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Research Article

Weighted Composition Operators from Logarithmic Bloch-Type Spaces to Bloch-Type Spaces

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The boundedness and compactness of the weighted composition operators from logarithmic Bloch-type spaces to Bloch-type spaces are studied here.

1. Introduction

Let $\mathbb{D}$ be the unit disc in the complex plane $\mathbb{C}$, $dm(z)$ the normalized Lebesgue area measure on $\mathbb{D}$, $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$, and $H^\infty(\mathbb{D})$ the space of bounded holomorphic functions on $\mathbb{D}$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}}|f(z)|$.

The logarithmic Bloch-type space $\mathcal{B}^\alpha_{\log^\beta} = \mathcal{B}^\alpha_{\log^\beta}(\mathbb{D})$, $\alpha > 0$, $\beta \geq 0$, was recently introduced in [1]. The space consists of all $f \in H(\mathbb{D})$ such that

$$b_{\alpha,\beta}(f) := \sup_{z\in\mathbb{D}}(1-|z|)^\alpha\left(\ln\frac{e^{\beta/\alpha}}{1-|z|}\right)^\beta \left|f'(z)\right| < \infty. \tag{1.1}$$

The norm on $\mathcal{B}^\alpha_{\log^\beta}$ is introduced as follows:

$$\|f\|_{\mathcal{B}^\alpha_{\log^\beta}} = \left|f(0)\right| + b_{\alpha,\beta}(f). \tag{1.2}$$

When $\beta = 0$, $\mathcal{B}^\alpha_{\log^\beta}$ becomes the α-Bloch space $\mathcal{B}^\alpha$. For α-Bloch and other Bloch-type spaces, see, for example, [1–9], as well as the related references therein. For $\alpha = \beta = 1$, $\mathcal{B}^\alpha_{\log^\beta}$ is the logarithmic Bloch space, which appeared in characterizing the multipliers of the Bloch space (see [3, 9]).

The little logarithmic Bloch-type space $\mathcal{B}^{\alpha}_{\log^{\beta},0} = \mathcal{B}^{\alpha}_{\log^{\beta},0}(\mathbb{D})$, $\alpha > 0$, $\beta \geq 0$, consists of all $f \in \mathcal{B}^{\alpha}_{\log^{\beta}}$ such that

$$\lim_{|z|\to 1-0}(1-|z|)^{\alpha}\left(\ln\frac{e^{\beta/\alpha}}{1-|z|}\right)^{\beta}\left|f'(z)\right| = 0. \tag{1.3}$$

The following theorem summarizes the basic properties of the logarithmic Bloch-type spaces. Here, as usual, for fixed $r \in [0,1)$, $f_r(z) = f(rz)$, $z \in \mathbb{D}$.

Theorem A (see [1]). *The following statements are true.*

(a) *The logarithmic Bloch-type space $\mathcal{B}^{\alpha}_{\log^{\beta}}$ is Banach with the norm given in* (1.2).

(b) *$\mathcal{B}^{\alpha}_{\log^{\beta},0}$ is a closed subset of $\mathcal{B}^{\alpha}_{\log^{\beta}}$.*

(c) *Assume $f \in \mathcal{B}^{\alpha}_{\log^{\beta}}$. Then $f \in \mathcal{B}^{\alpha}_{\log^{\beta},0}$ if and only if* $\lim_{r\to 1^-}\|f - f_r\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}} = 0$.

(d) *The set of all polynomials is dense in $\mathcal{B}^{\alpha}_{\log^{\beta},0}$.*

(e) *Assume $f \in \mathcal{B}^{\alpha}_{\log^{\beta}}$, then for each $r \in [0,1)$, $f_r \in \mathcal{B}^{\alpha}_{\log^{\beta},0}$. Moreover*

$$\|f_r\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}} \leq \|f\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}}. \tag{1.4}$$

A positive continuous function μ on $\mathbb{D}$ is called *weight*.

The Bloch-type space $\mathcal{B}_{\mu} = \mathcal{B}_{\mu}(\mathbb{D})$ consists of all $f \in H(\mathbb{D})$ such that

$$B_{\mu}(f) = \sup_{z\in\mathbb{D}}\mu(z)\left|f'(z)\right| < \infty, \tag{1.5}$$

where μ is a weight. With the norm

$$\|f\|_{\mathcal{B}_{\mu}} = \left|f(0)\right| + B_{\mu}(f), \tag{1.6}$$

the Bloch-type space becomes a Banach space.

The little Bloch-type space $\mathcal{B}_{\mu,0} = \mathcal{B}_{\mu,0}(\mathbb{D})$ is a subspace of $\mathcal{B}_{\mu}$ consisting of all f such that

$$\lim_{|z|\to 1}\mu(z)\left|f'(z)\right| = 0. \tag{1.7}$$

Let φ be a holomorphic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. For $f \in H(\mathbb{D})$ the corresponding weighted composition operator is defined by

$$(uC_{\varphi})(f)(z) = u(z)f(\varphi(z)), \quad z \in \mathbb{D}. \tag{1.8}$$

It is of interest to provide function-theoretic characterizations for when φ and u induce bounded or compact weighted composition operators on spaces of holomorphic functions.

For some classical results mostly on composition operators, see, for example, [10]. For some recent related results, mostly in $\mathbb{C}^n$ or related to Bloch-type or weighted-type spaces, see, for example, [4, 10–46] and the references therein.

Here we study the boundedness and compactness of the weighted composition operator from the logarithmic Bloch-type space and the little logarithmic Bloch-type space to the Bloch-type or the little Bloch-type space.

In this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \preccurlyeq b$ means that there is a positive constant C such that $a \le Cb$. We say that $a \asymp b$, if both $a \preccurlyeq b$ and $b \preccurlyeq a$ hold.

2. Auxiliary Results

In this section we quote several auxiliary results which will be used in the proofs of the main results.

Lemma 2.1. *Assume $\alpha > 0$, $\beta \ge 0$, then the following statements are true.*

(a) *Assume $\gamma \ge \beta/\alpha + \ln 2$, then the function*

$$h(x) = x^{\alpha}\left(\ln \frac{e^{\gamma}}{x}\right)^{\beta} \tag{2.1}$$

is increasing on the interval $(0, 2]$.

(b) *The function*

$$h_1(x) = x^{\alpha}\left(\ln \frac{e^{\beta/\alpha}}{x}\right)^{\beta} \tag{2.2}$$

is increasing on the interval $(0, 1]$.

Proof. (a) We have

$$h'(x) = x^{\alpha-1}\left(\ln \frac{e^{\gamma}}{x}\right)^{\beta-1}\left(\alpha \ln \frac{e^{\gamma}}{x} - \beta\right). \tag{2.3}$$

Since $x^{\alpha-1}(\ln(e^{\gamma}/x))^{\beta-1} > 0$, when $x \in (0, 2)$ and $\gamma \ge \beta/\alpha + \ln 2$, and the function $H(x) = \alpha \ln(e^{\gamma}/x) - \beta$ is decreasing on the interval $(0, 2]$, we have

$$\alpha \ln \frac{e^{\gamma}}{x} - \beta > \alpha \ln \frac{e^{\gamma}}{2} - \beta = \alpha\left(\gamma - \ln 2 - \frac{\beta}{\alpha}\right) \ge 0, \quad x \in (0, 2), \tag{2.4}$$

from which this statement follows.

