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Research Article

An Estimate of the Essential Norm of a Composition Operator from $F(p, q, s)$ to $\mathcal{B}^\alpha$ in the Unit Ball

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Let B_n be the unit ball of $\mathbb{C}^n$ and $\phi = (\phi_1, \ldots, \phi_n)$ a holomorphic self-map of B_n. Let $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $\alpha > 0$, and let C_ϕ be the composition operator between the space $F(p, q, s)$ and α-Bloch space $\mathcal{B}^\alpha$ induced by ϕ. This paper gives an estimate of the essential norm of C_ϕ. As a consequence, a necessary and sufficient condition for the composition operator C_ϕ to be compact from $F(p, q, s)$ to $\mathcal{B}^\alpha$ is obtained.

1. Introduction

Throughout the paper, dv denotes the *Lebesegue* measure on the unit ball B_n of $\mathbb{C}^n$ normalized so that $v(B_n) = 1$, $d\sigma$ denotes the normalized rotation invariant measure on the boundary ∂B_n of B_n, and $H(B_n)$ denotes the class of all holomorphic functions on B_n.

For $a \in B_n$, let $g(z, a) = \log|\varphi_a(z)|^{-1}$ be *Green's* function on B_n with logarithmic singularity at a, where φ_a is the *Möbius* transformation of B_n with $\varphi_a(0) = a$, $\varphi_a(a) = 0$ and $\varphi_a = \varphi_a^{-1}$.

Let $0 < p, s < \infty$, $-n-1 < q < \infty$, and $q+s > -1$. We say that f is a function of $F(p, q, s)$ if $f \in H(B_n)$ and

$$\|f\|_{F(p,q,s)} = |f(0)| + \left\{\sup_{a\in B_n}\int_{B_n} |\nabla f(z)|^p \left(1-|z|^2\right)^q g^s(z, a)\,dv(z)\right\}^{1/p} < \infty, \tag{1.1}$$

where $\nabla f(z) = (\partial f/\partial z_1, \ldots, \partial f/\partial z_n)$ denotes the complex gradient of f.

For $\alpha > 0$, we say that $f \in H(B_n)$ is an α-Bloch function on B_n, if

$$\|f\|_{\alpha,1} = \sup_{z\in B_n}\left(1-|z|^2\right)^{\alpha}\left|\nabla f(z)\right| < \infty. \tag{1.2}$$

The class of all α-Bloch functions on B_n is called α-Bloch space on B_n and denoted by $\mathcal{B}^{\alpha}$. It is easy to prove that $\mathcal{B}^{\alpha}$ is a Banach space with the norm

$$\|f\|_{\mathcal{B}^{\alpha}} = \left|f(0)\right| + \|f\|_{\alpha,1}. \tag{1.3}$$

When $\alpha = 1$, we obtain the classical Bloch functions and Bloch space.

It is proved by Yang and Ouyang [1] that the norm $\|f\|_{\alpha,1}$ is equivalent to the norm

$$\|f\|_{\alpha,2} = \sup_{z\in B_n}\left(1-|z|^2\right)^{\alpha}\left|Rf(z)\right|, \tag{1.4}$$

where $Rf(z) = \nabla f(z)z = \langle \nabla f(z), \overline{z}\rangle$ is the inner product of $\nabla f(z)$ and $\overline{z}$. For $\alpha = 1$, Timoney [2] proved that the above two norms are equivalent to the third norm:

$$\|f\|_{1,3} = \sup\left\{\frac{\left|\nabla f(z)u\right|}{H_z^{1/2}(u,u)} : z \in B_n,\ u \in \mathbb{C}^n \setminus \{0\}\right\}, \tag{1.5}$$

where $\nabla f(z)u = \langle \nabla f(z), \overline{u}\rangle$, and $H_z(u,u)$ is the Bergman metric defined by

$$H_z(u,u) = \frac{n+1}{2}\frac{\left(1-|z|^2\right)|u|^2 + |\langle u,z\rangle|^2}{\left(1-|z|^2\right)^2} \quad \text{for } z \in B_n,\ u \in \mathbb{C}^n \setminus \{0\}. \tag{1.6}$$

On this basis, Zhang and Xu [3] defined another norm $\|f\|_{\alpha,3}$ as follows:

$$\|f\|_{\alpha,3} = \sup_{\substack{u\in\mathbb{C}^n\setminus\{0\}\\ z\in B_n}} \frac{(1-|z|^2)^{\alpha}\left|\langle \nabla f(z), \overline{u}\rangle\right|}{\{G_z(u,u)\}^{1/2}}, \tag{1.7}$$

where

$$G_z(u,u) = \begin{cases} \left(1-|z|^2\right)|u|^2 + |\langle u,z\rangle|^2, & \alpha > \dfrac{1}{2}, \\ \left(1-|z|^2\right)|u|^2\log^2\dfrac{2}{1-|z|^2} + |\langle u,z\rangle|^2, & \alpha = \dfrac{1}{2}, \\ \left(1-|z|^2\right)^{2\alpha}|u|^2 + |\langle u,z\rangle|^2, & 0 < \alpha < \dfrac{1}{2}. \end{cases} \tag{1.8}$$

They proved that this norm is equivalent to $\|f\|_{\alpha,1}$ and $\|f\|_{\alpha,2}$ for any $\alpha > 0$. We give their result as Lemma 2.3 in this paper. For more details, we recommend the readers refer to [3].

Let $\phi(z) = (\phi_1(z), \ldots, \phi_n(z))$ be a holomorphic self-map of B_n; the composition operator C_ϕ induced by ϕ is defined by

$$(C_\phi f)(z) = f(\phi(z)). \tag{1.9}$$

In recent years, many specialists have devoted themselves to the research of composition operators which includes boundedness, compactness, and spectra. Concerning these results, we also recommend the interested readers refer to [2, 4–7].

Another hot topic is the essential norm of composition operators. First, we recall that the essential norm of a continuous linear operator T is the distance from T to the compact operators, that is,

$$\|T\|_e = \inf\{\|T - K\| : K \text{ is compact}\}. \tag{1.10}$$

Notice that $\|T\|_e = 0$ if and only if T is compact, so that estimates on $\|T\|_e$ lead to conditions for T to be compact.

In 1987, J. H. Shapiro calculated the essential norm of a composition operator on Hilbert spaces of analytic functions (Hardy and weighted Bergman spaces) in terms of natural counting functions associated with ϕ. In [8], Gorkin and MacCluer obtained the estimates for the essential norm of a composition operator acting from the Hardy space H^p to H^q, $p > q$, in one or several variables. In [9], Montes-Rodríguez gave the exact essential norm of a composition operator on the Bloch space in the disc. After that, Zhou and Shi generalized Alfonso's result to the polydisc in [10, 11]. This paper, with fundamental ideas of the proof following Zhou and Shi, gives an estimate of composition operator from $F(p,q,s)$ to $\mathcal{B}^\alpha$ in the unit ball. In addition, we get a similar estimate of composition operators between different Bloch type spaces and obtain some necessary and sufficient conditions for the composition operators C_ϕ to be compact for $F(p,q,s)$ to $\mathcal{B}^\alpha$.

In the following, we will use the symbols c, c_1, and c_2 to denote a finite positive number which does not depend on variables z, a, w and may depend on some norms and parameters p, q, s, n, α, x, f, and so forth, not necessarily the same at each occurrence.

Our main result is the following.

