

Research Article

Stability of a Bi-Jensen Functional Equation II

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We investigate the stability of the bi-Jensen functional equation II $f((x+y)/2, z) - f(x, z) - f(y, z) = 0$, $2f(x, (y+z)/2) - f(x, y) - f(x, z) = 0$ in the spirit of Gävruta.

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1. Introduction

In 1940, Ulam [1] raised a question concerning the stability of homomorphisms. Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta \quad (1.1)$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \rightarrow G_2$ with

$$d(h(x), H(x)) < \varepsilon \quad (1.2)$$

for all $x \in G_1$. The case of approximately additive mappings was solved by Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. In 1978, Rassias [3] gave a generalization of Hyers' theorem by allowing the Cauchy difference to be controlled by a sum of powers like

$$\|h(x+y) - h(x) - h(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p). \quad (1.3)$$

Găvruta [4] provided a further generalization of Rassias' theorem in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general function.

Throughout this paper, let X and Y be a normed space and a Banach space, respectively. A mapping $g : X \rightarrow Y$ is called a Cauchy mapping (resp., a Jensen mapping) if g satisfies the functional equation $g(x+y) = g(x) + g(y)$ (resp., $2g((x+y)/2) = g(x) + g(y)$).

For a given mapping $f : X \times X \rightarrow Y$, we define

$$\begin{aligned} J_1 f(x, y, z) &:= 2f\left(\frac{x+y}{2}, z\right) - f(x, z) - f(y, z), \\ J_2 f(x, y, z) &:= 2f\left(x, \frac{y+z}{2}\right) - f(x, y) - f(x, z) \end{aligned} \quad (1.4)$$

for all $x, y, z \in X$. A mapping $f : X \times X \rightarrow Y$ is called a bi-Jensen mapping if f satisfies the functional equations $J_1 f = 0$ and $J_2 f = 0$.

Bae and Park [5] obtained the generalized Hyers-Ulam stability of a bi-Jensen mapping. Jun et al. [6] improved the results of Bae and Park in the spirit of Rassias.

In this paper, we investigate the stability of a bi-Jensen functional equation $J_1 f = 0$, $J_2 f = 0$ in the spirit of Găvruta.

2. Stability of a Bi-Jensen Functional Equation

Jun et al. [7] established the basic properties of a bi-Jensen mapping in the following lemma.

Lemma 2.1. *Let $f : X \times X \rightarrow Y$ be a bi-Jensen mapping. Then, the following equalities hold:*

$$\begin{aligned} f(x, y) &= \frac{1}{4^n} f(2^n x, 2^n y) + \left(\frac{1}{2^n} - \frac{1}{4^n}\right) (f(2^n x, 0) + f(0, 2^n y)) + \left(1 - \frac{1}{2^n}\right)^2 f(0, 0) \\ &= \frac{f(2^n x, 2^n y)}{2^n} + \frac{2^n - 1}{2^{2n+1}} (f(2^n x, -2^n y) + f(-2^n x, 2^n y)) + \left(1 - \frac{1}{2^n}\right)^2 f(0, 0), \\ f(x, y) &= \frac{1}{4^n} f(2^n x, 2^n y) + (2^n - 1) \left(f\left(\frac{x}{2^n}, 0\right) + f\left(0, \frac{y}{2^n}\right)\right) - \left(2^{n+1} - 3 + \frac{1}{4^n}\right) f(0, 0), \\ f(x, y) &= 4^n f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + (2^n - 4^n) \left(f\left(\frac{x}{2^n}, 0\right) + f\left(0, \frac{y}{2^n}\right)\right) + (2^n - 1)^2 f(0, 0), \\ f(x, y) &= \frac{1}{2^n} f(2^n x, y) + \frac{1}{2^n} \left(1 - \frac{1}{2^n}\right) f(0, 2^n y) + \left(1 - \frac{1}{2^n}\right)^2 f(0, 0) \end{aligned} \quad (2.1)$$

for all $x, y \in X$ and $n \in \mathbb{N}$.

Now we have the stability of a bi-Jensen mapping.