The proof of (b) is similar, hence it is omitted. □

The next lemma regarding the point evaluation functional on $\mathcal{B}^{\alpha}_{\log^{\beta}}$ follows from [1, Lemma 3] and some elementary asymptotic relationship, such as

$$(1-|z|)^{\alpha-1}\left(\ln\frac{e^{\beta/\alpha}}{1-|z|}\right)^{\beta} \asymp \left(1-|z|^2\right)^{\alpha-1}\left(\ln\frac{e^{\beta/\alpha}}{1-|z|^2}\right)^{\beta}, \quad \alpha>1,\ \beta\geq 0. \tag{2.5}$$

Lemma 2.2. *Let $f\in\mathcal{B}^{\alpha}_{\log^{\beta}}(\mathbb{D})$. Then*

$$|f(z)|\leq C\begin{cases}\|f\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}}, & \alpha\in(0,1)\ \text{or}\ \alpha=1,\ \beta>1,\\ |f(0)|+\|f\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}}\ln\ln\dfrac{e^{\beta/\alpha}}{1-|z|^2}, & \alpha=\beta=1,\\ |f(0)|+\|f\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}}\left(\ln\dfrac{e^{\beta/\alpha}}{1-|z|^2}\right)^{1-\beta}, & \alpha=1,\ \beta\in(0,1),\\ |f(0)|+\dfrac{\|f\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}}}{\left(1-|z|^2\right)^{\alpha-1}\left(\ln\left(e^{\beta/\alpha}/\left(1-|z|^2\right)\right)\right)^{\beta}}, & \alpha>1,\ \beta\geq 0,\end{cases} \tag{2.6}$$

for some $C>0$ independent of f.

The proof of the following lemma is similar to [25, Lemma 2.1], so we omit it.

Lemma 2.3. *Assume μ is a weight. A closed set K in $\mathcal{B}_{\mu,0}$ is compact if and only if it is bounded and*

$$\lim_{|z|\to 1}\sup_{f\in K}\mu(z)\left|f'(z)\right|=0. \tag{2.7}$$

Remark 2.4. If in Lemma 2.3 we assume that K is not closed, then the word *compact* can be replaced by *relatively compact*.

The next characterization of compactness is proved in a standard way (see, e.g., the proofs of the corresponding lemmas in [10, 30, 47–49]). Hence we omit it.

Lemma 2.5. *Assume that $u\in H(\mathbb{D})$, φ is a holomorphic self-map of $\mathbb{D}$, and μ is a weight. Let X be one of the following spaces $\mathcal{B}^{\alpha}_{\log^{\beta}}$, $\mathcal{B}^{\alpha}_{\log^{\beta},0}$, and Y one of the spaces $\mathcal{B}_{\mu}$, $\mathcal{B}_{\mu,0}$. Then the operator $uC_{\varphi}:X\to Y$ is compact if and only if $uC_{\varphi}:X\to Y$ is bounded and for every bounded sequence $(f_k)_{k\in\mathbb{N}}\subset X$ converging to 0 uniformly on compacts of $\mathbb{D}$ one has*

$$\lim_{k\to\infty}\left\|uC_{\varphi}f_k\right\|_{Y}=0. \tag{2.8}$$

Some concrete examples of the functions belonging to logarithmic Bloch-type spaces can be found in the next lemma.

Lemma 2.6. *The following statements are true.*

(a) *Assume that $\alpha \neq 1$ and $\beta \geq 0$, then*

$$f_w(z) = \frac{1}{(1 - z\overline{w})^{\alpha-1}(\ln(e^{\gamma}/(1 - z\overline{w})))^{\beta}}, \quad w \in \mathbb{D}, \tag{2.9}$$

where $\gamma \geq \beta/\alpha + \ln 2$ and $f_w(0) = 1/\gamma^{\beta}$ is a nonconstant function belonging to $\mathcal{B}^{\alpha}_{\log^{\beta}}$.

(b) *Assume that $\alpha = 1$ and $\beta \in [0, \infty) \setminus \{1\}$, then*

$$f_w^{(1)}(z) = \left(\ln \frac{e^{\gamma}}{1 - z\overline{w}}\right)^{1-\beta}, \quad w \in \mathbb{D}, \tag{2.10}$$

where $\gamma \geq \beta + \ln 2$ and $f_w^{(1)}(0) = \gamma^{1-\beta}$ is a nonconstant function belonging to $\mathcal{B}^{\alpha}_{\log^{\beta}}$.

(c) *Assume that $\alpha = \beta = 1$, then*

$$f_w^{(2)}(z) = \ln\ln \frac{e^{\gamma}}{1 - z\overline{w}}, \quad w \in \mathbb{D}, \tag{2.11}$$

where $\gamma \geq 1 + \ln 2$ and $f_w^{(2)}(0) = \ln\gamma$ is a nonconstant function belonging to $\mathcal{B}^{\alpha}_{\log^{\beta}}$.

Moreover, for each $w \in \mathbb{D}$, it holds that f_w, $f_w^{(1)}$, $f_w^{(2)}$ belong to the corresponding $\mathcal{B}^{\alpha}_{\log^{\beta},0}$ space, and for fixed α and β

$$\sup_{w\in\mathbb{D}} \|f_w\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}} \leq C, \qquad \sup_{w\in\mathbb{D}} \left\|f_w^{(1)}\right\|_{\mathcal{B}^{1}_{\log^{\beta}}} \leq C, \qquad \sup_{w\in\mathbb{D}} \left\|f_w^{(2)}\right\|_{\mathcal{B}^{1}_{\log^{1}}} \leq C. \tag{2.12}$$

Proof. (a) Let $w \in \mathbb{D}$ be fixed. Then we have

$$\begin{aligned}
&(1-|z|)^{\alpha}\left(\ln \frac{e^{\beta/\alpha}}{1-|z|}\right)^{\beta}\left|f_w'(z)\right| \\
&\quad = (1-|z|)^{\alpha}\left(\ln \frac{e^{\beta/\alpha}}{1-|z|}\right)^{\beta} \\
&\quad\quad \times \left|\frac{(\alpha-1)\overline{w}}{(1-z\overline{w})^{\alpha}(\ln(e^{\gamma}/(1-z\overline{w})))^{\beta}} - \frac{\beta\overline{w}}{(1-z\overline{w})^{\alpha}(\ln(e^{\gamma}/(1-z\overline{w})))^{\beta+1}}\right| \qquad (2.13)\\
&\quad \le |\alpha-1|\frac{(1-|z|)^{\alpha}\left(\ln\left(e^{\beta/\alpha}/(1-|z|)\right)\right)^{\beta}}{|1-z\overline{w}|^{\alpha}(\ln(e^{\gamma}/|1-z\overline{w}|))^{\beta}} + \beta\frac{(1-|z|)^{\alpha}\left(\ln\left(e^{\beta/\alpha}/(1-|z|)\right)\right)^{\beta}}{|1-z\overline{w}|^{\alpha}(\ln(e^{\gamma}/|1-z\overline{w}|))^{\beta+1}} \\
&\quad \le \left(|\alpha-1| + \frac{\beta}{\ln(e^{\gamma}/2)}\right)\frac{(1-|z|)^{\alpha}(\ln(e^{\gamma}/(1-|z|)))^{\beta}}{|1-z\overline{w}|^{\alpha}(\ln(e^{\gamma}/|1-z\overline{w}|))^{\beta}} \\
&\quad \le |\alpha-1| + \frac{\beta}{\ln(e^{\gamma}/2)}, \qquad (2.14)
\end{aligned}$$

where in (2.13) we have used that $\gamma > \beta/\alpha$ and in (2.14) we have used the fact that the function in (2.1) is increasing on the interval $(0,2]$.

