(t) = \max\{t/2, t-1/2\}$(see [9]), then

$$\|x+ty\|^p + \|x-ty\|^p \le 1+(1+t)^p, \quad \forall x, y \in S(X). \tag{3.13}$$

In fact, if $\|x+ty\| \le 1$, then the inequality holds obviously. If $\|x+ty\| = h(1 \le h \le 1+t)$, then we have

$$\|x+ty\|^p + \|x-ty\|^p \le h^p + [2(\rho_X(t)+1)-h]^p. \tag{3.14}$$

(1) If $0 < t \le 1$, then $\|x+ty\|^p + \|x-ty\|^p \le h^p + (2+t-h)^p := f(h)$. Note that the function $f(h)$ attain is its maximum at $h = 1$; thus we obtain the above inequality. Put $x = (1,1)$, $y = (0,1)$, then

$$\|x+ty\|^p + \|x-ty\|^p = 1+(1+t)^p. \tag{3.15}$$

Finally we have $J_{X,p}(t) = [1+(1+t)^p/2]^{1/p}$.

(2) If $1 < t < \infty$, then $\|x+ty\|^p + \|x-ty\|^p \le h^p + (2t+1-h)^p := f(h)$. Note that the function $f(h)$ attain is its maximum at $h = 1+t$; thus we obtain

$$\|x+ty\|^p + \|x-ty\|^p \le t^p + (1+t)^p. \tag{3.16}$$

Put $x = (1,1)$, $y = (0,1)$, then

$$\|x+ty\|^p + \|x-ty\|^p = t^p + (1+t)^p. \tag{3.17}$$

Thus we have $J_{X,p}(t) = [t^p + (1+t)^p/2]^{1/p}$. □

4. The Constant and the Property of Fixed Point

In 1997, García-Falset introduced the following coefficient:

$$R(X) := \sup\left\{\liminf_{n\to\infty} \|x_n + x\|\right\}, \tag{4.1}$$

where the supremum is taken over all weakly null sequences in $B(X)$ and all $x \in S(X)$. He proved that a reflexive Banach space X with $R(X) < 2$ enjoys the fixed property (see [10]). In [11], B. Sims defined the coefficient of weak orthogonality,

$$\omega(X) := \sup\left\{\lambda : \lambda \liminf_{n\to\infty}\|x_n + x\| \le \liminf_{n\to\infty}\|x_n - x\|\right\}, \tag{4.2}$$

where the supremum is taken over all $x \in X$ and all weakly null sequences $\{x_n\}$. In [12], the relation between the coefficient of weak orthogonality, the García-Falset coefficient, and James and von Neumann-Jordan constant is given in the following Theorem.

Theorem 4.1. *Let X be a Banach space. Then*

(1) *$R(X)\omega(X) \le J(X)$, and*

(2) *$(R(X))^2(1 + (\omega(X))^2) \le 4C_{NJ}(X)$.*

Similarly, one can get the relation between the coefficient of weak orthogonality, the García-Falset coefficient, and the $J^p_{X,p}(1)$ in the following Theorem.

Theorem 4.2. *Let X be a Banach space. Then*

$$2J^p_{X,p}(1) \ge \left(1 + (\omega(X))^p\right)[R(X)]^p. \tag{4.3}$$

Proof. For any $\epsilon > 0$ there exist $x \in S(X)$ and (x_n) in $B(X)$ such that

$$\liminf_{n\to\infty}\|x_n + x\| \ge R(X) - \epsilon. \tag{4.4}$$

Without loss of generality we may assume that $\lim_{n\to\infty}\|x_n+x\| \ge R(X)-\epsilon$ and $\lim_{n\to\infty}\|x_n-x\|$ exist. Now we have

$$\begin{aligned} 2J^p_{X,p}(1) &\ge \lim_{n\to\infty}\left(\|x_n + x\|^p + \|x_n - x\|^p\right) \\ &\ge (1 + (\omega(X))^p)\lim_{n\to\infty}\|x_n + x\|^p \\ &\ge (1 + (\omega(X))^p)(R(X) - \epsilon)^p. \end{aligned} \tag{4.5}$$

Letting $\epsilon \to 0$ gives the results. □

Corollary 4.3. *If $J_{X,p}(1) < 2^{1-1/p}(1 + \omega(X)^p)^{1/p}$. Then R(X) < 2.*

Proof. This is a direct result of Theorem 4.2. □

Remark 4.4. In particular $p = 1$, we get that $\rho_X(1) = \rho_{X^*}(1) < \omega(X)$; then $R(X) < 2$ and $R(X^*) < 2$. (Note that the fact that $\omega(X) = \omega(X^*)$ whenever X is reflexive is proved in [13].)