Theorem 1.1. *Let $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ be a holomorphic self-map of B_n and let $\|C_\phi\|_e$ be the essential norm of a bounded composition operator $C_\phi : F(p,q,s) \to \mathcal{B}^\alpha$; then there are $c_1, c_2 > 0$, independent of w, such that*

$$c_1 \lim_{\delta \to 0} \sup_{\operatorname{dist}(\phi(w), \partial B_n) < \delta} X(w,w) \le \|C_\phi\|_e \le c_2 \lim_{\delta \to 0} \sup_{\operatorname{dist}(\phi(w), \partial B_n) < \delta} X(w,w), \tag{1.11}$$

where

$$X(w,w) = \frac{\left(1 - |w|^2\right)^\alpha}{\left(1 - |\phi(w)|^2\right)^{(n+1+q)/p}} \left\{G_{\phi(w)}\left(R\phi(w), R\phi(w)\right)\right\}^{1/2}, \tag{1.12}$$

and when $0 < (n+1+q)/p < 1/2$,

$$G_{\phi(w)}\left(R\phi(w), R\phi(w)\right) = (1 - \left|\phi(w)\right|^2)^{2(n+1+q)/p}\left|R\phi(w)\right|^2 + \left|\langle R\phi(w), \phi(w)\rangle\right|^2; \tag{1.13}$$

when $0 < (n+1+q)/p = 1/2$,

$$G_{\phi(w)}\left(R\phi(w), R\phi(w)\right) = \left(1 - \left|\phi(w)\right|^2\right)\log^2\frac{1}{1 - \left|\phi(w)\right|^2}\left|R\phi(w)\right|^2 + \left|\langle R\phi(w), \phi(w)\rangle\right|^2; \tag{1.14}$$

when $(n+1+q)/p > 1/2$,

$$G_{\phi(w)}\left(R\phi(w), R\phi(w)\right) = \left(1 - \left|\phi(w)\right|^2\right)\left|R\phi(w)\right|^2 + \left|\langle R\phi(w), \phi(w)\rangle\right|^2. \tag{1.15}$$

2. Some Lemmas

In order to prove the main result, we will give some lemmas first.

Lemma 2.1 (see [12, Lemma 2.2]). *Let* $\alpha > 0$. *Then there is a constant* $c > 0$, *and for all* $f \in \mathcal{B}^\alpha$ *and* $w \in B_n$, *the estimate*

$$\left|f(w)\right| \leq cG_\alpha(w)\left\|f\right\|_{\mathcal{B}^\alpha} \tag{2.1}$$

holds, where the function G_α *has been defined as follows.*

(i) *If* $0 < \alpha < 1$, *then* $G_\alpha(w) = 1$.

(ii) *If* $\alpha = 1$, *then* $G_\alpha(w) = \ln(4/(1 - |w|^2))$.

(iii) *If* $\alpha > 1$, *then* $G_\alpha(w) = 1/(1 - |w|^2)^{\alpha-1}$.

Lemma 2.2 (see [12, Lemma 2.1]). *If* $0 < p, s < +\infty$, $-n-1 < q < +\infty$, $q+s > -1$, *then* $F(p,q,s) \subset \mathcal{B}^{(n+1+q)/p}$ *and there exists* $c > 0$ *such that for all* $f \in F(p,q,s)$, $\|f\|_{\mathcal{B}^{(n+1+q)/p}} \leq c\|f\|_{F(p,q,s)}$.

Lemma 2.3 (see [3, Theorem 2]). *Let* $0 < \alpha < +\infty$, $f \in \mathcal{B}^\alpha$. *Then* $\|f\|_{\alpha,1}$, $\|f\|_{\alpha,2}$, *and* $\|f\|_{\alpha,3}$ *are equivalent.*

In [12], Zhou and Chen characterize the boundedness of weighted composition operator $W_{\psi,\phi}$ between $F(p,q,s)$ and $\mathcal{B}^\alpha$. Take $\psi = 1$ in [12, Theorem 1.2, page 902] and by similar proof we can get the following lemma.

Lemma 2.4. *For* $0 < p, s < +\infty$, $-n-1 < q < +\infty$, $q+s > -1$, $\alpha > 0$, *let* ϕ *be a holomorphic self-map of* B_n. *Then* $C_\phi : F(p,q,s) \to \mathcal{B}^\alpha$ *is bounded if and only if*

$$\sup_{w\in B_n} X(w,w) < \infty, \tag{2.2}$$

where $X(w,w)$ *has been defined at* (1.12).

Lemma 2.5 (see [12, Lemma 2.5]). *For $0 < p, s < +\infty$, $-n-1 < q < +\infty$, $q + s > -1$, there exists $c > 0$ such that*

$$\sup_{a\in B_n}\int_{B_n}\frac{\left(1-|w|^2\right)^p}{|1-\langle z,w\rangle|^{n+1+q+p}}\left(1-|z|^2\right)^q g^s(z,a)\,dv(z)\le c, \tag{2.3}$$

for every $w \in B_n$.

Lemma 2.6 (see [12, Lemma 2.7]). *Suppose $0 < p, s < +\infty$ and $s + p > n$, then one has the following.*

(i) *If $s > n$, then there is a constant $c > 0$, for all $w \in B_n$*

$$\sup_{a\in B_n}\int_{B_n}\left(\log\frac{1}{1-|z^2|}\right)^{-p}\left|\log\frac{1}{1-\langle z,w\rangle}\right|^p\frac{\left(1-|z|^2\right)^{p-n-1}}{|1-\langle z,w\rangle|^p}g^s(z,a)\,dv(z)<c. \tag{2.4}$$

(ii) *If $s \le n$, then when one chooses x which satisfies $\max\{1, n/p\} < x < n/(n-s)$, (if $n = s$, just let $x > \max\{1, n/p\}$), then*

$$\sup_{a\in B_n}\int_{B_n}\left(\log\frac{1}{1-|z^2|}\right)^{-2/x}\left|\log\frac{1}{1-\langle z,w\rangle}\right|^{2/x}\frac{\left(1-|z|^2\right)^{p-n-1}}{(|1-\langle z,w\rangle|)^p}g^s(z,a)\,dv(z)<c. \tag{2.5}$$

Lemma 2.7. *If $\{f_k\}$ is a bounded sequence in $F(p,q,s)$, then there exists a subsequence $\{f_{k_j}\}$ of $\{f_k\}$ which converges uniformly on compact subsets of B_n to a holomorphic function $f \in F(p,q,s)$.*

Proof. Choose a bounded sequence $\{f_k\}$ from $F(p,q,s)$ with $\|f_k\|_{F(p,q,s)} \le c$. By Lemma 2.1, $\{f_k\}$ is uniformly bounded on compact subsets of B_n. By Montel's theorem, we may extract subsequence $\{f_{k_j}\}$ which converges uniformly on compact subsets of B_n to a holomorphic function f. By Weierstrass's theorem we have $f \in H(B_n)$ and $\partial f_{k_j}/\partial z_l \to \partial f/\partial z_l$ for each $l \in \{1,2,\ldots,n\}$ on every compact subsets of B_n. It follows that $\nabla f_{k_j} \to \nabla f$ uniformly on compact subsets of B_n.

Let $B_m = \{z \in \mathbb{C}^n : |z| < 1 - 1/m\} \subset B_n$ $(m = 1,2,\ldots)$; then

$$\begin{aligned}\int_{B_n}|\nabla f|^p\left(1-|z|^2\right)^q g^s(z,a)\,dv(z)&=\lim_{m\to+\infty}\int_{B_m}\lim_{j\to+\infty}\left|\nabla f_{k_j}\right|^p\left(1-|z|^2\right)^q g^s(z,a)\,dv(z)\\&=\lim_{m\to+\infty}\lim_{j\to+\infty}\int_{B_m}\left|\nabla f_{k_j}\right|^p\left(1-|z|^2\right)^q g^s(z,a)\,dv(z).\end{aligned} \tag{2.6}$$

But $\|f_{k_j}\|_{F(p,q,s)} \le c$, then

$$\int_{B_m}\left|\nabla f_{k_j}\right|^p\left(1-|z|^2\right)^q g^s(z,a)\,dv(z)\le c^p, \tag{2.7}$$

and therefore

$$\int_{B_n} |\nabla f|^p (1-|z|^2)^q g^s(z,a)\,dv(z) \le c^p. \tag{2.8}$$

So $\|f\|_F(p,q,s) \le c^p$, which implies $f \in F(p,q,s)$. □

Lemma 2.8 (see [10, 11, Lemma 2.6]). *Let Ω be a domain in $\mathbb{C}^n$, $f \in H(\Omega)$. If a compact set K and its neighborhood G satisfy $K \subset G \subset\subset \Omega$ and $\rho = \operatorname{dist}(K, \partial G) > 0$, then*

$$\sup_{z\in K}\left|\frac{\partial f}{\partial z_j}(z)\right| \le \frac{\sqrt{n}}{\rho}\sup_{z\in G}|f(z)| \quad (j = 1,\ldots,n). \tag{2.9}$$

3. The Proof of Theorem 1.1

To obtain the lower estimate we first prove the following proposition.