Theorem 2.2. Let $\varphi, \psi : X \times X \times X \rightarrow [0, \infty)$ be two functions satisfying

$$\sum_{j=1}^{\infty} \frac{\varphi(2^j x, 2^j y, 2^j z)}{2^j} < \infty, \quad (2.2)$$

$$\sum_{j=1}^{\infty} \frac{\psi(2^j x, 2^j y, 2^j z)}{2^j} < \infty \quad (2.3)$$

for all $x, y, z \in X$. Let $f : X \times X \rightarrow Y$ be a mapping such that

$$\|J_1 f(x, y, z)\| \leq \varphi(x, y, z), \quad (2.4)$$

$$\|J_2 f(x, y, z)\| \leq \psi(x, y, z)$$

for all $x, y, z \in X$. Then, there exists a unique bi-Jensen mapping $F : X \times X \rightarrow Y$ such that

$$\|f(x, y) - F(x, y)\| \leq \sum_{j=1}^{\infty} \frac{\Phi(2^j x, 2^j y)}{4^j} + \sum_{j=1}^{\infty} \frac{\varphi(0, 0, 2^j y) + \varphi(2^j x, 0, 0)}{2^j} \quad (2.5)$$

for all $x, y \in X$ with $F(0, 0) = f(0, 0)$, where

$$\Phi(x, y) = \frac{\varphi(x, 0, y) + \varphi(x, 0, y)}{2} + \varphi\left(\frac{x}{2}, 0, y\right) + \varphi\left(x, 0, \frac{y}{2}\right) + \frac{3\varphi(0, 0, y) + 3\varphi(x, 0, 0)}{2}. \quad (2.6)$$

The mapping $F : X \times X \rightarrow Y$ is given by

$$F(x, y) := \lim_{j \rightarrow \infty} \left[\frac{f(2^j x, 2^j y)}{4^j} + \frac{f(2^j x, 0) + f(0, 2^j y)}{2^j} \right] + f(0, 0) \quad (2.7)$$

for all $x, y \in X$.

Proof. Let f', f'', f''' be the maps defined by

$$f'(x, y) = f(x, y) - f(0, y),$$

$$f''(x, y) = f(x, y) - f(x, 0), \quad (2.8)$$

$$f'''(x, y) = f(x, y) - f(x, 0) - f(0, y) + f(0, 0)$$

for all $x, y \in X$. By (2.4), we get

$$\begin{aligned}
 & \left\| \frac{f'(2^j x, 0)}{2^j} - \frac{f'(2^{j+1} x, 0)}{2^{j+1}} \right\| = \left\| \frac{J_1 f(2^{j+1} x, 0, 0)}{2^{j+1}} \right\| \leq \frac{\varphi(2^{j+1} x, 0, 0)}{2^{j+1}}, \\
 & \left\| \frac{f''(0, 2^j y)}{2^j} - \frac{f''(0, 2^{j+1} y)}{2^{j+1}} \right\| = \left\| \frac{J_2 f(0, 0, 2^{j+1} y)}{2^{j+1}} \right\| \leq \frac{\varphi(0, 0, 2^{j+1} y)}{2^{j+1}}, \\
 & \left\| \frac{f'(2^j x, 2^j y)}{4^j} - \frac{f'(2^{j+1} x, 2^{j+1} y)}{4^{j+1}} \right\| \\
 &= \left\| \frac{J_1 f(2^{j+1} x, 0, 2^{j+1} y) + J_2 f(2^{j+1} x, 0, 2^{j+1} y) + 2J_2 f(2^j x, 0, 2^{j+1} y)}{2 \cdot 4^{j+1}} \right. \\
 & \quad \left. + \frac{2J_1 f(2^{j+1} x, 0, 2^j y) - 3J_2 f(0, 0, 2^{j+1} y) - 3J_1 f(2^{j+1} x, 0, 0)}{2 \cdot 4^{j+1}} \right\| \\
 & \leq \frac{\Phi(2^{j+1} x, 2^{j+1} y)}{4^{j+1}}
 \end{aligned} \tag{2.9}$$