From (2.13), since $1-|w| \le |1-z\overline{w}|$, $z, w \in \mathbb{D}$, and by Lemma 2.1(a), we have that

$$\begin{aligned}
&(1-|z|)^{\alpha}\left(\ln \frac{e^{\beta/\alpha}}{1-|z|}\right)^{\beta}\left|f_w'(z)\right| \\
&\quad \le \left(|\alpha-1| + \frac{\beta}{\ln(e^{\gamma}/2)}\right)\frac{(1-|z|)^{\alpha}(\ln(e^{\gamma}/(1-|z|)))^{\beta}}{(1-|w|)^{\alpha}(\ln(e^{\gamma}/(1-|w|)))^{\beta}} \longrightarrow 0,
\end{aligned} \qquad (2.15)$$

as $|z| \to 1-0$, from which it follows that $f_w \in \mathcal{B}^{\alpha}_{\log^{\beta},0}$, as desired.

(b) For fixed $w \in \mathbb{D}$, we have

$$\begin{aligned}
(1-|z|)\left(\ln \frac{e^{\beta}}{1-|z|}\right)^{\beta}\left|\left(f_w^{(1)}\right)'(z)\right| &= (1-|z|)\left(\ln \frac{e^{\beta}}{1-|z|}\right)^{\beta}\left|\frac{(1-\beta)\overline{w}}{(1-z\overline{w})(\ln(e^{\gamma}/(1-z\overline{w})))^{\beta}}\right| \\
&\le \left|\beta-1\right|\frac{(1-|z|)(\ln(e^{\gamma}/(1-|z|)))^{\beta}}{|1-z\overline{w}|(\ln(e^{\gamma}/|1-z\overline{w}|))^{\beta}} \qquad (2.16)\\
&\le \left|\beta-1\right|, \qquad (2.17)
\end{aligned}$$

where in (2.16) we have used the assumption $\gamma > \beta$, while in (2.17), as in (a), we have used the fact that the function in (2.1) is increasing on the interval $(0,2]$.

From (2.16), and by Lemma 2.1(a), we obtain

$$(1-|z|)\left(\ln\frac{e^{\beta}}{1-|z|}\right)^{\beta}\left|\left(f_w^{(1)}\right)'(z)\right| \le |\beta-1|\frac{(1-|z|)(\ln(e^{\gamma}/(1-|z|)))^{\beta}}{(1-|w|)(\ln(e^{\gamma}/(1-|w|)))^{\beta}} \longrightarrow 0, \tag{2.18}$$

as $|z| \to 1-0$. Hence $f_w^{(1)} \in \mathcal{B}^1_{\log^{\beta},0}$, finishing the proof of this statement.

(c) We have

$$\begin{aligned}(1-|z|)\left(\ln\frac{e}{1-|z|}\right)\left|\left(f_w^{(2)}\right)'(z)\right| &= (1-|z|)\left(\ln\frac{e}{1-|z|}\right)\left|\frac{\overline{w}}{(1-z\overline{w})\ln(e^{\gamma}/(1-z\overline{w}))}\right| \\ &\le \frac{(1-|z|)\ln(e/(1-|z|))}{|1-z\overline{w}|\ln(e^{\gamma}/|1-z\overline{w}|)} \end{aligned}\tag{2.19}$$

$$\le \frac{(1-|z|)\ln(e^{\gamma}/(1-|z|))}{(1-|z|)\ln(e^{\gamma}/(1-|z|))} \le 1, \tag{2.20}$$

where we have used the assumption $\gamma > 1$ and the fact that function (2.1) is increasing on $(0,2]$.

From (2.19), Lemma 2.1(a), and since $\gamma > 1$ we obtain

$$(1-|z|)\left(\ln\frac{e}{1-|z|}\right)\left|f_w'(z)\right| \le \frac{(1-|z|)(\ln(e^{\gamma}/(1-|z|)))}{(1-|w|)(\ln(e^{\gamma}/(1-|w|)))} \longrightarrow 0, \tag{2.21}$$

as $|z| \to 1^-$, that is, $f_w^{(2)} \in \mathcal{B}^1_{\log^1,0}$.

Estimations (2.12) follow from (2.14), (2.17), (2.20) and by using the following facts

$$\begin{aligned} f_w(0) &= \frac{1}{\gamma^{\beta}}, \quad \alpha \ne 1,\ \beta \ge 1, \\ f_w^{(1)}(0) &= \gamma^{1-\beta}, \quad \alpha = 1,\ \beta \in (0,1), \\ f_w^{(2)}(0) &= \ln\gamma, \quad \alpha = \beta = 1, \end{aligned}\tag{2.22}$$

we finish the proof of the lemma. □

Remark 2.7. Note that from Lemmas 2.2 and 2.6 the functions f_w, $f_w^{(1)}$, $f_w^{(2)}$ defined in (2.9)–(2.11) have maximal growths in the corresponding logarithmic Bloch-type spaces.

3. Boundedness and Compactness of the Operator $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta}(\mathbb{D})$ (or $\mathcal{B}^\alpha_{\log^\beta,0}(\mathbb{D})) \to \mathcal{B}_\mu(\mathbb{D})$

This section studies the boundedness and compactness of the weighted composition operator $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta}(\mathbb{D})$ (or $\mathcal{B}^\alpha_{\log^\beta,0}(\mathbb{D})) \to \mathcal{B}_\mu(\mathbb{D})$.

Case 1. $\alpha > 1$, $\beta \geq 0$.

Theorem 3.1. *Assume $\alpha > 1$, $\beta \geq 0$, φ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$, and μ is a weight. Then the operator $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta}$ (or $\mathcal{B}^\alpha_{\log^\beta,0}$) $\to \mathcal{B}_\mu$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \mu(z)\left|u'(z)\right|\left(1+\frac{1}{\left(1-|\varphi(z)|^2\right)^{\alpha-1}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(z)|^2\right)\right)\right)^\beta}\right)<\infty, \tag{3.1}$$

$$\sup_{z\in\mathbb{D}}\frac{\mu(z)|u(z)|\left|\varphi'(z)\right|}{\left(1-|\varphi(z)|^2\right)^{\alpha}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(z)|^2\right)\right)\right)^\beta}<\infty. \tag{3.2}$$

Proof. First assume that (3.1) and (3.2) hold. Then, by Lemma 2.2 and the definition of $\mathcal{B}^\alpha_{\log^\beta}$, we have

$$\begin{aligned}\left\|uC_\varphi f\right\|_{\mathcal{B}_\mu} &= \left|u(0)f(\varphi(0))\right| + \sup_{z\in\mathbb{D}}\mu(z)\left|u'(z)f(\varphi(z))+u(z)f'(\varphi(z))\varphi'(z)\right| &(3.3)\\
&\leq C|u(0)|\|f\|_{\mathcal{B}^\alpha_{\log^\beta}}\left(1+\frac{1}{\left(1-|\varphi(0)|^2\right)^{\alpha-1}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(0)|^2\right)\right)\right)^\beta}\right)\\
&\quad + C\|f\|_{\mathcal{B}^\alpha_{\log^\beta}}\sup_{z\in\mathbb{D}}\left(\mu(z)\left|u'(z)\right|+\frac{\mu(z)|u'(z)|}{\left(1-|\varphi(z)|^2\right)^{\alpha-1}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(z)|^2\right)\right)\right)^\beta}\right)\\
&\quad + \|f\|_{\mathcal{B}^\alpha_{\log^\beta}}\sup_{z\in\mathbb{D}}\frac{\mu(z)|u(z)|\left|\varphi'(z)\right|}{\left(1-|\varphi(z)|^2\right)^{\alpha}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(z)|^2\right)\right)\right)^\beta}. &(3.4)\end{aligned}$$

Applying (3.1) and (3.2) in (3.4), the boundedness of $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta}$ (or $\mathcal{B}^\alpha_{\log^\beta,0}$) $\to \mathcal{B}_\mu$ follows.