Proposition 3.1. *If $C_\phi : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded, then for all $w \in B_n$ which satisfies $|\phi(w)| > \sqrt{2/3}$, there is a function $g_w \in F(p,q,s)$ such that*

(i) *there exists $c_1, c_2 > 0$, independent of w, such that*

$$c_1 \le \|g_w\|_{F(p,q,s)} \le c_2; \tag{3.1}$$

(ii) *$\{g_w\}$ converges to zero uniformly for z on compact subsets of B_n when $|\phi(w)| \to 1$;*

(iii) *there is a constant $c > 0$, for all $w \in B_n$,*

$$(1-|w|^2)^\alpha |\nabla(g_w \circ \phi)(w)| > cX(w,w), \tag{3.2}$$

where $X(w,w)$ is the same as Theorem 1.1.

Proof. For all $w \in B_n$ with $|\phi(w)| > \sqrt{2/3}$, we suppose $\phi(w) = r_w e_1$, where $r_w = |\phi(w)|$, e_1 is the vector $(1,0,\ldots,0)$.

Next we break the proof into two cases.

(1) Assume that

$$G_{\phi(w)}(R\phi(w), R\phi(w)) \le 2|\langle R\phi(w), \phi(w)\rangle|^2. \tag{$\star$}$$

Let

$$g_w(z) = \frac{(z_1 - r_w)(1-r_w^2)}{(1-r_w z_1)^{(n+1+q)/p+1}}. \tag{3.3}$$

Then

$$\frac{\partial g_w(z)}{\partial z_1} = \frac{1-r_w^2}{(1-r_w z_1)^{(n+1+q)/p+1}}\left(1+\frac{n+1+q}{p}\frac{(z_1-r_w)r_w}{1-r_w z_1}\right),$$
$$\frac{\partial g_w(z)}{\partial z_k} = 0, \quad k = 2,\dots,n. \tag{3.4}$$

Therefore

$$\begin{aligned}\left|\nabla g_w(z)\right| &= \frac{1-r_w^2}{|1-r_w z_1|^{(n+1+q)/p+1}}\left|1+\frac{n+1+q}{p}\frac{(z_1-r_w)r_w}{1-r_w z_1}\right| \\ &\le \left(1+\frac{n+1+q}{p}\right)\frac{1-r_w^2}{|1-r_w z_1|^{(n+1+q)/p+1}}.\end{aligned} \tag{3.5}$$

By Lemma 2.5, $g_w \in F(p,q,s)$, and there exists $c_2 > 0$ independent of w such that $\|g_w\|_{F(p,q,s)} \le c_2$.

On the other hand, taking $z_0 = (z_1^0, 0,\dots,0) = (r_w, 0,\dots,0) \in B_n$; then

$$(1-|z_0|^2)^{(n+1+q)/p}\left|\nabla g_w(z_0)\right| = \left(1-|r_w|^2\right)^{(n+1+q)/p}(1-|r_w|^2)^{1/((n+1+q)/p)} = 1. \tag{3.6}$$

So

$$\|g_w\|_{\mathcal{B}^{(n+1+q)/p}} = \left|g_w(0)\right| + \sup_{z\in B_n}(1-|z|^2)^{(n+1+q)/p}\left|\nabla g_w(z)\right| \ge 1 + r_w^3 - r_w > \left(\sqrt{\frac{2}{3}}\right)^3. \tag{3.7}$$

By Lemma 2.2, $g_w \in \mathcal{B}^{(n+1+q)/p}$, and $\|g_w\|_{F(p,q,s)} \ge c\|g_w\|_{\mathcal{B}^{(n+1+q)/p}}$, we have

$$\|g_w\|_{F(p,q,s)} \ge c\left(\sqrt{\frac{2}{3}}\right)^3 = c_1. \tag{3.8}$$

By the discussion above we get

$$c_1 \le \|g_w\|_{F(p,q,s)} \le c_2. \tag{3.9}$$

At the same time, for fixed $z \in B_n$, it is clear that $\lim_{r_w\to 1}|g_w(z)| \to 0$ uniformly for z on compact subsets of B_n. This shows that (i) and (ii) hold.

By simple calculation it is easy to get that $G_w(w,w) < 2$; so by Lemma 2.3 we have

$$\begin{aligned}(1-|w|^2)^{\alpha}\left|\nabla(g_w\circ\phi)(w)\right| &\geq c\frac{(1-|w|^2)^{\alpha}\left|\nabla g_w(\phi(w))R\phi(w)\right|}{\sqrt{G_w(w,w)}}\\ &\geq c(1-|w|^2)^{\alpha}\left|\nabla g_w(\phi(w))R\phi(w)\right|.\end{aligned}\tag{3.10}$$

Notice that $\nabla g_w(\phi(w)) = ((1-r_w^2)/(1-r_w^2)^{(n+1+q)/p})e_1$. Therefore, from our assumption $(\star)$, we get

$$\begin{aligned}\left(1-|w|^2\right)^{\alpha}\left|\nabla(g_w\circ\phi)(w)\right| &\geq c\frac{\left(1-|w|^2\right)^{\alpha}}{r_w(1-r_w^2)^{(n+1+q)/p}}\left|e_1 r_w R\phi(w)\right|\\ &\geq c\frac{\left(1-|w|^2\right)^{\alpha}}{(1-r_w^2)^{(n+1+q)/p}}\left|\langle R\phi(w),\phi(w)\rangle\right|\\ &\geq c\frac{\left(1-|w|^2\right)^{\alpha}}{(1-r_w^2)^{(n+1+q)/p}}\left\{G_{\phi(w)}(R\phi(w),R\phi(w))\right\}^{1/2}\\ &= cX(w,w).\end{aligned}\tag{3.11}$$

(2) Assume that

$$G_{\phi(w)}(R\phi(w),R\phi(w)) > 2\left|\langle R\phi(w),\phi(w)\rangle\right|^2.\tag{$\star\star$}$$

Let $R\phi(w) = (\xi_1,\ldots,\xi_n)^T$. For $j = 2,\ldots,n$, let $\theta_j = \arg\xi_j$ and $a_j = e^{-i\theta_j}$ if $\xi_j \neq 0$, or let $a_j = 0$ if $\xi_j = 0$.

In Case $(n+1+q)/p > 1/2$, take

$$g_w(z) = \frac{(a_2z_2+\cdots+a_nz_n)(1-r_w^2)^{3/2}}{(1-r_wz_1)^{(n+1+q)/p+1}},\tag{3.12}$$

where $r_w = |\phi(w)|$. Then

$$\frac{\partial g_w(z)}{\partial z_1} = \frac{((n+1+q)/p+1) r_w (1-r_w^2)^{3/2}}{(1-r_w z_1)^{(n+1+q)/p+2}} (a_2 z_2 + \cdots + a_n z_n),$$
$$\frac{\partial g_w(z)}{\partial z_k} = \frac{a_k (1-r_w^2)^{3/2}}{(1-r_w z_1)^{(n+1+q)/p+1}}, \quad k = 2, \ldots, n. \tag{3.13}$$

