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Research Article

Nonlinear Retarded Integral Inequalities with Two Variables and Applications

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We consider some new nonlinear retarded integral inequalities with two variables, which extend the results in the work of W.-S. Wang (2007), and the one in the work of Y.-H. kim (2009). These inequalities include not only a nonconstant term outside the integrals but also more than one distinct nonlinear integrals without assumption of monotonicity. Finally, we give some applications to the boundary value problem of a partial differential equation for boundedness and uniqueness.

1. Introduction

Integral inequalities that give explicit bounds on unknown functions provide a very useful and important device in the study of many qualitative as well as quantitative properties of solutions of partial differential equations, integral equations, and integrodifferential equation. One of the best known and widely used inequalities in the study of nonlinear differential equations is Gronwall inequality [1], which states that if u and f are nonnegative continuous functions on the interval $[a,b]$ satisfying

$$u(t) \le c + \int_{\alpha}^{t} f(s)u(s)ds, \quad t \in [a,b], \tag{1.1}$$

where c is a nonnegative constant, then we have

$$u(t) \le c\exp\left(\int_{\alpha}^{t} f(s)ds\right), \quad t \in [a,b]. \tag{1.2}$$

Since the inequality (1.2) provides an explicit bound of the unknown function u it furnishes a handy tool in the study of various properties of solutions of differential equations. Because of its fundamental importance, several generalizations and analogous results of Gronwall inequality [1, 2] and its applications have attracted great interests of many mathematicians (e.g., [3–5]). Some recent works can be found, for example, in [6–17] and some references therein. In 2005, Agarwal et al. [6] investigated the inequality

$$u(t) \leq a(t) + \sum_{i=1}^{n} \int_{b_i(t_0)}^{b_i(t)} g_i(t,s) w_i(u(s)) ds, \quad t_0 \leq t < t_1. \tag{1.3}$$

In 2006, Cheung [9] studied the inequality

$$\begin{aligned} u^p(x,y) \leq a &+ \frac{p}{p-q} \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} g_1(s,t) u^q(s,t) dt\, ds \\ &+ \frac{p}{p-q} \int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} g_2(s,t) u^q(s,t) \varphi(u(s,t)) dt\, ds, \end{aligned} \tag{1.4}$$

for all $(x,y) \in [x_0, X) \times [y_0, Y)$, where a is a constant.

In 2007, Wang [16] discussed the retarded integral inequality

$$u^p(x,y) \leq a(x,y) + \sum_{i=1}^{k} \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} f_i(x,y,s,t) \varphi_i(u(s,t)) ds\, dt, \tag{1.5}$$

for all $(x,y) \in [x_0, x_1) \times [y_0, y_1)$.

In 2008, Agarwal et al. [7] discussed the retarded integral inequality

$$\varphi(u(t)) \leq c + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} u^q(s) \left[f_i(s) \varphi(u(s)) + g_i(s) \right] ds, \tag{1.6}$$

for all $t \in [t_0, T)$, where c is a constant.

In 2009, Kim [12] obtained the explicit bound of the unknown function of the following inequality:

$$\begin{aligned} \varphi(u(x,y)) \leq a(x,y) &+ c(x,y) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} u^q(s,t) g_i(s,t)\; ds\, dt \\ &+ c(x,y) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} u^q(s,t) f_i(s,t) \varphi(u(s,t)) ds\, dt, \end{aligned} \tag{1.7}$$

for all $(x,y) \in [x_0, X) \times [y_0, Y)$.

The purpose of the present paper is to establish some new nonlinear retarded integral inequalities of Gronwall-Bellman type with two variables. We can demonstrate that inequalities (1.4), (1.5), and (1.7), considered in [9, 12, 16], respectively, can also be solved with our results. We also apply our results to study the boundedness and uniqueness of the solutions of the boundary value problem of a partial differential equation.