Now assume the operator $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta}$ (or $\mathcal{B}^\alpha_{\log^\beta,0}$) $\to \mathcal{B}_\mu$ is bounded. By taking the test functions $f(z)\equiv 1$ and $f(z)\equiv z$ (which obviously belong to $\mathcal{B}^\alpha_{\log^\beta,0}$), we obtain

$$\sup_{z\in\mathbb{D}}\mu(z)\left|u'(z)\right|<\infty, \tag{3.5}$$

$$\sup_{z\in\mathbb{D}}\mu(z)\left|u'(z)\varphi(z)+u(z)\varphi'(z)\right|<\infty. \tag{3.6}$$

From (3.5) and (3.6), and since the function φ is bounded, it follows that

$$\sup_{z\in\mathbb{D}} \mu(z)\left|u(z)\varphi'(z)\right| < \infty. \tag{3.7}$$

For $w \in \mathbb{D}$, set

$$g_w(z) = \frac{\left(1-|w|^2\right)}{(1-\overline{w}z)^{\alpha}\left(\ln\left(e^{\beta/\alpha}/(1-\overline{w}z)\right)\right)^{\beta}} - \frac{\left(1-|w|^2\right)^2}{(1-\overline{w}z)^{\alpha+1}\left(\ln\left(e^{\beta/\alpha}/(1-\overline{w}z)\right)\right)^{\beta}}, \quad z \in \mathbb{D}. \tag{3.8}$$

We have that $g_w(w) = 0$,

$$g_w'(w) = -\frac{\overline{w}}{\left(1-|w|^2\right)^{\alpha}\left(\ln\left(e^{\beta/\alpha}/\left(1-|w|^2\right)\right)\right)^{\beta}}, \tag{3.9}$$

and as an easy consequence of Lemma 2.6(a), $\sup_{w\in\mathbb{D}}\|g_w\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}} \le C$ and $g_w \in \mathcal{B}^{\alpha}_{\log^{\beta},0}$ for each $w \in \mathbb{D}$.

Using these facts and the boundedness of $uC_\varphi : \mathcal{B}^{\alpha}_{\log^{\beta}}$ (or $\mathcal{B}^{\alpha}_{\log^{\beta},0}$) $\to \mathcal{B}_\mu$, for the test functions $g_{\varphi(w)}$, where $w \in \mathbb{D}$ and $\varphi(w) \ne 0$, we get

$$\frac{\mu(w)\left|u(w)\varphi'(w)\overline{\varphi(w)}\right|}{\left(1-|\varphi(w)|^2\right)^{\alpha}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(w)|^2\right)\right)\right)^{\beta}} \le \left\|uC_\varphi g_{\varphi(w)}\right\|_{\mathcal{B}_\mu} \le C\left\|uC_\varphi\right\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}\to\mathcal{B}_\mu}. \tag{3.10}$$

From (3.10) it follows that

$$\sup_{|\varphi(w)|>1/2} \frac{\mu(w)\left|u(w)\varphi'(w)\right|}{\left(1-|\varphi(w)|^2\right)^{\alpha}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(w)|^2\right)\right)\right)^{\beta}} \le 2C\left\|uC_\varphi\right\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}\to\mathcal{B}_\mu}. \tag{3.11}$$

On the other hand, by using (3.7) and Lemma 2.1(b), we have

$$\sup_{|\varphi(w)|\le 1/2} \frac{\mu(w)\left|u(w)\varphi'(w)\right|}{\left(1-|\varphi(w)|^2\right)^{\alpha}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(w)|^2\right)\right)\right)^{\beta}} < \sup_{|w|<1} \frac{\mu(w)|u(w)|\left|\varphi'(w)\right|}{(3/4)^{\alpha}\ln^{\beta}\left(4e^{\beta/\alpha}/3\right)} < \infty. \tag{3.12}$$

Hence, (3.11) and (3.12) imply (3.2).

Let

$$F_w(z) = \frac{(\alpha+1)\left(1-|w|^2\right)}{(1-\overline{w}z)^{\alpha}\left(\ln\left(e^{\beta/\alpha}/(1-\overline{w}z)\right)\right)^{\beta}} - \frac{\alpha\left(1-|w|^2\right)^2}{(1-\overline{w}z)^{\alpha+1}\left(\ln\left(e^{\beta/\alpha}/(1-\overline{w}z)\right)\right)^{\beta}}. \tag{3.13}$$

Then

$$F_w(w) = \frac{1}{\left(1-|w|^2\right)^{\alpha-1}\left(\ln\left(e^{\beta/\alpha}/\left(1-|w|^2\right)\right)\right)^{\beta}},$$
$$F'_w(w) = -\frac{\beta\overline{w}}{\left(1-|w|^2\right)^{\alpha}\left(\ln\left(e^{\beta/\alpha}/\left(1-|w|^2\right)\right)\right)^{\beta+1}},\tag{3.14}$$

and by Lemma 2.6(a) we get $\sup_{w\in\mathbb{D}}\|F_w\|_{\mathcal{B}^{\alpha}_{\log^\beta}} \le C$, and $F_w \in \mathcal{B}^{\alpha}_{\log^\beta,0}$ for every $w \in \mathbb{D}$. Using the boundedness of $uC_\varphi : \mathcal{B}^{\alpha}_{\log^\beta}$ (or $\mathcal{B}^{\alpha}_{\log^\beta,0}$) $\to \mathcal{B}_\mu$, the test functions $F_{\varphi(w)}$, and equalities (3.14) we get

$$\frac{\mu(w)|u'(w)|}{\left(1-|\varphi(w)|^2\right)^{\alpha-1}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(w)|^2\right)\right)\right)^{\beta}}$$
$$\le \left\|uC_\varphi F_{\varphi(w)}\right\|_{\mathcal{B}_\mu} + \frac{\beta\mu(w)|u(w)|\left|\varphi'(w)\right|\left|\varphi(w)\right|}{\left(1-|\varphi(w)|^2\right)^{\alpha}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(w)|^2\right)\right)\right)^{\beta+1}}\tag{3.15}$$

for each $\varphi(w) \ne 0$, $w \in \mathbb{D}$.