Therefore

$$\begin{aligned}
\left|\nabla g_w(z)\right| &= \sqrt{\left|\frac{\partial g_w(z)}{\partial z_1}\right|^2 + \left|\frac{\partial g_w(z)}{\partial z_2}\right|^2 + \cdots + \left|\frac{\partial g_w(z)}{\partial z_1}\right|^2} \\
&= \sqrt{\frac{((n+1+q)/p+1)^2 r_w^2 (1-r_w^2)^3 |a_2 z_2 + \cdots + a_n z_n|^2}{|1-r_w z_1|^{2((n+1+q)/p+2)}} + \frac{(n-1)(1-r_w^2)^3}{|1-r_w z_1|^{2((n+1+q)/p+1)}}} \\
&\le \sqrt{\frac{(n-1)((n+1+q)/p+1)^2 r_w^2 (1-r_w^2)^3 \left(|z_2|^2 + \cdots + |z_n|^2\right)}{|1-r_w z_1|^{2((n+1+q)/p+2)}} + \frac{(n-1)(1-r_w^2)^3}{|1-r_w z_1|^{2((n+1+q)/p+1)}}} \\
&\le \frac{\sqrt{(n-1)}(1-r_w^2)^{3/2}}{|1-r_w z_1|^{(n+1+q)/p+1}} \sqrt{\frac{((n+1+q)/p+1)^2 r_w^2 \left(1-|z_1|^2\right)}{|1-r_w z_1|^2} + 1} \\
&\le \frac{\sqrt{(n-1)}(1-r_w^2)}{|1-r_w z_1|^{(n+1+q)/p+1}} \left|1 - r_w^2 + r_w^2 \left(\frac{n+1+q}{p} + 1\right)^2\right|^{1/2} \\
&\le c \frac{1-r_w^2}{|1-r_w z_1|^{(n+1+q)/p+1}}.
\end{aligned} \tag{3.14}$$

It follows from Lemma 2.5 that $g_w \in F(p,q,s)$, and there exists $c_2 > 0$ independent of w such that $\|g_w\|_{F(p,q,s)} \le c_2$.

On the other hand, taking

$$z_0 = \left(z_1^{(0)}, \ldots, z_n^{(0)}\right) = \left(r_w, \frac{1}{\sqrt{2}}\sqrt{1-r_w^2}, 0, \ldots, 0\right), \tag{3.15}$$

then

$$|z_0|^2 = r_w^2 + \frac{1}{2}\left(1-r_w^2\right) = \frac{1}{2}\left(1+r_w^2\right) < 1. \tag{3.16}$$

Thus $z_0 \in B_n$. Notice that $1 \geq r_w \geq \sqrt{2/3}$ and by Lemma 2.2 we have

$$\begin{aligned}
\|g_w\|_{F(p,q,s)} &\geq c\|g_w\|_{\mathcal{B}^{(n+1+q)/p}} \\
&\geq c(1-|z_0|^2)^{(n+1+q)/p}|\nabla g_w(z_0)| \geq c\left(1-|z_0|^2\right)^{(n+1+q)/p}\left|\frac{\partial g_w(z_0)}{\partial z_1}\right| \\
&= c(1-r_w^2)^{(n+1+q)/p}\sqrt{\frac{((n+1+q)/p+1)^2 r_w^2(1-r_w^2)^3\left|a_2 z_2^{(0)}+\cdots+a_n z_n^{(0)}\right|^2}{\left|1-r_w z_1^{(0)}\right|^{2((n+1+q)/p+2)}}} \\
&= c(1-r_w^2)^{(n+1+q)/p}\sqrt{\frac{((n+1+q)/p+1)^2 r_w^2(1-r_w^2)^3\left(1/\sqrt{2}\right)\sqrt{1-r_w^2}}{(1-r_w^2)^{2((n+1+q)/p+2)}}} \\
&= c\left(\frac{n+1+q}{p}+1\right)r_w(1-r_w^2)^{-1/4} \\
&\geq c_1.
\end{aligned} \tag{3.17}$$

By the discussion above we get that $c_1 \leq \|g_w\|_{F(p,q,s)} \leq c_2$. At the same time, it is also clear that $\lim_{r_w\to 1}|g_w(z)| \to 0$; so (i) and (ii) hold.

Next we show that (iii) holds. First, by (3.13) and $\phi(w) = (r_w, 0, \ldots, 0)$ it is easy to get that

$$\nabla g_w(\phi(w)) = \frac{(1-r_w^2)^{1/2}}{(1-r_w^2)^{(n+1+q)/p}}(0, a_2, \ldots, a_n). \tag{3.18}$$

Notice that $R\phi(w) = (\xi_1, \ldots, \xi_n)^T$ and $a_i\xi_i = |\xi_i|$ $(i = 2, \ldots, n)$; so we have

$$|\nabla g_w(\phi(w))R\phi(w)| = \frac{(1-r_w^2)^{1/2}}{(1-r_w^2)^{(n+1+q)/p}}(|\xi_2| + \cdots + |\xi_n|). \tag{3.19}$$

Second, since $|\phi(w)| > \sqrt{2/3}$ and $(n+1+q)/p > 1/2$, it is clear that

$$(1-r_w^2)^{(n+1+q)/p}|R\phi(w)| > (1-r_w^2)^{1/2}|R\phi(w)| > |\langle\phi(w), R\phi(w)\rangle|, \tag{3.20}$$

and it follows that

$$\sqrt{3\left(1-|\phi(w)|^2\right)\left(|\xi_2|^2+\cdots+|\xi_n|^2\right)} > |\xi_1|. \tag{3.21}$$

Then

$$|\xi_2|^2+\cdots+|\xi_n|^2 \geq \frac{1}{2}\left(|\xi_1|^2+\cdots+|\xi_n|^2\right). \tag{3.22}$$

On the other hand, when $(n+1+q)/p > 1/2$,

$$G_{\phi(w)}(R\phi(w),R\phi(w)) = \left(1-|\phi(w)|^2\right)|R\phi(w)|^2+|\langle R\phi(w),\phi(w)\rangle|^2. \tag{3.23}$$

So by our assumption $(\star\star)$ we get

$$(1-|\phi(w)|^2)^{1/2}|R\phi(w)| > \sqrt{\frac{1}{2}}\{G_{\phi(w)}(R\phi(w),R\phi(w))\}^{1/2}, \tag{3.24}$$

and it follows that

$$(1-|\phi(w)|^2)^{(n+1+q)/p}|R\phi(w)| > \sqrt{\frac{1}{2}}\{G_{\phi(w)}(R\phi(w),R\phi(w))\}^{1/2}. \tag{3.25}$$

Combining (3.19), (3.22), and (3.25), it follows from $G_w(w,w) < 2$ and Lemma 2.3 that

$$\begin{aligned}
(1-|w|^2)^{\alpha}|\nabla(g_w\circ\phi)(w)| &\geq c\frac{(1-|w|^2)^{\alpha}|\nabla g_w(\phi(w))R\phi(w)|}{\sqrt{G_w(w,w)}}\\
&\geq c(1-|w|^2)^{\alpha}|\nabla g_w(\phi(w))R\phi(w)|\\
&= c\frac{\left(1-|w|^2\right)^{\alpha}}{(1-r_w^2)^{(n+1+q)/p}}(1-r_w^2)^{1/2}(|\xi_2|+\cdots+|\xi_n|)\\
&\geq c\frac{\left(1-|w|^2\right)^{\alpha}}{(1-r_w^2)^{(n+1+q)/p}}(1-r_w^2)^{1/2}\sqrt{|\xi_2|^2+\cdots+|\xi_n|^2}\\
&\geq c\frac{\left(1-|w|^2\right)^{\alpha}}{(1-r_w^2)^{(n+1+q)/p}}\left(1-r_w^2\right)^{(n+1+q)/p}\sqrt{|\xi_1|^2+\cdots+|\xi_n|^2}\\
&= c\frac{\left(1-|w|^2\right)^{\alpha}}{(1-r_w^2)^{(n+1+q)/p}}\left(1-r_w^2\right)^{(n+1+q)/p}|R\phi(w)|\\
&\geq c\frac{\left(1-|w|^2\right)^{\alpha}}{(1-r_w^2)^{(n+1+q)/p}}\{G_{\phi(w)}(R\phi(w),R\phi(w))\}^{1/2}.
\end{aligned} \tag{3.26}$$

This is (iii).