2. Main Result

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers, and $x_0, y_0, x_1, y_1 \in \mathbb{R}$ are given numbers. $\mathbb{R}_+ := [0, \infty)$, $I := [x_0, x_1)$, $J := [y_0, y_1)$ are the subsets of $\mathbb{R}$ and $\Lambda := I \times J \subset \mathbb{R}^2$. For any $(s, t) \in \Lambda$, let $\Lambda_{(s,\, t)}$ denote the subset $[x_0, s) \times [y_0, t) \cap \Lambda$ of Λ. $C^1(U, V)$ denotes the set of continuous differentiable functions of U into V.

Consider the following inequality:

$$\begin{aligned} &\psi(u(x,y)) \\ &\quad\le a(x,y) + \sum_{i=1}^{n}\left\{\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} u^q(s,t) g_i(x,y,s,t)\,ds\,dt \right. \\ &\qquad\qquad \left. + \int_{\delta_i(x_0)}^{\delta_i(x)}\int_{\gamma_i(y_0)}^{\gamma_i(y)} u^q(s,t) f_i(x,y,s,t)\varphi_i(u(s,t))\,ds\,dt\right\}, \quad \forall (x,y) \in \Lambda. \end{aligned} \tag{2.1}$$

Our inequality (2.1) not only includes a nonconstant term outside the integrals but also more than one distinct nonlinear integral without assumption of monotonicity. When $f_i(x, y, s, t) = c(x, y) f_i(s, t)$, $g_i(x, y, s, t) = c(x, y) g_i(s, t)$, $\alpha_i(x) = \delta_i(x)$, $\beta_i(y) = \gamma_i(y)$, and $\varphi_i(u) = \varphi(u)$, our inequality (2.1) reduces to (1.7) studied in [12]. When $u^q(s, t) = 1$, $g_i(x, y, s, t) = 0$, and $\psi(u) = u^q$, our inequality (2.1) reduces to (1.5) studied in [16].

Suppose that

(H_1) ψ is a strictly increasing continuous function on $\mathbb{R}_+$, $\psi(0) = 0$;

(H_2) all φ_i, $(i = 1, 2, \ldots, n)$ are continuous functions on $\mathbb{R}_+$ and positive on $(0, \infty)$;

(H_3) $a(x, y) \ge 0$ on Λ, and a is nondecreasing in each variable;

(H_4) $\alpha_i, \delta_i \in C^1(I, I)$ and $\beta_i, \gamma_i \in C^1(J, J)(i = 1, 2, \ldots, n)$ are nondecreasing such that $\alpha_i(x) \le x$ and $\delta_i(x) \le x$ on I, $\beta_i(y) \le y$ and $\gamma_i(y) \le y$ on J;

(H_5) $q > 0$ is a constant;

(H_6) all f_i, g_i $(i = 1, 2, \ldots, n)$ are nonnegative functions on $\Lambda \times \Lambda$.

Firstly, we technically consider a sequence of functions $w_i(s)$, which can be calculated recursively by

$$\begin{aligned} w_1(u) &:= \max_{\tau\in[0,\, u]} \varphi_1(\tau), \quad u > 0, \\ w_{i+1}(u) &:= \max_{\tau\in[0,\, u]}\left\{\frac{\varphi_{i+1}(\tau)}{w_i(\tau)}\right\} w_i(u), \quad u > 0,\ i = 1, 2, \ldots, n-1. \end{aligned} \tag{2.2}$$

Moreover, we define the following functions:

$$\Psi(u) := \int_0^u \frac{ds}{\left(\varphi^{-1}(s)\right)^q}, \quad u \geq 0, \tag{2.3}$$

$$W_i(u) := \int_0^u \frac{ds}{w_i\left(\varphi^{-1}\left(\Psi^{-1}(s)\right)\right)}, \quad u \geq 0, \ i = 1, \ 2, \ldots, n. \tag{2.4}$$

Obviously, both Ψ and W_i are strictly increasing and continuous functions. Letting Ψ^{-1}, W_i^{-1} denote Ψ, W_i inverse function, respectively, then both Ψ^{-1} and W_i^{-1} are also continuous and increasing functions.