for all $x, y \in X$ and $j \in \mathbb{N}$. For given integers l, m ($0 \leq l < m$),

$$\begin{aligned}
 & \left\| \frac{f'(2^l x, 0)}{2^l} - \frac{f'(2^m x, 0)}{2^m} \right\| \leq \sum_{j=l}^{m-1} \frac{\varphi(2^{j+1} x, 0, 0)}{2^{j+1}}, \\
 & \left\| \frac{f''(0, 2^l y)}{2^l} - \frac{f''(0, 2^m y)}{2^m} \right\| \leq \sum_{j=l}^{m-1} \frac{\varphi(0, 0, 2^{j+1} y)}{2^{j+1}}, \\
 & \left\| \frac{f'''(2^l x, 2^l y)}{4^l} - \frac{f'''(2^m x, 2^m y)}{4^m} \right\| \leq \sum_{j=l}^{m-1} \frac{\Phi(2^{j+1} x, 2^{j+1} y)}{4^{j+1}}
 \end{aligned} \tag{2.10}$$

for all $x, y \in X$. By (2.2) and (2.3), the sequences $\{(1/2^j)(f(2^j x, 0) - f(0, 0))\}$, $\{(1/2^j)(f(0, 2^j y) - f(0, 0))\}$, and $\{(1/4^j)f'(2^j x, 2^j y)\}$ are Cauchy sequences for all $x, y \in X$. Since Y is complete, the sequences $\{(1/2^j)(f(2^j x, 0) - f(0, 0))\}$, $\{(1/2^j)(f(0, 2^j y) - f(0, 0))\}$, and $\{(1/4^j)f'(2^j x, 2^j y)\}$ converge for all $x, y \in X$. Define $F_1, F_2, F_3 : X \times X \rightarrow Y$ by

$$\begin{aligned}
 F_1(x, y) &:= \lim_{j \rightarrow \infty} \frac{f(2^j x, 0)}{2^j}, \\
 F_2(x, y) &:= \lim_{j \rightarrow \infty} \frac{f(0, 2^j y)}{2^j}, \\
 F_3(x, y) &:= \lim_{j \rightarrow \infty} \frac{f'(2^j x, 2^j y)}{4^j}
 \end{aligned} \tag{2.11}$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \rightarrow \infty$ in (2.10), one can obtain the inequalities

$$\begin{aligned} \|f(x, 0) - f(0, 0) - F_1(x, y)\| &\leq \sum_{j=1}^{\infty} \frac{\varphi(2^j x, 0, 0)}{2^j}, \\ \|f(0, y) - f(0, 0) - F_2(x, y)\| &\leq \sum_{j=1}^{\infty} \frac{\varphi(0, 0, 2^j y)}{2^j}, \\ \|f(x, y) - f(x, 0) - f(0, y) + f(0, 0) - F_3(x, y)\| &\leq \sum_{j=1}^{\infty} \frac{\Phi(2^j x, 2^j y)}{4^j} \end{aligned} \quad (2.12)$$

for all $x, y \in X$. By (2.4) and the definitions of F_1 and F_2 , we get

$$\begin{aligned} J_1 F_1(x, y, z) &= \lim_{j \rightarrow \infty} \frac{1}{2^j} J_1 f(2^j x, 2^j y, 0) = 0, \\ J_2 F_1(x, y, z) &= 0, \\ J_1 F_2(x, y, z) &= 0, \\ J_2 F_2(x, y, z) &= \lim_{j \rightarrow \infty} \frac{1}{2^j} J_2 f(0, 2^j y, 2^j z) = 0, \\ J_1 F_3(x, y, z) &= \lim_{j \rightarrow \infty} \frac{J_1 f(2^j x, 2^j y, 2^j z)}{4^j} = 0, \\ J_2 F_3(x, y, z) &= \lim_{j \rightarrow \infty} \frac{J_2 f(2^j x, 2^j y, 2^j z)}{4^j} = 0 \end{aligned} \quad (2.13)$$

for all $x, y, z \in X$. So F is a bi-Jensen mapping satisfying (2.5), where F is given by