In Case $(n+1+q)/p = 1/2$ and $s > n$, take

$$g_w(z) = (a_2z_2 + \cdots + a_nz_n)\log^{-1}\frac{1}{1-r_w^2}\log^2\frac{1}{1-r_wz_1}. \tag{3.27}$$

In Case $(n+1+q)/p = 1/2$ and $s \le n$, take

$$g_w(z) = (a_2z_2 + \cdots + a_nz_n)\left(\log\frac{1}{1-r_w^2}\right)^{-2/px}\left(\log\frac{1}{1-r_wz_1}\right)^{1+2/px}, \tag{3.28}$$

where x is the one used in Lemma 2.6.

In Case $0 < (n+1+q)/p < 1/2$, take

$$g_w(z) = (a_2z_2 + \cdots + a_nz_n)\left\{1 - \frac{(1-r_w)^{3/2}}{(1-r_wz_1)((n+1+q)/p)+1}\right\}. \tag{3.29}$$

According to Lemmas 2.5 and 2.6, and the discussion of the case of $(n+1+q)/p > 1/2$, we can see that the functions above are just what we want.

In the general situation, or when $\phi(w) \ne |\phi(w)|e_1$, we use the unitary transformation U_w which satisfies the equation $\phi(w) = r_we_1U_w$, where $r_w = |\phi(w)|$. Then $f_w = g_w \circ U_w^{-1}$ is the desired function.

In fact, by $\nabla f_w(z) = \nabla(g_w \circ U_w^{-1})(z) = (\nabla g_w)(zU_w^{-1})(U_w^{-1})^T$ and $|zU_w^{-1}| = |z|$, we have

$$\begin{aligned}
&\int_{B_n} |\nabla f_w(z)|^p\left(1-|z|^2\right)^q g^s(z,a)\,dv(z)\\
&\quad= \int_{B_n}\left|(\nabla g_w)(zU_w^{-1})\left(U_w^{-1}\right)^T\right|^p(1-|z|^2)^q g^s(z,a)\,dv(z)\\
&\quad= \int_{B_n}|\nabla g_w(z)|^p(1-|z|^2)^q g^s(z,a)\,dv(z),
\end{aligned} \tag{3.30}$$

where in the last equation we use the linear coordinate translation $z = zU_w^{-1}$ and the fact that $F(p,q,s)$ is invariant under *möbius* translation. So

$$\|f_w\|_{F(p,q,s)} = \|g_w\|_{F(p,q,s)}. \tag{3.31}$$

Then we can prove the same result in the same way, and we omit the details here. □

Now, we are ready to prove Theorem 1.1. We begin by proving *the lower estimate*. Let

$$F_w(z) = \frac{g_w(z)}{\|g_w\|_{F(p,q,s)}}, \tag{3.32}$$

where $g_w(z)$ is defined as Proposition 3.1. It is clear that $\|F_w\|_{F(p,q,s)} = 1$ and $F_w(z)$ converges to zero uniformly on compact subsets of B_n when $|\phi(w)| \to 1$. Suppose that $K : F(p,q,s) \to \mathcal{B}^\alpha$ is compact, then $\|KF_w\|_{\mathcal{B}^\alpha} \to 0$ uniformly for z in compact subsets of B_n when $|\phi(w)| \to 1$ (in the following, it is clear that $|\phi(w)| \to 1$ when $\delta \to 0$); so we have

$$\begin{aligned}\|C_\phi - K\| &= \sup_{\|f\|_{F(p,q,s)}=1} \|(C_\phi - K)f\|_{\mathcal{B}^\alpha} \\ &\geq \sup_{\|f\|_{F(p,q,s)}=1} \left(\|C_\phi f\|_{\mathcal{B}^\alpha} - \|Kf\|_{\mathcal{B}^\alpha}\right) \\ &\geq \sup_{\operatorname{dist}(\phi(w),\partial B_n)<\delta} \left(\|C_\phi F_w\|_{\mathcal{B}^\alpha} - \|KF_w\|_{\mathcal{B}^\alpha}\right) \\ &\geq \sup_{\operatorname{dist}(\phi(w),\partial B_n)<\delta} \|C_\phi F_w\|_{\mathcal{B}^\alpha} - \sup_{\operatorname{dist}(\phi(w),\partial B_n)<\delta} \|KF_w\|_{\mathcal{B}^\alpha}.\end{aligned} \tag{3.33}$$

On the other hand, by (i) in Proposition 3.1, for $|\phi(w)| > \sqrt{2/3}$ we get

$$\begin{aligned}\sup_{\operatorname{dist}(\phi(w),\partial B_n)<\delta} \frac{\|g_w \circ \phi\|_{\mathcal{B}^\alpha}}{\|g_w\|_{F(p,q,s)}} &\geq \frac{1}{c_2} \sup_{\operatorname{dist}(\phi(w),\partial B_n)<\delta} \|g_w \circ \phi\|_{\mathcal{B}^\alpha} \\ &\geq \frac{1}{c_2} \sup_{\operatorname{dist}(\phi(w),\partial B_n)<\delta} \sup_{z\in B_n} \left(1 - |z|^2\right)^\alpha \left|\nabla\left(g_w \circ \phi\right)(z)\right| \\ &\geq \frac{1}{c_2} \sup_{\operatorname{dist}(\phi(w),\partial B_n)<\delta} \left(1 - |w|^2\right)^\alpha \left|\nabla\left(g_w \circ \phi\right)(w)\right|.\end{aligned} \tag{3.34}$$

By (iii) in Proposition 3.1, when $|\phi(w)| > \sqrt{2/3}$ we have

$$\left(1 - |w|^2\right)^\alpha \left|\nabla\left(g_w \circ \phi\right)(w)\right| \geq c \cdot X(w,w). \tag{3.35}$$

Therefore

$$\|C_\phi - K\| \geq \frac{c}{c_2} \sup_{\operatorname{dist}\left(\phi(w),\partial B_n\right)<\delta} X(w,w) - \sup_{\operatorname{dist}\left(\phi(w),\partial B_n\right)<\delta} \|KF_w\|_{\mathcal{B}^\alpha}. \tag{3.36}$$

Let $\delta \to 0$, we get

$$\|C_\phi - K\| \geq \frac{c}{c_2} \lim_{\delta\to 0} \sup_{\operatorname{dist}(\phi(w),\partial B_n)<\delta} X(w,w). \tag{3.37}$$

It follows from the definition of $\|C_\phi\|_e$ that

$$\begin{aligned}\|C_\phi\|_e &= \inf\{\|C_\phi - K\| : K \text{ is compact}\}\\ &\geq \frac{c}{c_2}\lim_{\delta\to 0}\sup_{\operatorname{dist}(\phi(w),\partial B_n)<\delta} X(w,w)\\ &= c_1\lim_{\delta\to 0}\sup_{\operatorname{dist}(\phi(w),\partial B_n)<\delta} X(w,w).\end{aligned} \tag{3.38}$$

This is the lower estimate.

To obtain the upper estimate in Theorem 1.1 we first prove the following proposition.

Proposition 3.2. *Let ϕ be a holomorphic self-map of B_n. For $m = 2, 3, \ldots$ one defines the operators as follows:*

$$K_m f(w) = f\left(\frac{m-1}{m}w\right), \quad f \in H(B_n),\ w \in B_n. \tag{3.39}$$

Then the operators K_m have the following properties.

(i) *For all $f \in H(B_n), K_m f \in F(p,q,s)$.*

(ii) *For fixed m, K_m is compact on $F(p,q,s)$.*

(iii) *If $C_\phi : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded, then $C_\phi K_m f \in \mathcal{B}^\alpha$ and $C_\phi K_m : F(p,q,s) \to \mathcal{B}^\alpha$ is compact.*

(iv) $\|I - K_m\| \leq 2$.