From (3.2), (3.5), (3.15), and using the fact that

$$\sup_{x\in[0,1)}\left(\ln\frac{e^{\beta/\alpha}}{1-x^2}\right)^{-1} \le \frac{\alpha}{\beta}, \quad \text{when } \beta > 0,\tag{3.16}$$

condition (3.1) follows. □

Theorem 3.2. *Assume $\alpha > 1$, $\beta \ge 0$, φ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$, and μ is a weight. Then the operator $uC_\varphi : \mathcal{B}^{\alpha}_{\log^\beta}$ (or $\mathcal{B}^{\alpha}_{\log^\beta,0}$) $\to \mathcal{B}_\mu$ is compact if and only if $uC_\varphi : \mathcal{B}^{\alpha}_{\log^\beta}$ (or $\mathcal{B}^{\alpha}_{\log^\beta,0}$) $\to \mathcal{B}_\mu$ is bounded*

$$\lim_{|\varphi(z)|\to 1}\mu(z)\left|u'(z)\right|\left(1+\frac{1}{\left(1-|\varphi(z)|^2\right)^{\alpha-1}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(z)|^2\right)\right)\right)^{\beta}}\right) = 0,\tag{3.17}$$

$$\lim_{|\varphi(z)|\to 1}\frac{\mu(z)|u(z)|\left|\varphi'(z)\right|}{\left(1-|\varphi(z)|^2\right)^{\alpha}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(z)|^2\right)\right)\right)^{\beta}} = 0.\tag{3.18}$$

Proof. Suppose that $uC_\varphi : \mathcal{B}^{\alpha}_{\log^\beta}$ (or $\mathcal{B}^{\alpha}_{\log^\beta,0}$) $\to \mathcal{B}_\mu$ is compact. Then it is clear that $uC_\varphi : \mathcal{B}^{\alpha}_{\log^\beta}$ (or $\mathcal{B}^{\alpha}_{\log^\beta,0}$) $\to \mathcal{B}_\mu$ is bounded. If $\|\varphi\|_\infty < 1$, then (3.17) and (3.18) are vacuously satisfied.

Hence assume that $\|\varphi\|_\infty = 1$. Let $(z_m)_{m\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_m)| \to 1$ as $m \to \infty$, and $g_m(z) = g_{\varphi(z_m)}(z)$, $m \in \mathbb{N}$, where g_w is defined in (3.8). Then $\sup_{m\in\mathbb{N}}\|g_m\|_{\mathcal{B}^\alpha_{\log^\beta}} < \infty$, $g_m \to 0$ uniformly on compacts of $\mathbb{D}$ as $m \to \infty$, $g_m(\varphi(z_m)) = 0$, and

$$g_m'(\varphi(z_m)) = -\frac{\overline{\varphi(z_m)}}{(1-|\varphi(z_m)|^2)^\alpha\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(z_m)|^2\right)\right)\right)^\beta}. \tag{3.19}$$

Hence from (3.10) and Lemma 2.5 we have that

$$\frac{\mu(z_m)\left|u(z_m)\varphi'(z_m)\overline{\varphi(z_m)}\right|}{(1-|\varphi(z_m)|^2)^\alpha\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(z_m)|^2\right)\right)\right)^\beta} \le \left\|uC_\varphi g_m\right\|_{\mathcal{B}_\mu} \longrightarrow 0 \quad \text{as } m \longrightarrow \infty \tag{3.20}$$

from which (3.18) follows.

Let $F_m = F_{\varphi(z_m)}$, $m \in \mathbb{N}$ where F_w is defined in (3.13). Then $\sup_{m\in\mathbb{N}}\|F_m\|_{\mathcal{B}^\alpha_{\log^\beta}} < \infty$ and $F_m \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $m \to \infty$. Since $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta}$ (or $\mathcal{B}^\alpha_{\log^\beta,0}$) $\to \mathcal{B}_\mu$ is compact, we see that

$$\lim_{m\to\infty}\|uC_\varphi F_m\|_{\mathcal{B}_\mu} = 0. \tag{3.21}$$

From (3.15) we have

$$\begin{aligned}&\frac{\mu(z_m)|u'(z_m)|}{(1-|\varphi(z_m)|^2)^{\alpha-1}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(z_m)|^2\right)\right)\right)^\beta}\\ &\quad\le \|uC_\varphi F_m\|_{\mathcal{B}_\mu} + \frac{\beta\mu(z_m)\left|u(z_m)\varphi'(z_m)\overline{\varphi(z_m)}\right|}{(1-|\varphi(z_m)|^2)^\alpha\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(z_m)|^2\right)\right)\right)^{\beta+1}},\end{aligned} \tag{3.22}$$

which along with (3.16), (3.18), and (3.21) implies

$$\lim_{|\varphi(z)|\to 1}\frac{\mu(z)|u'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(z)|^2\right)\right)\right)^\beta} = 0. \tag{3.23}$$

On the other hand, we have

$$\frac{\mu(z)|u'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(z)|^2\right)\right)\right)^\beta} \ge C\mu(z)\left|u'(z)\right|, \tag{3.24}$$

for some positive C. From (3.23) and (3.24), equality (3.17) follows.

Conversely, assume that $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta}$ (or $\mathcal{B}^\alpha_{\log^\beta,0}$) $\to \mathcal{B}_\mu$ is bounded and (3.17) and (3.18) hold. From the proof of Theorem 3.1 we know that

$$B_\mu(u) = \sup_{z\in\mathbb{D}} \mu(z)\left|u'(z)\right| < \infty, \qquad K_2 = \sup_{z\in\mathbb{D}} \mu(z)\left|\varphi'(z)\right||u(z)| < \infty. \tag{3.25}$$

On the other hand, from (3.17) and (3.18) we have that, for every $\varepsilon > 0$, there is a $\delta \in (0,1)$, such that

$$\begin{gathered} \mu(z)\left|u'(z)\right|\left(1+\frac{1}{\left(1-\left|\varphi(z)\right|^2\right)^{\alpha-1}\left(\ln\left(e^{\beta/\alpha}/\left(1-\left|\varphi(z)\right|^2\right)\right)\right)^\beta}\right) < \varepsilon, \\ \frac{\mu(z)|u(z)|\left|\varphi'(z)\right|}{(1-|\varphi(z)|^2)^\alpha\left(\ln\left(e^{\beta/\alpha}/\left(1-\left|\varphi(z)\right|^2\right)\right)\right)^\beta} < \varepsilon \end{gathered} \tag{3.26}$$

whenever $\delta < |\varphi(z)| < 1$.

Assume $(f_m)_{m\in\mathbb{N}}$ is a sequence in $\mathcal{B}^\alpha_{\log^\beta}$ (or $\mathcal{B}^\alpha_{\log^\beta,0}$) such that $\sup_{m\in\mathbb{N}}\|f_m\|_{\mathcal{B}^\alpha_{\log^\beta}} \le L$ and f_m converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $m \to \infty$. Let $K = \{z \in \mathbb{D} : |\varphi(z)| \le \delta\}$. Then from (3.25), (3.26), and by Lemma 2.2, it follows that

$$\begin{aligned} &\sup_{z\in\mathbb{D}} \mu(z)\left|\left(uC_\varphi f_m\right)'(z)\right| \\ &\quad\le \sup_{z\in K}\mu(z)\left|\varphi'(z)\right||u(z)|\left|f_m'(\varphi(z))\right| + \sup_{z\in K}\mu(z)\left|u'(z)\right|\left|f_m(\varphi(z))\right| \\ &\qquad + \sup_{z\in\mathbb{D}\setminus K}\mu(z)\left|\varphi'(z)\right||u(z)|\left|f_m'(\varphi(z))\right| + \sup_{z\in\mathbb{D}\setminus K}\mu(z)\left|u'(z)\right|\left|f_m(\varphi(z))\right| \\ &\quad\le K_2\sup_{|w|\le\delta}\left|f_m'(w)\right| + C\sup_{z\in\mathbb{D}\setminus K}\mu(z)\left|u'(z)\right| \\ &\qquad \times\left(1+\frac{1}{\left(1-\left|\varphi(z)\right|^2\right)^{\alpha-1}\left(\ln\left(e^{\beta/\alpha}/\left(1-\left|\varphi(z)\right|^2\right)\right)\right)^\beta}\right)\|f_m\|_{\mathcal{B}^\alpha_{\log^\beta}} \\ &\qquad + B_\mu(u)\sup_{|w|\le\delta}\left|f_m(w)\right| + C\sup_{z\in\mathbb{D}\setminus K}\frac{\mu(z)|u(z)|\left|\varphi'(z)\right|}{(1-|\varphi(z)|^2)^\alpha\left(\ln\left(e^{\beta/\alpha}/\left(1-\left|\varphi(z)\right|^2\right)\right)\right)^\beta}\|f_m\|_{\mathcal{B}^\alpha_{\log^\beta}} \\ &\quad\le K_2\sup_{|w|\le\delta}\left|f_m'(w)\right| + B_\mu(u)\sup_{|w|\le\delta}\left|f_m(w)\right| + 2C\varepsilon\|f_m\|_{\mathcal{B}^\alpha_{\log^\beta}}. \end{aligned} \tag{3.27}$$