$$F(x, y) = F_1(x, y) + F_2(x, y) + F_3(x, y) + f(0, 0). \quad (2.14)$$

Now, let $F' : X \times X \rightarrow Y$ be another bi-Jensen mapping satisfying (2.5) with $F'(0, 0) = f(0, 0)$. By Lemma 2.1, we have

$$\begin{aligned} &\|F(x, y) - F'(x, y)\| \\ &= \left\| \frac{(F - F')(2^n x, 2^n y)}{4^n} + \left(\frac{1}{2^n} - \frac{1}{4^n} \right) ((F - F')(2^n x, 0) + (F - F')(0, 2^n y)) \right\| \\ &\leq \left\| \frac{(F - f)(2^n x, 2^n y)}{4^n} \right\| + \left\| \frac{(F - f)(0, 2^n y)}{2^n} \right\| + \left\| \frac{(F - f)(2^n x, 0)}{2^n} \right\| \\ &\quad + \left\| \frac{(f - F')(2^n x, 2^n y)}{4^n} \right\| + \left\| \frac{(f - F')(0, 2^n y)}{2^n} \right\| + \left\| \frac{(f - F')(2^n x, 0)}{2^n} \right\| \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=n+1}^{\infty} \frac{\Phi(2^j x, 2^j y)}{2^{j-1}} + \sum_{j=n+1}^{\infty} \frac{\psi(0, 0, 2^j y) + \varphi(2^j x, 0, 0)}{2^{j-2}} \\
&\quad + \sum_{j=n+1}^{\infty} \frac{\Phi(2^j x, 0) + \Phi(0, 2^j y)}{2^{j-1}} + \frac{\varphi(0, 0, 0)}{2^{n-1}} + \frac{\psi(0, 0, 0)}{2^{n-1}}
\end{aligned} \tag{2.15}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. As $n \rightarrow \infty$, we may conclude that $F(x, y) = F'(x, y)$ for all $x, y \in X$. Thus such a bi-Jensen mapping $F : X \times X \rightarrow Y$ is unique. \square

Remark 2.3. Let $\varphi, \psi : X \times X \times X \rightarrow [0, \infty)$ be the functions defined by

$$\varphi(x, y, z) = \psi(x, y, z) := \frac{\varepsilon}{3} \tag{2.16}$$

for all $x, y, z \in X$. Let $f, F, F' : X \times X \rightarrow Y$ be the bi-Jensen mappings defined by

$$f(x, y) := 0, \quad F(x, y) := \varepsilon, \quad F'(x, y) := -\varepsilon \tag{2.17}$$

for all $x, y \in X$. Then, φ , ψ , and f satisfy (2.2), (2.3), (2.4) for all $x, y, z \in X$. In addition, f , F satisfy (2.5) for all $x, y \in X$ and f , F' also satisfy (2.5) for all $x, y \in X$. But we get $F' \neq F$. Hence, the condition $F(0, 0) = f(0, 0)$ is necessary to show that the mapping F is unique.

We have another stability result applying for several cases.

Theorem 2.4. Let $\varphi, \psi : X \times X \times X \rightarrow [0, \infty)$ be two functions satisfying

$$\sum_{j=0}^{\infty} 4^j \left(\varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) + \psi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) \right) < \infty \tag{2.18}$$

for all $x, y, z \in X$. Let $f : X \times X \rightarrow Y$ be a mapping satisfying (2.4) for all $x, y, z \in X$. Then, there exists a unique bi-Jensen mapping $F : X \times X \rightarrow Y$ satisfying

$$\|f(x, y) - F(x, y)\| \leq \sum_{j=0}^{\infty} \left(4^j \Phi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) + 2^j \psi\left(0, 0, \frac{y}{2^j}\right) + 2^j \varphi\left(\frac{x}{2^j}, 0, 0\right) \right) \tag{2.19}$$

for all $x, y \in X$. The mapping $F : X \times X \rightarrow Y$ is given by

$$F(x, y) := \lim_{j \rightarrow \infty} \left[4^j f\left(\frac{x}{2^j}, \frac{y}{2^j}\right) - (4^j - 2^j) \left(f\left(\frac{x}{2^j}, 0\right) + f\left(0, \frac{y}{2^j}\right) \right) + (2^j - 1)^2 f(0, 0) \right] \quad (2.20)$$

for all $x, y \in X$.