(v) *$(I - K_m)f$ tends to zero uniformly on compact subsets of B_n, when $m \to \infty$.*

Proof. (i) Since $f \in H(B_n)$, there exists a $M > 0$ (only depending on f) such that

$$\left|\frac{\partial f}{\partial z_k}\left(\frac{m-1}{m}w\right)\right| \leq M, \quad k = 1,\ldots,n, \tag{3.40}$$

where $z = (z_1,\ldots,z_n) = ((m-1)/m)(w_1,\ldots,w_n)$; therefore

$$\left|\nabla(K_m f)(w)\right| \leq \frac{m-1}{m}\sum_{k=1}^{n}\left|\frac{\partial f}{\partial z_k}\left(\frac{m-1}{m}w\right)\right| \leq \frac{m-1}{m}nM. \tag{3.41}$$

By Lemma 2.5 we have

$$\int_{B_n}(1-|w|^2)^q g^s(w,a)\,dv(w) < \infty. \tag{3.42}$$

So

$$\begin{aligned}
&\int_{B_n} \left|\nabla (K_m f)(w)\right|^p (1-|w|^2)^q g^s(w,a)\,dv(w) \\
&\quad\le \left(\frac{m-1}{m} nM\right)^p \int_{B_n} (1-|w|^2)^q g^s(w,a)\,dv(w) < \infty.
\end{aligned} \tag{3.43}$$

This shows that $K_m f \in F(p,q,s)$.

(ii) Choose a bounded sequence $\{f_j\}$ from $F(p,q,s)$. By Lemma 2.7, we know that there exists a subsequence of $\{f_j\}$ (we still denote it by $\{f_j\}$ here) which converges to a function $f \in F(p,q,s)$ uniformly on compact subsets of B_n and $\{\partial f_j/\partial w_i\}$ $(i = 1,\dots,n)$ also converges uniformly on compact subsets of B_n to holomorphic function $\partial f/\partial w_i$. So when j is large enough, for any $\epsilon > 0$, $z \in E_1 = \{((m-1)/m)z : z \in B_n\}$, and $l = 1,\dots,n$, we have

$$\left|\frac{\partial(f_j - f)}{\partial z_l}(z)\right| < \epsilon. \tag{3.44}$$

So when $j \to \infty$, we get

$$\begin{aligned}
\sup_{w\in B_n}\left|\nabla (K_m f_j - K_m f)(w)\right| &= \sup_{w\in B_n}\left|\nabla (f_j - f)\left(\frac{m-1}{m} w\right)\right| \\
&\le \sup_{z\in E_1}\frac{m-1}{m}\sum_{l=1}^{n}\left|\frac{\partial(f_j - f)}{\partial z_l}(z)\right| \\
&\le \frac{m-1}{m} n\epsilon.
\end{aligned} \tag{3.45}$$

Therefore

$$\begin{aligned}
\left\|K_m f_j - K_m f\right\|_{F(p,q,s)} &= \left|f_j(0) - f(0)\right| + \sup_{a\in B_n}\int_{B_n} \left|\nabla (K_m f_j - K_m f)(w)\right|^p (1-|w|^2)^q g^s(w,a)\,dv(w) \\
&\le \left|f_j(0) - f(0)\right| + \left(\frac{m-1}{m} n\epsilon\right)^p \sup_{a\in B_n}\int_{B_n} (1-|w|^2)^q g^s(w,a)\,dv(w) \\
&\le \left|f_j(0) - f(0)\right| + c\left(\frac{m-1}{m} n\epsilon\right)^p \longrightarrow 0.
\end{aligned} \tag{3.46}$$

This shows that $\{K_m f_j\}$ converges to $g = K_m f \in F(p,q,s)$. So (ii) holds.

(iii) By (i) and the fact that C_ϕ is bounded, the former is obvious. By (ii) and noting that C_ϕ is bounded, we get that $C_\phi K_m$ is compact.

(iv) First, for all $f \in \mathcal{B}^{(n+1+q)/p}$, we have $(I-K_m)f(0)=0$; therefore

$$\begin{aligned}
\|(I-K_m)f\|_{\mathcal{B}^{(n+1+q)/p}} &= \sup_{w\in B_n}(1-|w|^2)^{(n+1+q)/p}\left|\nabla\left[(I-K_m)f\right](w)\right| \\
&\le \sup_{w\in B_n}(1-|w|^2)^{(n+1+q)/p}\left(\left|\nabla f(w)\right|+\left|\nabla(K_m f)(w)\right|\right) \\
&\le \|f\|_{\mathcal{B}^{(n+1+q)/p}}+\frac{m-1}{m}\sup_{w\in B_n}\left(1-\left|\frac{m-1}{m}w\right|^2\right)^{(n+1+q)/p}\left|\nabla f\left(\frac{m-1}{m}w\right)\right| \\
&\le 2\|f\|_{\mathcal{B}^{(n+1+q)/p}},
\end{aligned} \tag{3.47}$$

which implies that $\|I-K_m\|\le 2$.

(v) For any compact subset $E\subset B_n$, there exists r $(0<r<1)$ such that $E\subset rB_n\subset B_n$. On the other hand, for all $z\in E$, write $r_m=(m-1)/m$:

$$\begin{aligned}
\left|(I-K_m)f(z)\right| &= \left|f(z)-f_m(z)\right| \\
&= \left|f(z)-f(r_m z)\right| \\
&= \left|\int_{r_m}^1 \frac{d}{dt}(f(tz))dt\right| \\
&= \left|\int_{r_m}^1 \sum_{k=1}^n \frac{\partial f}{\partial w_k}(tz)\cdot z_k dt\right| \\
&\le \sum_{k=1}^n \int_{r_m}^1 \left|\frac{\partial f}{\partial w_k}(tz)\right| dt.
\end{aligned} \tag{3.48}$$

When $t\in[r_m,1]$, $|tz|=t|z|<|z|<r$ for all $z\in E$. But $(\partial f/\partial w_k)(w)$ is bounded uniformly on $r\overline{B_n}$; therefore for all $z\in E$, $|(\partial f/\partial w_k)(tz)|\le M$. So when $m\to\infty$, we have

$$\left|(I-K_m)f(z)\right|\le nM(1-r_m)\longrightarrow 0. \tag{3.49}$$

Thus $(I-K_m)f$ tends to zero uniformly on compact subsets of B_n. The proof is completed. □

Let us now return to the proof of *the upper estimate*.

First, for some $\delta>0$ we denote that

$$\begin{aligned}
G_1 &:= \left\{w\in B_n : \operatorname{dist}\left(\phi(w),\partial B_n\right)<\delta\right\}, \\
G_2 &:= \left\{w\in B_n : \operatorname{dist}\left(\phi(w),\partial B_n\right)\ge\delta\right\}, \\
G_2' &:= \{z\in B_n : \operatorname{dist}(z,\partial B_n)\ge\delta\}.
\end{aligned} \tag{3.50}$$

Then $G_1 \cup G_2 = B_n$ and G_2' is a compact set of B_n, and $z = \phi(w) \in G_2'$ if and only if $w \in G_2$. For any $f \in F(p,q,s)$, write $\|f\|_F = \|f\|_{F(p,q,s)}$, then by Lemma 2.2 and (iv) of Proposition 3.2 we have

$$\begin{aligned}
\left\|C_\phi\right\|_e &\leq \left\|C_\phi - C_\phi K_m\right\| \\
&= \left\|C_\phi (I - K_m)\right\| \\
&= \sup_{\|f\|_F=1} \left\|C_\phi (I - K_m) f\right\|_{\mathcal{B}^\alpha} \\
&= \sup_{\|f\|_F=1} \left\{ \sup_{w\in B_n} (1-|w|^2)^\alpha \left|\nabla\left[(I-K_m)f\circ\phi\right](w)\right| + \left|\left[(I-K_m)f\right](\phi(0))\right| \right\} \\
&= \sup_{\|f\|_F=1} \left\{ \sup_{w\in B_n} X(w,w) \frac{\left(1-|\phi(w)|^2\right)^{(n+1+q)/p} \left|\nabla\left[(I-K_m)f\circ\phi\right](w)\right|}{\sqrt{G_{\phi(w)}\left(R\phi(w), R\phi(w)\right)}} \right. \\
&\qquad \left. + \left|\left[(I-K_m)f\right](\phi(0))\right| \right\} \\
&\leq c_2 \sup_{\|f\|_F=1} \left\{ \sup_{w\in B_n} X(w,w) \left(1-|\phi(w)|^2\right)^{(n+1+q)/p} \left|\nabla\left[(I-K_m)f\right]\phi(w)\right| \right. \\
&\qquad \left. + \left|\left[(I-K_m)f\right](\phi(0))\right| \right\} \\
&\leq c_2 \|I-K_m\| \sup_{w\in G_1} X(w,w) \\
&\quad + c_2 \sup_{\|f\|_F=1} \sup_{w\in G_2} X(w,w) \left(1-|\phi(w)|^2\right)^{(n+1+q)/p} \left|\nabla\left[(I-K_m)f\right](\phi(w))\right| \\
&\quad + c_2 \sup_{\|f\|_F=1} \left|\left[(I-K_m)f\right](\phi(0))\right| \\
&\leq c_2 \sup_{w\in G_1} X(w,w) + I + II.
\end{aligned} \tag{3.51}$$

By (v) of Proposition 3.2 we know that $[(I-K_m)f](z)$ converges to zero uniformly on G_2', and so $[(I-K_m)f](\phi(w))$ also converges to zero uniformly on G_2 for every fixed f. Next we prove that for any $w \in G_2$ and $\|f\|_F = 1$, $I, II \to 0$ when $m \to \infty$ and $\delta \to 0$.