Let

$$\tilde{f}_i(x, y, s, t) := \max_{(\tau, \xi) \in [x_0, x] \times [y_0, y]} f_i(\tau, \xi, s, t), \tag{2.5}$$

$$\tilde{g}_i(x, y, s, t) := \max_{(\tau, \xi) \in [x_0, x] \times [y_0, y]} g_i(\tau, \xi, s, t), \ i = 1, \ 2, \ \ldots, n. \tag{2.6}$$

Then $\tilde{f}_i(x, y, s, t)$ and $\tilde{g}_i(x, y, s, t)$ are nonnegative and nondecreasing in x, y for each fixed (s, t) and satisfy $\tilde{f}_i(x, y, s, t) \geq f_i(x, y, s, t)$, $\tilde{g}_i(x, y, s, t) \geq g_i(x, y, s, t)$, $i = 1, 2, \ldots, n$.

Theorem 2.1. *Suppose that* $(H_1$–$H_6)$ *hold and* $u(x, y)$ *is a nonnegative function on* Λ *satisfying* (2.1). *Then*

$$u(x, y) \leq \varphi^{-1}\left\{\Psi^{-1}\left[W_n^{-1}\left(W_n(\Xi_n(x, y)) + \int_{\delta_n(x_0)}^{\delta_n(x)} \int_{\gamma_n(y_0)}^{\gamma_n(y)} \tilde{f}_n(x, y, s, t) ds\, dt\right)\right]\right\}, \tag{2.7}$$

for $(x, y) \in \Lambda_{(X_1, Y_1)}$, *where*

$$\Xi_1(x, y) := \Psi(a(x, y)) + \sum_{i=1}^n \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} \tilde{g}_i(x, y, s, t) ds\, dt,$$

$$\Xi_i(x, y) := W_{i-1}^{-1}\left(W_{i-1}(\Xi_{i-1}(x, y)) + \int_{\delta_{i-1}(x_0)}^{\delta_{i-1}(x)} \int_{\gamma_{i-1}(y_0)}^{\gamma_{i-1}(y)} \tilde{f}_{i-1}(x, y, s, t) ds dt\right), \quad i = 2, 3, \ldots, n. \tag{2.8}$$

$(X_1, Y_1) \in \Lambda$ *is arbitrarily given on the boundary of the planar region*

$$\mathcal{R}_1 := \Bigg\{ (x,y) \in \Lambda : W_i(\Xi_i(x,y)) + \int_{\delta_i(x_0)}^{\delta_i(x)} \int_{\gamma_i(y_0)}^{\gamma_i(y)} \widetilde{f}_i(x,y,s,t)\,ds\,dt \le \int_0^\infty \frac{ds}{w_i(\varphi^{-1}(\Psi^{-1}(s)))}, \quad i = 1,2,\ldots,n, \\ W_n^{-1}\left(W_n(\Xi_n(x,y)) + \int_{\delta_n(x_0)}^{\delta_n(x)} \int_{\gamma_n(y_0)}^{\gamma_n(y)} \widetilde{f}_n(x,y,s,t)\,ds\,dt \right) \le \int_0^\infty \frac{ds}{(\varphi^{-1}(s))^q} \Bigg\}. \tag{2.9}$$

Corollary 2.2. *Let $u, a, f_i, g_i, \alpha_i, \beta_i, \delta_i, \gamma_i$ and $\varphi_i(u)$ $i = 1,2,\ldots,n$ be as defined in Theorem 2.1. Suppose that $p > q > 0$ are constants. If*

$$u^p(x,y) \le a(x,y) + \sum_{i=1}^n \Bigg\{ \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} u^q(s,t) g_i(x,y,s,t)\,ds\,dt \\ + \int_{\delta_i(x_0)}^{\delta_i(x)} \int_{\gamma_i(y_0)}^{\gamma_i(y)} u^q(s,t) f_i(x,y,s,t) \varphi_i(u(s,t))\,ds\,dt \Bigg\}, \tag{2.10}$$