Proof. By (2.4) and the similar method in Theorem 2.2, we define the maps $F_1, F_2, F_3 : X \times X \rightarrow Y$ by

$$\begin{aligned} F_1(x, y) &:= \lim_{j \rightarrow \infty} 2^j \left(f\left(\frac{x}{2^j}, 0\right) - f(0, 0) \right), \\ F_2(x, y) &:= \lim_{j \rightarrow \infty} 2^j \left(f\left(0, \frac{y}{2^j}\right) - f(0, 0) \right), \\ F_3(x, y) &:= \lim_{j \rightarrow \infty} 4^j \left[f\left(\frac{x}{2^j}, \frac{y}{2^j}\right) - f\left(\frac{x}{2^j}, 0\right) - f\left(0, \frac{y}{2^j}\right) - f(0, 0) \right] \end{aligned} \quad (2.21)$$

for all $x, y \in X$. By (2.4) and the definitions of F_1, F_2 , and F_3 , we get

$$\begin{aligned} J_1 F_1(x, y, z) &= \lim_{j \rightarrow \infty} 2^j J_1 f\left(\frac{x}{2^j}, \frac{y}{2^j}, 0\right) = 0, \\ J_2 F_1(x, y, z) &= 0, \\ J_1 F_2(x, y, z) &= 0, \\ J_2 F_2(x, y, z) &= \lim_{j \rightarrow \infty} 2^j J_2 f\left(0, \frac{y}{2^j}, \frac{z}{2^j}\right) = 0, \\ J_1 F_3(x, y, z) &= \lim_{j \rightarrow \infty} 4^j \left[J_1 f\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) - J_1 f\left(\frac{x}{2^j}, \frac{y}{2^j}, 0\right) \right] = 0, \\ J_2 F_3(x, y, z) &= \lim_{j \rightarrow \infty} 4^j \left[J_2 f\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) - J_2 f\left(0, \frac{y}{2^j}, \frac{z}{2^j}\right) \right] = 0 \end{aligned} \quad (2.22)$$

for all $x, y, z \in X$. By the similar method in Theorem 2.2, F is a bi-Jensen mapping satisfying (2.19), where F is given by

$$F(x, y) = F_1(x, y) + F_2(x, y) + F_3(x, y) + f(0, 0). \quad (2.23)$$

Now, let $F' : X \times X \rightarrow Y$ be another bi-Jensen mapping satisfying (2.19). Using Lemma 2.1, $\varphi(0, 0, 0) = \psi(0, 0, 0) = 0$, and $F'(0, 0) = f(0, 0) = F(0, 0)$, we have

$$\begin{aligned} & \|F(x, y) - F'(x, y)\| \\ &= \left\| 4^n (F - F') \left(\frac{x}{2^n}, \frac{y}{2^n} \right) + (2^n - 4^n) \left[(F - F') \left(\frac{x}{2^n}, 0 \right) + (F - F') \left(0, \frac{y}{2^n} \right) \right] \right\| \\ &\leq 4^n \left[\left\| (F - f) \left(\frac{x}{2^n}, \frac{y}{2^n} \right) \right\| + \left\| (f - F') \left(\frac{x}{2^n}, \frac{y}{2^n} \right) \right\| + \left\| (F - f) \left(\frac{x}{2^n}, 0 \right) \right\| \right. \\ &\quad \left. + \left\| (f - F') \left(\frac{x}{2^n}, 0 \right) \right\| + \left\| (F - f) \left(0, \frac{y}{2^n} \right) \right\| + \left\| (f - F') \left(0, \frac{y}{2^n} \right) \right\| \right] \\ &\leq 2 \sum_{j=n}^{\infty} 4^j \left(\Phi \left(\frac{x}{2^j}, \frac{y}{2^j} \right) + 2\psi \left(0, 0, \frac{y}{2^j} \right) + 2\varphi \left(\frac{x}{2^j}, 0, 0 \right) + \Phi \left(\frac{x}{2^j}, 0 \right) + \Phi \left(0, \frac{y}{2^j} \right) \right) \end{aligned} \quad (2.24)$$