Since

$$\left|\left[(I-K_m)f\right](\phi(0))\right| = \left| f(\phi(0)) - f\left(\frac{m-1}{m}\phi(0)\right) \right|, \tag{3.52}$$

let $F(t) = f(t\phi(0) + (1-t)((m-1)/m)\phi(0))$. Thus

$$\begin{aligned}
\left|\left[(I-K_m)f\right](\phi(0))\right| &= \left|\int_0^1 F'(t)dt\right| \\
&\leq \int_0^1 \sum_{k=1}^{n}\left|\frac{\partial f}{\partial \zeta_k}\left(t\phi(0) + (1-t)\frac{m-1}{m}\phi(0)\right)\left(\phi_k(0) - \frac{m-1}{m}\phi_k(0)\right)\right|dt \\
&\leq n\int_0^1 \left|\nabla f\left(t\phi(0) + (1-t)\frac{m-1}{m}\phi(0)\right)\right| \cdot \frac{1}{m}|\phi(0)|dt \\
&\leq \frac{n}{m}\int_0^1 \left|\nabla f\left(t\phi(0) + (1-t)\frac{m-1}{m}\phi(0)\right)\right|dt.
\end{aligned} \tag{3.53}$$

Since $f \in F(p,q,s) \subset \mathcal{B}^{(n+1+q)/p}$, $(1-|z|^2)^{(n+1+q)/p}|\nabla f(z)| \leq \|f\|_{\mathcal{B}^{(n+1+q)/p}} \leq c$, we get $|\nabla f(z)| \leq c(1-|z|^2)^{-(n+1+q)/p}$. On the other hand, when $0 < t < 1$, we have

$$\left(1 - \left|t\phi(0) + (1-t)\frac{m-1}{m}\phi(0)\right|^2\right)^{-(n+1+q)/p} \leq (1-|\phi(0)|)^{-(n+1+q)/p}. \tag{3.54}$$

So

$$\begin{aligned}
\left|\left[(I-K_m)f\right](\phi(0))\right| &\leq c\frac{n}{m}\int_0^1 \left(1 - \left|t\phi(0) + (1-t)\frac{m-1}{m}\phi(0)\right|^2\right)^{-(n+1+q)/p} dt \\
&\leq c\frac{n}{m}(1-|\phi(0)|)^{-(n+1+q)/p} \longrightarrow 0 \quad (m \longrightarrow \infty).
\end{aligned} \tag{3.55}$$

Let $m \to \infty$; we get $II \to 0$.

Let $w \in G_2$ and $\phi(w) = z = (z_1, \ldots, z_n)$; then

$$\begin{aligned}
I &= c_2 \sup_{\|f\|_F=1}\sup_{w\in G_2} X(w,w)\left(1-|z|^2\right)^{(n+1+q)/p}\left|\nabla\left[(I-K_m)f\right](z)\right| \\
&= c_2 \sup_{\|f\|_F=1}\sup_{w\in G_2} X(w,w)\left(1-|z|^2\right)^{(n+1+q)/p}\left|\nabla f(z) - \frac{m-1}{m}\nabla f\left(\frac{m-1}{m}z\right)\right| \\
&\leq c_2 \sup_{\|f\|_F=1}\sup_{w\in G_2} X(w,w)(1-|z|^2)^{(n+1+q)/p}\left|\nabla f(z) - \nabla f\left(\frac{m-1}{m}z\right)\right| \\
&\quad + \frac{c_2}{m}\sup_{\|f\|_F=1}\sup_{w\in G_2} X(w,w)(1-|z|^2)^{(n+1+q)/p}\left|\nabla f\left(\frac{m-1}{m}z\right)\right|
\end{aligned}$$

$$\begin{aligned}
&\le c_2 \sup_{\|f\|_F=1}\sup_{w\in G_2} X(w,w)\left(1-|z|^2\right)^{(n+1+q)/p}\left|\nabla f(z)-\nabla f\left(\frac{m-1}{m}z\right)\right| \\
&\quad+\frac{c_2}{m}\sup_{\|f\|_F=1}\sup_{w\in G_2} X(w,w)\left(1-\left|\frac{m-1}{m}z\right|^2\right)^{(n+1+q)/p}\left|\nabla f\left(\frac{m-1}{m}z\right)\right| \\
&\le c_2 \sup_{\|f\|_F=1}\sup_{w\in G_2} X(w,w)\sum_{l=1}^{n}\left|\frac{\partial f}{\partial z_l}(z)-\frac{\partial f}{\partial z_l}\left(\frac{m-1}{m}z\right)\right| \\
&\quad+\frac{c_2}{m}\sup_{\|f\|=1}\sup_{w\in G_2} X(w,w)\|f\|_{\mathcal{B}^{(n+1+q)/p}} \\
&= I_1+I_2.
\end{aligned} \tag{3.56}$$

By Lemma 2.4 we get $\sup_{w\in G_2} X(w,w)<\infty$, and noticing that $\|f\|_{\mathcal{B}^{(n+1+q)/p}}\le c$, so it is easy to get that $I_2\to 0$ when $m\to\infty$.

For I_1, first we have

$$\begin{aligned}
&\left|\frac{\partial f}{\partial z_l}(z)-\frac{\partial f}{\partial z_l}\left(\left(1-\frac{1}{m}\right)z\right)\right| \\
&\quad=\left|\frac{\partial f}{\partial z_l}(z)-\frac{\partial f}{\partial z_l}\left(\left(1-\frac{1}{m}\right)z_1,z_2,\ldots,z_n\right)+\frac{\partial f}{\partial z_l}\left(\left(1-\frac{1}{m}\right)z_1,z_2,\ldots,z_n\right)\right. \\
&\qquad\left.-\cdots+\frac{\partial f}{\partial z_l}\left(\left(1-\frac{1}{m}\right)z_1,\ldots,\left(1-\frac{1}{m}\right)z_{n-1},z_n\right)-\frac{\partial f}{\partial z_l}\left(\left(1-\frac{1}{m}\right)z\right)\right| \\
&\quad\le\sum_{j=2}^{n}\left|\frac{\partial f}{\partial z_l}\left(\left(1-\frac{1}{m}\right)z_1,\ldots,\left(1-\frac{1}{m}\right)z_{j-1},z_j,\ldots,z_n\right)\right. \\
&\qquad\left.-\frac{\partial f}{\partial z_l}\left(\left(1-\frac{1}{m}\right)z_1,\ldots,\left(1-\frac{1}{m}\right)z_j,z_{j+1},\ldots,z_n\right)\right| \\
&\qquad+\left|\frac{\partial f}{\partial z_l}(z_1,z_2,\ldots,z_n)-\frac{\partial f}{\partial z_l}\left(\left(1-\frac{1}{m}\right)z_1,z_2,\ldots,z_n\right)\right| \\
&\quad=\sum_{j=2}^{n}\left|\int_{(1-1/m)z_j}^{z_j}\frac{\partial^2 f}{\partial z_l\partial z_j}\left(\left(1-\frac{1}{m}\right)z_1,\ldots,\left(1-\frac{1}{m}\right)z_{j-1},\zeta,z_{j+1},\ldots,z_n\right)d\zeta\right| \\
&\qquad+\left|\int_{(1-1/m)z_1}^{z_1}\frac{\partial^2 f}{\partial z_l\partial z_1}(\zeta,z_2,\ldots,z_n)d\zeta\right| \\
&\quad\le\frac{1}{m}\sum_{j=1}^{n}\sup_{z\in G_2'}\left|\frac{\partial^2 f}{\partial z_l\partial z_j}(z)\right| \\
&\quad=\frac{1}{m}\sum_{j=1}^{n}\sup_{w\in G_2}\left|\frac{\partial^2 f}{\partial z_l\partial z_j}(z)\right|.
\end{aligned} \tag{3.57}$$