for all $(x,y) \in \Lambda$, then

$$u(x,y) \le \left\{ G_n^{-1}\left[G_n(B_n(x,y)) + \frac{p-q}{p} \int_{\delta_n(x_0)}^{\delta_n(x)} \int_{\gamma_n(y_0)}^{\gamma_n(y)} \widetilde{f}_n(X,Y,s,t) \right] \right\}^{1/(p-q)}, \tag{2.11}$$

for all $(x,y) \in \Lambda_{(X_2,Y_2)}$, where $\widetilde{f}_i$ and $\widetilde{g}_i$ are defined by (2.5) *and* (2.6), *and*

$$B_1(x,y) := a(x,y)^{(p-q)/p} + \frac{p-q}{p} \sum_{i=1}^n \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i y_0}^{\beta_i(y)} \widetilde{g}_i(x,y,s,t)\,ds\,dt, \\ B_i(x,y) := G_{i-1}^{-1}\left(G_{i-1}(B_{i-1}(x,y)) + \frac{p-q}{p} \int_{\delta_{i-1}(x_0)}^{\delta_{i-1}(x)} \int_{\gamma_{i-1}(y_0)}^{\gamma_{i-1}(y)} \widetilde{f}_{i-1}(x,y,s,t)\,ds\,dt \right), \\ i = 2,3,\ldots,n, \\ G_i(x,y) := \int_0^u \frac{ds}{w_i(s^{((1/p)-q)})}, \quad u \ge 0,\ i = 1,2,\ldots,n. \tag{2.12}$$

G_i^{-1} denotes the inverse function of G_i, and $(X_2, Y_2) \in \Lambda$ lies on the boundary of the planar region

$$\mathcal{R}_2 := \left\{ (x,y)\Lambda : G_i(B_i(x,y)) + \frac{p-q}{p}\int_{\delta_i(x_0)}^{\delta_i(x)}\int_{\gamma_i(y_0)}^{\gamma_i(y)} \widetilde{f}_i(x,y,s,t)\,ds\,dt \right.$$
$$\left. \leq \int_0^\infty \frac{ds}{w_i\left(s^{(1/p)-q}\right)},\ i = 1,2,\ldots,n \right\}. \tag{2.13}$$

Corollary 2.3. *Let $u, a, q, f_i, g_i, \delta_i, \gamma_i$ and $\varphi_i(u) i = 1,2,\ldots,n$ be as defined in Theorem 2.1. Supposing that*

$$\varphi(u(x,y)) \leq a(x,y) + \sum_{i=1}^{n}\int_{\delta_i(x_0)}^{\delta_i(x)}\int_{\gamma_i(y_0)}^{\gamma_i(y)} u^q(s,t) f_i(x,y,s,t)\varphi_i(u(s,t))\,ds\,dt, \tag{2.14}$$

for all $(x,y) \in \Lambda$, then

$$u(x,y) \leq \varphi^{-1}\left\{\Psi^{-1}\left[W_n^{-1}\left(W_n(K_n(x,y)) + \int_{\delta_n(x_0)}^{\delta_n(x)}\int_{\gamma_n(y_0)}^{\gamma_n(y)} \widetilde{f}_n(x,y,s,t)\,ds\,dt\right)\right]\right\}, \tag{2.15}$$

for all $(x,y) \in \Lambda_{(X_3,Y_3)}$, where

$$K_1(x,y) := \Psi(a(x,y)),$$
$$K_i(x,y) := W_{i-1}^{-1}\left(W_{i-1}(K_{i-1}(x,y)) + \int_{\delta_{i-1}(x_0)}^{\delta_{i-1}(x)}\int_{\gamma_{i-1}(y_0)}^{\gamma_{i-1}(y)} \widetilde{f}_{i-1}(x,y,s,t)\,ds\,dt\right), \tag{2.16}$$
$$i = 2,3,\ldots,n.$$