for all $x, y \in X$ and $n \in \mathbb{N}$. As $n \rightarrow \infty$, we may conclude that $F(x, y) = F'(x, y)$ for all $x, y \in X$. Thus such a bi-Jensen mapping $F : X \times X \rightarrow Y$ is unique. \square

Theorem 2.5. Let $\varphi, \psi : X \times X \times X \rightarrow [0, \infty)$ be two functions satisfying

$$\sum_{j=0}^{\infty} 2^j \varphi \left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j} \right) + \sum_{j=0}^{\infty} \frac{\varphi(2^j x, 2^j y, 2^j z)}{4^j} < \infty, \quad (2.25)$$

$$\sum_{j=0}^{\infty} 2^j \psi \left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j} \right) + \sum_{j=0}^{\infty} \frac{\psi(2^j x, 2^j y, 2^j z)}{4^j} < \infty \quad (2.26)$$

for all $x, y, z \in X$. Let $f : X \times X \rightarrow Y$ be a mapping satisfying (2.4) for all $x, y, z \in X$. Then, there exists a bi-Jensen mapping $F : X \times X \rightarrow Y$ satisfying

$$\|f(x, y) - F(x, y)\| \leq \sum_{j=1}^{\infty} \frac{\Phi(2^j x, 2^j y)}{4^j} + \sum_{j=0}^{\infty} 2^j \left(\psi \left(0, 0, \frac{y}{2^j} \right) + \varphi \left(\frac{x}{2^j}, 0, 0 \right) \right) \quad (2.27)$$

for all $x, y \in X$, where the mapping $F : X \times X \rightarrow Y$ is given by

$$\begin{aligned} F(x, y) &:= \lim_{j \rightarrow \infty} \frac{1}{4^j} [f(2^j x, 2^j y) - f(2^j x, 0) - f(0, 2^j y)] \\ &\quad + \lim_{j \rightarrow \infty} \left[2^j \left(f \left(\frac{x}{2^j}, 0 \right) + f \left(0, \frac{y}{2^j} \right) \right) - (2^{j+1} - 1) f(0, 0) \right] \end{aligned} \quad (2.28)$$

for all $x, y \in X$.

Proof. We can obtain F_1 , F_2 as in Theorem 2.4 and F_3 as in Theorem 2.2. Hence, F is a bi-Jensen mapping satisfying (2.27), where F is given by

$$F(x, y) = F_1(x, y) + F_2(x, y) + F_3(x, y) + f(0, 0). \quad (2.29)$$

□

Theorem 2.6. Let $\varphi, \psi : X \times X \times X \rightarrow [0, \infty)$ be two functions satisfying (2.2) and (2.26) for all $x, y, z \in X$. Let $f : X \times X \rightarrow Y$ be a mapping satisfying (2.4) for all $x, y, z \in X$. Then, there exists a bi-Jensen mapping $F : X \times X \rightarrow Y$ satisfying

$$\|f(x, y) - F(x, y)\| \leq \sum_{j=1}^{\infty} \frac{\Phi(2^j x, 2^j y)}{4^j} + \sum_{j=0}^{\infty} 2^j \psi\left(0, 0, \frac{y}{2^j}\right) + \sum_{j=1}^{\infty} \frac{\varphi(2^j x, 0, 0)}{2^j} \quad (2.30)$$

for all $x, y \in X$, where the mapping $F : X \times X \rightarrow Y$ is given by

$$\begin{aligned} F(x, y) := & \lim_{j \rightarrow \infty} \frac{1}{4^j} [f(2^j x, 2^j y) - f(2^j x, 0) - f(0, 2^j y)] \\ & + \lim_{j \rightarrow \infty} \left[\frac{1}{2^j} f(2^j x, 0) + 2^j f\left(0, \frac{y}{2^j}\right) - (2^j - 1)f(0, 0) \right] \end{aligned} \quad (2.31)$$

for all $x, y \in X$.