Denote $G_3 := \{z \in B_n : \operatorname{dist}(z, \partial B_n) > \delta/2\}$, then $G_2' \subset G_3 \subset\subset B_n$. Since $\operatorname{dist}(G_2', \partial G_3) = \delta/2$, by Lemma 2.8, when $z \in G_2'$ (i.e, $w \in G_2$) we get

$$\left|\frac{\partial f}{\partial z_l}(z) - \frac{\partial f}{\partial z_l}\left(\left(1 - \frac{1}{m}\right)z\right)\right| \leq \frac{2n\sqrt{n}}{m\delta} \sup_{z \in G_3}\left|\frac{\partial f}{\partial z_l}(z)\right|. \tag{3.58}$$

On the other hand, it follows from $\|f\|_{\mathcal{B}^{(n+1+q)/p}} \leq c$ that

$$\sup_{z \in G_3}(1 - |z|^2)^{(n+1+q)/p}|\nabla f(z)| \leq \|f\|_{\mathcal{B}^{(n+1+q)/p}} \leq c. \tag{3.59}$$

By (3.59) and the definition of G_3, we get

$$\sup_{z \in G_3}|\nabla f(z)| \leq c\left(1 - \left(\frac{\delta}{2}\right)^2\right)^{-(n+1+q)/p}. \tag{3.60}$$

Therefore

$$\sup_{z \in G_3}\left|\frac{\partial f}{\partial z_l}(z)\right| \leq \sup_{z \in G_3}|\nabla f(z)| \leq c\left(1 - \left(\frac{\delta}{2}\right)^2\right)^{-(n+1+q)/p}. \tag{3.61}$$

Combining (3.58) and (3.61), we have

$$\begin{aligned} I_1 &\leq c_2 c \sup_{\|f\|_F = 1} \sup_{w \in G_2} X(w, w) \cdot \left(1 - \left(\frac{\delta}{2}\right)^2\right)^{-(n+1+q)/p} \cdot \sum_{l=1}^{n} \frac{2n\sqrt{n}}{m\delta} \\ &= c_2 c \sup_{\|f\|_F = 1} \sup_{w \in G_2} X(w, w) \cdot \left(1 - \left(\frac{\delta}{2}\right)^2\right)^{-(n+1+q)/p} \cdot \frac{2n^2\sqrt{n}}{m\delta}. \end{aligned} \tag{3.62}$$

By Lemma 2.4, $\sup_{w \in G_2} X(w, w) < \infty$, and so $\lim_{m \to \infty} I_1 = 0$.

Now, let $m \to \infty$ and $\delta \to 0$; we get the upper estimate:

$$\|C_\phi\|_e \leq c_2 \lim_{\delta \to 0} \sup_{\operatorname{dist}(\phi(w), \partial B_n) < \delta} X(w, w). \tag{3.63}$$

So, the proof of Theorem 1.1 is finished.

4. Two Corollaries

Lemma 2.2 tells us that $F(p,q,s) \subset \mathcal{B}^{(n+1+q)/p}$, as in the similar discussion of Theorem 1.1; so we can get an estimate of the essential norm of a composition operator between Bloch-type spaces. That is the following corollary.

Corollary 4.1. *Let $\alpha, \beta > 0$, let $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ be a holomorphic self-map of B_n, and let $\|C_\phi\|_e$ be the essential norm of a bounded composition operator $C_\phi : \mathcal{B}^\beta \to \mathcal{B}^\alpha$. Then there are $c_1, c_2 > 0$, independent of w, such that*

$$c_1 \lim_{\delta \to 0} \sup_{\operatorname{dist}(\phi(w), \partial B_n) < \delta} X(w,w) \leq \|C_\phi\|_e \leq c_2 \lim_{\delta \to 0} \sup_{\operatorname{dist}(\phi(w), \partial B_n) < \delta} X(w,w). \tag{4.1}$$

Remark 4.2. In Corollary 4.1, the quantity $X(w,w)$ is similar to Theorem 1.1, but we need to substitute $(n+1+q)/p$ with β.

It is well known that $\|T\|_e = 0$ if and only if T is compact; so the estimate on $\|C_\phi\|_e$ leads to conditions for C_ϕ to be compact. From Theorem 1.1 we get the following corollary.

Corollary 4.3. *Let $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ be a holomorphic self-map of B_n. Then a bounded composition operator $C_\phi : F(p,q,s) \to \mathcal{B}^\alpha$ is compact if and only if*

$$\lim_{\delta \to 0} \sup_{\operatorname{dist}(\phi(w), \partial B_n) < \delta} X(w,w) = 0, \tag{4.2}$$

where $X(w,w)$ is the same as Theorem 1.1.

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References

[1] W. Yang and C. Ouyang, "Exact location of α-Bloch spaces in L^p_α and H^p of a complex unit ball," *The Rocky Mountain Journal of Mathematics*, vol. 30, no. 3, pp. 1151–1169, 2000.

[2] R. M. Timoney, "Bloch functions in several complex variables. I," *The Bulletin of the London Mathematical Society*, vol. 12, no. 4, pp. 241–267, 1980.

[3] M. Z. Zhang and W. Xu, "Composition operators on α-Bloch spaces of the unit ball," *Acta Mathematica Sinica (English Series)*, vol. 23, no. 11, pp. 1991–2002, 2007.

[4] K. Madigan and A. Matheson, "Compact composition operators on the Bloch space," *Transactions of the American Mathematical Society*, vol. 347, no. 7, pp. 2679–2687, 1995.

[5] J. H. Shi and L. Luo, "Composition operators on the Bloch space of several complex variables," *Acta Mathematica Sinica. English Series*, vol. 16, no. 1, pp. 85–98, 2000.

[6] Z. H. Zhou and J. Shi, "Composition operators on the Bloch space in polydiscs," *Complex Variables: Theory and Application*, vol. 46, no. 1, pp. 73–88, 2001.

[7] Z. H. Zhou and J. H. Shi, "Compactness of composition operators on the Bloch space in classical bounded symmetric domains," *The Michigan Mathematical Journal*, vol. 50, no. 2, pp. 381–405, 2002.

[8] P. Gorkin and B. D. MacCluer, "Essential norms of composition operators," *Integral Equations and Operator Theory*, vol. 48, no. 1, pp. 27–40, 2004.
[9] A. Montes-Rodríguez, "The essential norm of a composition operator on Bloch spaces," *Pacific Journal of Mathematics*, vol. 188, no. 2, pp. 339–351, 1999.
[10] Z. H. Zhou and J. H. Shi, "The essential norm of a composition operator on the Bloch space in polydiscs," *Chinese Annals of Mathematics A*, vol. 24, no. 2, pp. 199–208, 2003.
[11] Z. H. Zhou and J. H. Shi, "The essential norm of a composition operator on the Bloch space in polydiscs," *Chinese Journal of Contemporary Mathematics*, vol. 24, no. 2, pp. 175–186, 2003.
[12] Z. H. Zhou and R.-Y. Chen, "Weighted composition operators from $F(p, q, s)$ to Bloch type spaces on the unit ball," *International Journal of Mathematics*, vol. 19, no. 8, pp. 899–926, 2008.