$\widetilde{f}_i$ is defined by (2.5), Ψ, W_i are as defined in (2.3) and (2.4), respectively, and $\Psi^{-1} W_i^{-1}$ denote the inverse functions of Ψ and W_i. $(X_3, Y_3) \in \Lambda$ lies on the boundary of the planar region

$$\mathcal{R}_3 := \left\{ (x,y) \in \Lambda : W_i(K_i(x,y)) + \int_{\delta_i(x_0)}^{\delta_i(x)}\int_{\gamma_i(y_0)}^{\gamma_i(y)} \widetilde{f}_i(x,y,s,t)\,ds\,dt \right.$$
$$\leq \int_0^\infty \frac{ds}{w_i(\varphi^{-1}(\Psi^{-1}(s)))},\ i = 1,2,\ldots,n,$$
$$W_n^{-1}\left(W_n(E_n(x,y)) + \int_{\delta_n(x_0)}^{\delta_n(x)}\int_{\gamma_n(y_0)}^{\gamma_n(y)} \widetilde{f}_n(x,y,s,t)\,ds\,dt\right)$$
$$\left. \leq \int_0^\infty \frac{ds}{(\varphi^{-1}(s))^q} \right\}. \tag{2.17}$$

Theorem 2.4. *Suppose that* $(H_1$–$H_6)$ *hold and* $u(x,y)$*is a nonnegative function on* Λ *satisfying*

$$\varphi(u(x,y)) \le a(x,y) + \sum_{i=1}^{n}\left\{\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} u^q(s,t)g_i(x,y,s,t)\,ds\,dt + \int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} u^q(s,t)f_i(x,y,s,t)\varphi_i(u(s,t))\,ds\,dt\right\}, \quad \forall (x,y)\in\Lambda. \tag{2.18}$$

Then

$$u(x,y) \le \varphi^{-1}\left\{\Psi^{-1}\left[W_n^{-1}\left(W_n(H_n(x,y)) + \int_{\alpha_n(x_0)}^{\alpha_n(x)}\int_{\beta_n(y_0)}^{\beta_n(y)} \tilde{f}_n(x,y,s,t)\,ds\,dt\right)\right]\right\}, \tag{2.19}$$

for $(x,y)\in\Lambda_{(X_4,\,Y_4)}$*, where*

$$\begin{aligned} H_1(x,y) &:= \Psi(a(x,y)) + \sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} \tilde{g}_i(x,y,s,t)\,ds\,dt, \\ H_i(x,y) &:= W_{i-1}^{-1}\left(W_{i-1}(H_{i-1}(x,y)) + \int_{\alpha_{i-1}(x_0)}^{\alpha_{i-1}(x)}\int_{\beta_{i-1}(y_0)}^{\beta_{i-1}(y)} \tilde{f}_{i-1}(x,y,s,t)\,ds\,dt\right), \end{aligned} \tag{2.20}$$

for $i = 2,3,\ldots,n$*, and* $(X_4,Y_4)\in\Lambda$ *is arbitrarily given on the boundary of the planar region*

$$\begin{aligned} \mathcal{R}_4 := \Bigg\{ &(x,y)\in\Lambda : W_i(H_i(x,y)) + \int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} \tilde{f}_i(x,y,s,t)\,ds\,dt \\ &\le \int_0^{\infty}\frac{ds}{w_i(\varphi^{-1}(\Psi^{-1}(s)))},\ i = 1,2,\ldots,n \\ &W_n^{-1}\left(W_n(H_n(x,y)) + \int_{\alpha_n(x_0)}^{\alpha_n(x)}\int_{\beta_n(y_0)}^{\beta_n(y)} \tilde{f}_n(x,y,s,t)\,ds\,dt\right) \\ &\le \int_0^{\infty}\frac{ds}{(\varphi^{-1}(s))^q}\Bigg\}. \end{aligned} \tag{2.21}$$