Therefore

$$\begin{aligned}\|uC_\varphi f_m\|_{\mathcal{B}_\mu} &= \left|f_m(\varphi(0))\right||u(0)| + \sup_{z\in\mathbb{D}}\mu(z)\left|(uC_\varphi f_m)'(z)\right| \\ &\leq K_2\sup_{|w|\leq\delta}\left|f_m'(w)\right| + B_\mu(u)\sup_{|w|\leq\delta}\left|f_m(w)\right| + 2CL\varepsilon + \left|f_m(\varphi(0))\right||u(0)|.\end{aligned} \tag{3.28}$$

Since $(f_m)_{m\in\mathbb{N}}$ converges to zero on compact subsets of $\mathbb{D}$ as $m \to \infty$, by the Weierstrass theorem it follows that the sequence $(f_m')_{m\in\mathbb{N}}$ also converges to zero on compact subsets of $\mathbb{D}$ as $m \to \infty$, in particular $\lim_{m\to\infty}\sup_{|w|\leq\delta}|f_m'(w)| = 0$ and $\lim_{m\to\infty}|f_m(\varphi(0))| = 0$. Using these facts and letting $m \to \infty$ in the last inequality, we obtain that

$$\limsup_{m\to\infty}\left\|uC_\varphi f_m\right\|_{\mathcal{B}_\mu} \leq 2CL\varepsilon. \tag{3.29}$$

Since ε is an arbitrary positive number it follows that the last limit is equal to zero. Applying Lemma 2.5, the implication follows. □

Theorem 3.3. *Assume $\alpha > 0$, $\beta \geq 0$, φ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$, and μ is a weight. Then $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta,0} \to \mathcal{B}_{\mu,0}$ is bounded if and only if $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta,0} \to \mathcal{B}_\mu$ is bounded*

$$\lim_{|z|\to 1}\mu(z)\left|u'(z)\right| = 0, \tag{3.30}$$

$$\lim_{|z|\to 1}\mu(z)|u(z)|\left|\varphi'(z)\right| = 0. \tag{3.31}$$

Proof. First assume that $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta,0} \to \mathcal{B}_{\mu,0}$ is bounded. Then, it is clear that $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta,0} \to \mathcal{B}_\mu$ is bounded, and as usual by taking the test functions $f(z) \equiv 1$ and $f(z) \equiv z$, and using the fact $\|\varphi\|_\infty \leq 1$, we obtain (3.30) and (3.31).

Conversely, assume that the operator $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta,0} \to \mathcal{B}_\mu$ is bounded, $u \in \mathcal{B}_{\mu,0}$, and condition (3.31) holds.

Then, for each polynomial p, we have

$$\begin{aligned}\mu(z)\left|(uC_\varphi p)'(z)\right| &\leq \mu(z)\left|u'(z)\right|\left|p(\varphi(z))\right| + \mu(z)\left|u(z)\varphi'(z)p'(\varphi(z))\right| \\ &\leq \mu(z)\left|u'(z)\right|\|p\|_\infty + \mu(z)\left|u(z)\varphi'(z)\right|\|p'\|_\infty,\end{aligned} \tag{3.32}$$

from which along with conditions (3.30) and (3.31) it follows that $uC_\varphi p \in \mathcal{B}_{\mu,0}$. Since according to Theorem A the set of all polynomials is dense in $\mathcal{B}^\alpha_{\log^\beta,0}$, we see that for every $f \in \mathcal{B}^\alpha_{\log^\beta,0}$ there is a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty}\left\|f - p_n\right\|_{\mathcal{B}^\alpha_{\log^\beta}} = 0. \tag{3.33}$$

From this and by the boundedness of the operator $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta,0} \to \mathcal{B}_\mu$ we have that

$$\left\|uC_\varphi f - uC_\varphi p_n\right\|_{\mathcal{B}_\mu} \le \left\|uC_\varphi\right\|_{\mathcal{B}^\alpha_{\log^\beta,0} \to \mathcal{B}_\mu} \left\|f - p_n\right\|_{\mathcal{B}^\alpha_{\log^\beta,0}} \longrightarrow 0 \tag{3.34}$$

as $n \to \infty$. Hence $uC_\varphi(\mathcal{B}^\alpha_{\log^\beta,0}) \subseteq \mathcal{B}_{\mu,0}$, and consequently $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta,0} \to \mathcal{B}_{\mu,0}$ is bounded. □

Remark 3.4. Note that Theorem 3.3 holds for all $\alpha > 0$ and $\beta \ge 0$.

Theorem 3.5. *Assume $\alpha > 1$, $\beta \ge 0$, φ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$, and μ is a weight. Then the operator $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta}$ (or $\mathcal{B}^\alpha_{\log^\beta,0}$) $\to \mathcal{B}_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \frac{\mu(z)|u'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(z)|^2\right)\right)\right)^\beta} = 0, \tag{3.35}$$

$$\lim_{|z|\to 1} \frac{\mu(z)|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(z)|^2\right)\right)\right)^\beta} = 0. \tag{3.36}$$

Proof. If $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta}$ (or $\mathcal{B}^\alpha_{\log^\beta,0}$) $\to \mathcal{B}_{\mu,0}$ is compact, then it is bounded so that conditions (3.30) and (3.31) hold. On the other hand, $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta}$ (or $\mathcal{B}^\alpha_{\log^\beta,0}$) $\to \mathcal{B}_\mu$ is compact, which implies that (3.17) and (3.18) hold.

By (3.18) we have that, for every $\varepsilon > 0$, there exists an $r \in (0,1)$ such that

$$\frac{\mu(z)|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(z)|^2\right)\right)\right)^\beta} < \varepsilon, \tag{3.37}$$

when $r < |\varphi(z)| < 1$. From (3.31), there exists a $\rho \in (0,1)$ such that

$$\mu(z)|u(z)||\varphi'(z)| < \varepsilon h_1\left(1-r^2\right) \tag{3.38}$$

when $\rho < |z| < 1$, and where h_1 is the function in Lemma 2.1(b).

Therefore, when $\rho < |z| < 1$ and $r < |\varphi(z)| < 1$, we have that

$$\frac{\mu(z)|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(z)|^2\right)\right)\right)^\beta} < \varepsilon. \tag{3.39}$$

On the other hand, if $\rho < |z| < 1$ and $|\varphi(z)| \le r$, from (3.38) and Lemma 2.1(b) we have

$$\frac{\mu(z)|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(z)|^2\right)\right)\right)^{\beta}} \le \frac{\mu(z)|u(z)||\varphi'(z)|}{h_1(1-r^2)} < \varepsilon. \tag{3.40}$$

Combining (3.39) and (3.40), we obtain (3.36). Similarly, from (3.17) and (3.30) is obtained (3.35), as claimed.