Theorem 2.7. Let $\varphi, \psi : X \times X \times X \rightarrow [0, \infty)$ be two functions satisfying (2.3) and (2.25) for all $x, y, z \in X$. Let $f : X \times X \rightarrow Y$ be a mapping satisfying (2.4) for all $x, y, z \in X$. Then, there exists a bi-Jensen mapping $F : X \times X \rightarrow Y$ satisfying

$$\|f(x, y) - F(x, y)\| \leq \sum_{j=1}^{\infty} \frac{\Phi(2^j x, 2^j y)}{4^j} + \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, 0, 0\right) \quad (2.32)$$

for all $x, y \in X$, where the mapping $F : X \times X \rightarrow Y$ is given by

$$\begin{aligned} F(x, y) := & \lim_{j \rightarrow \infty} \frac{1}{4^j} [f(2^j x, 2^j y) - f(2^j x, 0) - f(0, 2^j y)] \\ & + \lim_{j \rightarrow \infty} \left[2^j f\left(\frac{x}{2^j}, 0\right) + \frac{1}{2^j} f(0, 2^j y) - (2^j - 1)f(0, 0) \right] \end{aligned} \quad (2.33)$$

for all $x, y \in X$.

Applying Theorems 2.2–2.7, we easily get the following corollaries.

Corollary 2.8. Let $0 < p (\neq 1, 2)$ and $\varepsilon > 0$. Let $f : X \times X \rightarrow Y$ be a mapping such that

$$\begin{aligned} \|J_1 f(x, y, z)\| & \leq \varepsilon (\|x\|^p + \|y\|^p + \|z\|^p), \\ \|J_2 f(x, y, z)\| & \leq \varepsilon (\|x\|^p + \|y\|^p + \|z\|^p) \end{aligned} \quad (2.34)$$

for all $x, y, z \in X$. Then, there exists a unique bi-Jensen mapping $F : X \times X \rightarrow Y$ such that

$$\|f(x, y) - F(x, y)\| \leq \varepsilon \left(\frac{7 \cdot 2^p + 2}{2|4 - 2^p|} + \frac{2^p}{|2 - 2^p|} \right) (\|x\|^p + \|y\|^p) \quad (2.35)$$

for all $x, y \in X$.

Proof. Applying Theorem 2.2 (Theorems 2.4 and 2.5, resp.) for the case $0 < p < 1$ ($2 < p$ and $1 < p < 2$, resp.), we obtain the desired result. \square

Corollary 2.9. Let $0 < p, q < 2$ ($p, q \neq 1$), and $\varepsilon > 0$. Let $f : X \times X \rightarrow Y$ be a mapping such that

$$\begin{aligned} \|J_1 f(x, y, z)\| &\leq \varepsilon (\|x\|^p + \|y\|^p + \|z\|^p), \\ \|J_2 f(x, y, z)\| &\leq \varepsilon (\|x\|^q + \|y\|^q + \|z\|^q) \end{aligned} \quad (2.36)$$

for all $x, y, z \in X$. Then, there exists a unique bi-Jensen mapping $F : X \times X \rightarrow Y$ such that

$$\begin{aligned} &\|f(x, y) - F(x, y)\| \\ &\leq \varepsilon \left(\frac{3 \cdot 2^p}{4 - 2^p} \|x\|^p + \frac{2^q + 2}{2(4 - 2^q)} \|x\|^q + \frac{2^p}{|2 - 2^p|} \|x\|^p + \frac{2^p + 2}{2(4 - 2^p)} \|y\|^p + \frac{3 \cdot 2^q}{4 - 2^q} \|y\|^q + \frac{2^q}{|2 - 2^q|} \|y\|^q \right) \end{aligned} \quad (2.37)$$

for all $x, y \in X$.

Proof. Applying Theorems 2.6 and 2.7, we obtain the desired result. \square

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