The weakly convergent sequence coefficient $WCS(X)$ (see [14]) of X is the number

$$WCS(X) := \inf\left\{\frac{A(\{x_n\})}{r_a(\{x_n\})}\right\}, \tag{4.6}$$

where the infimum is taken over all sequences $\{x_n\}$ in X which are weakly (not strongly) convergent, $A(\{x_n\}) := \limsup_n \{\|x_i - x_j\| : i, j \geq n\}$ is the asymptotic diameter of $\{x_n\}$, and $r_a(\{x_n\}) := \inf\{\limsup_n \|x_n - y\| : y \in \overline{co}(\{x_n\})\}$ is the asymptotic radius of $\{x_n\}$. In this paper, we utilize the following equivalent definitions (see [15]):

$$WCS(X) = \inf\left\{ \lim_{n \neq m} \|x_n - x_m\| \right\}, \tag{4.7}$$

where the infimum is taken over all weakly null sequence $\{x_n\} \subset X$ with $\|x_n\| = 1$ for all n and $\lim_{n \neq m} \|x_n - x_m\|$ exist. It is known that $WCS(X) > 1$ implies that X has weak uniform normal structure (see [14]).

The following lemma can be found in [16].

Lemma 4.5. *Let X be a superreflexive Banach space. Denote that $WCS(X) = d$ and X does not have Schur property. Then, there exist $\widetilde{x}_1, \widetilde{x}_2 \in S(\widetilde{X})$ and $\widetilde{f}_1, \widetilde{f}_2 \in S(\widetilde{X}^*)$ such that*

(1) $\|\widetilde{x}_1 - \widetilde{x}_2\| = d$, $\|\widetilde{x}_1 + \widetilde{x}_2\| \leq R(X)$, *and* $\widetilde{f}_i(\widetilde{x}_j) = 0$ *for all* $i \neq j$;

(2) $\widetilde{f}_i(\widetilde{x}_i) = 1$ *for* $i = 1, 2$.

Theorem 4.6. *Suppose that a Banach space X fails the Schur property and $d = WCS(X)$. Then*

$$2J^p_{X^*,p}(t) \geq \frac{(1+t)^p}{d^p} + \frac{(1+t)^p}{R(X)^p}. \tag{4.8}$$

Specially when $p = 1$ and $t = 1$, $p = 2$, we have

$$\rho_{X^*}(t) + 1 \geq \frac{1+t}{2d} + \frac{1+t}{2R(X)}, \qquad E(X^*) \geq \frac{4}{d^2} + \frac{4}{R(X)^2}. \tag{4.9}$$

Proof. Using Lemma 4.5, we have

$$\begin{aligned} 2J^p_{X^*,p}(t) &\geq \left\|\widetilde{f}_2 - t\widetilde{f}_1\right\|^p + \left\|\widetilde{f}_2 + t\widetilde{f}_1\right\|^p \\ &\geq \left\|(\widetilde{f}_2 - t\widetilde{f}_1)\left(\frac{\widetilde{x}_2 - \widetilde{x}_1}{d}\right)\right\|^p + \left\|(\widetilde{f}_2 + t\widetilde{f}_1)\left(\frac{\widetilde{x}_2 + \widetilde{x}_1}{R(X)}\right)\right\|^p \\ &\geq \frac{(1+t)^p}{d^p} + \frac{(1+t)^p}{R(X)^p}. \end{aligned} \tag{4.10}$$

□

Corollary 4.7. *Let X be a Banach space.*

(1) *If $\rho_{X^*}(t) < ((t-1)R(X) + (1+t))/2R(X)$, then X has normal structure.*

(2) *If $E(X^*) < 4 + (4/R(X)^2)$, then X has normal structure.*

Remark 4.8. The inequality (2) in Corollary 4.7 become is equality when $X = l_{2,\infty}$ and $X = l_p$ where $2 \leq p < \infty$ (see [6, 17]). Therefore the condition (2) in Corollary 4.7 isstrict.

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