Conversely, assume that (3.35) and (3.36) hold. First note that (3.35) implies (3.30). Indeed if (3.30) did not hold then there would be a sequence $(z_n)_{n\in\mathbb{N}}$ and a $\delta > 0$ such that

$$\inf_{n\in\mathbb{N}} \mu(z_n)|u'(z_n)| \ge \delta \tag{3.41}$$

and $\lim_{n\to\infty}|\varphi(z_n)| = L \in \overline{\mathbb{D}}$. From this and the continuity of the function

$$h_2(x) = \frac{1}{(1-x^2)^{\alpha-1}\left(\ln\left(e^{\beta/\alpha}/(1-x^2)\right)\right)^{\beta}}, \quad x \in [0,1), \tag{3.42}$$

we would have that

$$\inf_{n\in\mathbb{N}} \frac{\mu(z_n)|u'(z_n)|}{(1-|\varphi(z_n)|^2)^{\alpha-1}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(z_n)|^2\right)\right)\right)^{\beta}} \ge \delta \inf_{[0,1)} h_2(x) > 0, \tag{3.43}$$

which is a contradiction with (3.35).

For any $f \in \mathcal{B}^{\alpha}_{\log^{\beta}}$, we have

$$\begin{aligned}\mu(z)\left|(uC_\varphi f)'(z)\right| \le C\|f\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}}\Bigg(&\mu(z)|u'(z)| + \frac{\mu(z)|u'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(z)|^2\right)\right)\right)^{\beta}}\\ &+\frac{\mu(z)|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha}\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(z)|^2\right)\right)\right)^{\beta}}\Bigg).\end{aligned} \tag{3.44}$$

Using conditions (3.30), (3.35), and (3.36) in (3.44), it follows that $uC_\varphi f \in \mathcal{B}_{\mu,0}$ for each $f \in \mathcal{B}^{\alpha}_{\log^{\beta}}$, moreover the set

$$uC_\varphi\left(\left\{f \in \mathcal{B}^{\alpha}_{\log^{\beta}}\ \left(\text{or } \mathcal{B}^{\alpha}_{\log^{\beta},0}\right) : \|f\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}} \le 1\right\}\right) \tag{3.45}$$

is bounded in $\mathcal{B}_{\mu,0}$.

Taking the supremum in (3.44) over the unit ball of the space $\mathcal{B}^{\alpha}_{\log^{\beta}}$ (or $\mathcal{B}^{\alpha}_{\log^{\beta},0}$), then letting $|z| \to 1$ and using conditions (3.30), (3.35), and (3.36), we obtain

$$\lim_{|z|\to 1}\sup_{\|f\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}}\le 1}\mu(z)\left|(uC_{\varphi}f)'(z)\right| = 0, \tag{3.46}$$

from which along with Lemma 2.3 the compactness of the operator $uC_{\varphi} : \mathcal{B}^{\alpha}_{\log^{\beta}}$ (or $\mathcal{B}^{\alpha}_{\log^{\beta},0}$) $\to \mathcal{B}_{\mu,0}$ follows. □

Case 2. $\alpha = 1,\ \beta \in (0,1)$.

Theorem 3.6. *Assume that φ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$, and μ is a weight. Then the operator $uC_{\varphi} : \mathcal{B}^{1}_{\log^{\beta}}$ (or $\mathcal{B}^{1}_{\log^{\beta},0}$) $\to \mathcal{B}_{\mu}$ is bounded if and only if*

$$\begin{gathered}\sup_{z\in\mathbb{D}}\mu(z)\left|u'(z)\right|\left(1+\left(\ln\frac{e^{\beta}}{1-|\varphi(z)|^{2}}\right)^{1-\beta}\right)<\infty,\\ \sup_{z\in\mathbb{D}}\frac{\mu(z)|u(z)|\left|\varphi'(z)\right|}{\left(1-|\varphi(z)|^{2}\right)\left(\ln\left(e^{\beta}/\left(1-|\varphi(z)|^{2}\right)\right)\right)^{\beta}}<\infty.\end{gathered} \tag{3.47}$$

Proof. The proof of the theorem is similar to the proof of Theorem 3.1. The sufficiency follows by using the triangle inequality in (3.3) and then the third inequality in Lemma 2.2 and the definition of the space $\mathcal{B}^{1}_{\log^{\beta}}$.

For the necessity it is enough to follow the lines of the corresponding part of the proof of Theorem 3.1 and use the test functions $f(z) \equiv 1,\ f(z) \equiv z$,

$$f_w(z) = \frac{\left(f^{(1)}_w(z)\right)^2}{f^{(1)}_w(w)} - \frac{\left(f^{(1)}_w(z)\right)^3}{\left(f^{(1)}_w(w)\right)^2}, \quad w \in \mathbb{D}, \tag{3.48}$$

$$g_w(z) = 3\frac{\left(f^{(1)}_w(z)\right)^2}{f^{(1)}_w(w)} - 2\frac{\left(f^{(1)}_w(z)\right)^3}{\left(f^{(1)}_w(w)\right)^2}, \quad w \in \mathbb{D}, \tag{3.49}$$

which belong to $\mathcal{B}^{1}_{\log^{\beta}}$ (for the functions in (3.48) and (3.49) it easily follows by Lemma 2.6(b)), where $f^{(1)}_w(z)$ is the function in (2.10). We omit the details. □

The proofs of the following two theorems are similar to the proofs of Theorems 3.2 and 3.5, where the test functions in (3.48) and (3.49) are used as well as the lemmas in Section 2. Hence their proofs are omitted.

Theorem 3.7. *Assume that φ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$, and μ is a weight. Then the operator $uC_\varphi : \mathcal{B}^1_{\log^\beta}$ (or $\mathcal{B}^1_{\log^\beta,0}$) $\to \mathcal{B}_\mu$ is compact if and only if $uC_\varphi : \mathcal{B}^1_{\log^\beta}$ (or $\mathcal{B}^1_{\log^\beta,0}$) $\to \mathcal{B}_\mu$ is bounded*

$$\lim_{|\varphi(z)|\to 1} \mu(z)\left|u'(z)\right|\left(1+\left(\ln\frac{e^\beta}{1-\left|\varphi(z)\right|^2}\right)^{1-\beta}\right)=0,$$
$$\lim_{|\varphi(z)|\to 1}\frac{\mu(z)|u(z)|\left|\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)\left(\ln\left(e^\beta/\left(1-\left|\varphi(z)\right|^2\right)\right)\right)^\beta}=0. \tag{3.50}$$

Theorem 3.8. *Assume that φ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$, and μ is a weight. Then the operator $uC_\varphi : \mathcal{B}^1_{\log^\beta}$ (or $\mathcal{B}^1_{\log^\beta,0}$) $\to \mathcal{B}_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \mu(z)\left|u'(z)\right|\left(1+\left(\ln\frac{e^\beta}{1-\left|\varphi(z)\right|^2}\right)^{1-\beta}\right)=0,$$
$$\lim_{|z|\to 1}\frac{\mu(z)|u(z)|\left|\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)\left(\ln\left(e^\beta/\left(1-\left|\varphi(z)\right|^2\right)\right)\right)^\beta}=0. \tag{3.51}$$

Case 3. $\alpha = \beta = 1$.

The following results were proved in [15]. Hence we quote them for the benefit of the reader, and without any proof.

Theorem 3.9. *Assume that φ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$, and μ is a weight. Then the operator $uC_\varphi : \mathcal{B}^1_{\log^1}$ (or $\mathcal{B}^1_{\log^1,0}$) $\to \mathcal{B}_\mu$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \mu(z)|u'(z)|\max\left\{1,\ln\ln\frac{e}{1-\left|\varphi(z)\right|^2}\right\}<\infty,$$
$$\sup_{z\in\mathbb{D}}\frac{\mu(z)|u(z)|\left|\varphi'(z)\right|}{\left(1-|\varphi(z)|^2\right)\ln\left(e/\left(1-|\varphi(z)|^2\right)\right)}<\infty. \tag{3.52}$$

Theorem 3.10. *Assume that φ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$, and μ is a weight. Then the operator $uC_\varphi : \mathcal{B}^1_{\log^1}$ (or $\mathcal{B}^1_{\log^1,0}$) $\to \mathcal{B}_\mu$ is compact if and only if $uC_\varphi : \mathcal{B}^1_{\log^1}$ (or $\mathcal{B}^1_{\log^1,0}$) $\to \mathcal{B}_\mu$ is bounded*

$$\lim_{|\varphi(z)|\to 1} \mu(z)\left|u'(z)\right|\max\left\{1,\ln\ln\frac{e}{1-\left|\varphi(z)\right|^2}\right\}=0,$$
$$\lim_{|\varphi(z)|\to 1}\frac{\mu(z)|u(z)|\left|\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)\ln\left(e/\left(1-|\varphi(z)|^2\right)\right)}=0. \tag{3.53}$$

Theorem 3.11. *Assume that φ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$, and μ is a weight. Then the operator $uC_\varphi : \mathcal{B}^1_{\log^1}$ (or $\mathcal{B}^1_{\log^1,0}$) $\to \mathcal{B}_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \mu(z)|u'(z)| \max\left\{1, \ln\ln\frac{e}{1-|\varphi(z)|^2}\right\} = 0,$$
$$\lim_{|z|\to 1} \frac{\mu(z)|u(z)||\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)\ln\left(e/\left(1-|\varphi(z)|^2\right)\right)} = 0. \tag{3.54}$$

Case 4. $\alpha \in (0,1)$, or $\alpha = 1$ and $\beta > 1$.

Here we consider the cases $\alpha \in (0,1)$, or $\alpha = 1$ and $\beta > 1$.

Theorem 3.12. *Assume that $\alpha \in (0,1)$, or $\alpha = 1$ and $\beta > 1$, $u \in H(\mathbb{D})$, μ is a weight, and φ is a holomorphic self-map of $\mathbb{D}$. Then $uC^g_\varphi : \mathcal{B}^\alpha_{\log^\beta}$ (or $\mathcal{B}^\alpha_{\log^\beta,0}$) $\to \mathcal{B}_\mu$ is bounded if and only if $u \in \mathcal{B}_\mu$ and condition* (3.2) *holds.*

Proof. The sufficiency follows by using the first inequality in Lemma 2.2 and the definition of the space $\mathcal{B}^\alpha_{\log^\beta}$ in (3.3).

For the necessity, by using the test functions $f(z) \equiv 1$, $f(z) \equiv z$ we first get conditions (3.5) and (3.7). To get (3.2) for the case $\alpha = 1$ and $\beta > 1$ we use the test functions

$$f_w(z) = 2\frac{1-|w|^2}{1-z\overline{w}} f^{(1)}_w(z) - \frac{\left(1-|w|^2\right)^2}{(1-z\overline{w})^2} f^{(1)}_w(z), \quad w \in \mathbb{D}. \tag{3.55}$$

Note that $f_w(w) = f^{(1)}_w(w)$,

$$f'_w(w) = \frac{(1-\beta)\overline{w}}{\left(1-|w|^2\right)\left(\ln\left(e^\gamma/\left(1-|w|^2\right)\right)\right)^\beta}, \tag{3.56}$$

and similar to Lemma 2.6(b), $\sup_{w\in\mathbb{D}}\|f_w\|_{\mathcal{B}^\alpha_{\log^\beta}} \le C$ and $f_w \in \mathcal{B}^\alpha_{\log^\beta,0}$ for each $w \in \mathbb{D}$.

Hence for the family $(f_{\varphi(w)})_{w\in\mathbb{D}}$, we get

$$\frac{(1-\beta)\mu(w)|u(w)||\varphi'(w)||\varphi(w)|}{\left(1-|\varphi(w)|^2\right)\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(w)|^2\right)\right)\right)^\beta}$$
$$\le C\left\|uC_\varphi f_{\varphi(w)}\right\|_{\mathcal{B}_\mu} + \frac{\mu(w)|u'(w)|}{\left(\ln\left(e^{\beta/\alpha}/\left(1-|\varphi(w)|^2\right)\right)\right)^{\beta-1}}, \tag{3.57}$$

from which along with (3.5) and the assumption $\beta > 1$, easily follows (3.2) in this case.

When $\alpha \in (0,1)$, condition (3.2) follows as in Theorem 3.1, by using the test functions in (3.8). □

Theorem 3.13. *Assume that* $\alpha \in (0,1)$, *or* $\alpha = 1$ *and* $\beta > 1$, $u \in H(\mathbb{D})$, μ *is a weight, and* φ *is a holomorphic self-map of* $\mathbb{D}$, *and* $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta}$ (*or* $\mathcal{B}^\alpha_{\log^\beta,0}$) $\to \mathcal{B}_\mu$ *is bounded. Then* $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta}$ (*or* $\mathcal{B}^\alpha_{\log^\beta,0}$) $\to \mathcal{B}_\mu$ *is compact if*

$$\lim_{|\varphi(z)|\to 1} \mu(z)\left|u'(z)\right| = 0, \tag{3.58}$$

and condition (3.18) *holds.*

Proof. The proof is similar to the corresponding parts of the proofs of Theorems 3.2 and 3.7, so is omitted. □

Remark 3.14. Note that if $\alpha \in (0,1)$, or $\alpha = 1$ and $\beta > 1$ and $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta}$ (or $\mathcal{B}^\alpha_{\log^\beta,0}$) $\to \mathcal{B}_\mu$ is compact, then condition (3.18) is proved as in Theorems 3.2 and 3.7, by using the test functions in (3.8) and (3.48). If $\|\varphi\|_\infty < 1$ then condition (3.58) is vacuously satisfied. At the moment, we are not sure if the compactness implies condition (3.58) in the case $\|\varphi\|_\infty = 1$. Hence for the interested readers we leave this as an open problem.

The following theorem is proved as the corresponding part of Theorem 3.5.

Theorem 3.15. *Assume that* $\alpha \in (0,1)$, *or* $\alpha = 1$ *and* $\beta > 1$, $u \in H(\mathbb{D})$, μ *is a weight, and* φ *is a holomorphic self-map of* $\mathbb{D}$. *Then the operator* $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta}$ (*or* $\mathcal{B}^\alpha_{\log^\beta,0}$) $\to \mathcal{B}_{\mu,0}$ *is compact if* $u \in \mathcal{B}_{\mu,0}$ *and condition* (3.36) *holds.*

Remark 3.16. Note that if $uC_\varphi : \mathcal{B}^\alpha_{\log^\beta}$ (or $\mathcal{B}^\alpha_{\log^\beta,0}$) $\to \mathcal{B}_{\mu,0}$ is compact, then clearly $u \in \mathcal{B}_{\mu,0}$.

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