Corollary 2.5. *Suppose that* $(H_1$–$H_5)$ *hold and* $u(x,y)$*is a nonnegative function on* Λ *satisfying*

$$\psi(u(x,y)) \le a(x,y) + \sum_{i=1}^{n}\left\{\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} u^q(s,t)g_i(s,t)ds\,dt + \int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} u^q(s,t)f_i(s,t)\varphi_i(u(s,t))ds\,dt\right\}, \quad \forall(x,y)\in\Lambda, \tag{2.22}$$

where f_i, g_i $(i = 2,3,\dots,n)$ *are nonnegative functions on* Λ*. Then*

$$u(x,y) \le \psi^{-1}\left\{\Psi^{-1}\left[W_n^{-1}\left(W_n(L_n(x,y)) + \int_{\alpha_n(x_0)}^{\alpha_n(x)}\int_{\beta_n(y_0)}^{\beta_n(y)} f_n(s,t)ds\,dt\right)\right]\right\}, \tag{2.23}$$

for $(x,y)\in\Lambda_{(X_5,\,Y_5)}$*, where*

$$\begin{aligned} L_1(x,y) &:= \Psi(a(x,y)) + \sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} g_i(s,t)ds\,dt, \\ L_i(x,y) &:= W_{i-1}^{-1}\left(W_{i-1}(L_{i-1}(x,y)) + \int_{\alpha_{i-1}(x_0)}^{\alpha_{i-1}(x)}\int_{\beta_{i-1}(y_0)}^{\beta_{i-1}(y)} f_{i-1}(s,t)ds\,dt\right), \end{aligned} \tag{2.24}$$

for $i = 2,\ 3,\dots,n$*, and* $(X_5, Y_5)\in\Lambda$ *is arbitrarily given on the boundary of the planar region*

$$\begin{aligned} \mathcal{R}_5 := \Bigg\{ &(x,y)\in\Lambda : W_i(L_i(x,y)) + \int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)ds\,dt \\ &\le \int_0^{\infty}\frac{ds}{w_i(\psi^{-1}(\Psi^{-1}(s)))},\ i = 1,2,\dots,n \\ &W_n^{-1}\left(W_n(L_n(x,y)) + \int_{\alpha_n(x_0)}^{\alpha_n(x)}\int_{\beta_n(y_0)}^{\beta_n(y)} f_n(s,t)ds\,dt\right) \\ &\le \int_0^{\infty}\frac{ds}{(\psi^{-1}(s))^q}\Bigg\}. \end{aligned} \tag{2.25}$$

3. Proofs and Remarks

Proof of Theorem 2.1. Obviously, the sequence $w_i(s)$ defined by $\varphi_i(s)$ in (2.2) is nondecreasing nonnegative functions and satisfies $w_i(s) \geq \varphi_i(s)$, $i = 1, 2, \ldots, n$. Moreover, the ratios $w_{i+1}(s)/w_i(s)$, $i = 1, 2, \ldots, n-1$ are all nondecreasing. From (2.1), (2.2) and (2.5), (2.6), we have

$$\psi(u(x,y)) \leq a(x,y) + \sum_{i=1}^{n} \left\{ \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} u^q(s,t)\tilde{g}_i(x,y,s,t)ds\,dt \right.$$
$$\left. + \int_{\delta_i(x_0)}^{\delta_i(x)} \int_{\gamma_i(y_0)}^{\gamma_i(y)} u^q(s,t)\tilde{f}_i(x,y,s,t)w_i(u(s,t))ds\,dt \right\}, \quad (3.1)$$
$$\forall (x,y) \in \Lambda.$$

We first discuss the case that $a(x,y) > 0$ for all $(x,y) \in \Lambda$. Consider the auxiliary inequality

$$\psi(u(x,y)) \leq a(X,Y) + \sum_{i=1}^{n} \left\{ \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} u^q(s,t)\tilde{g}_i(X,Y,s,t)ds\,dt \right.$$
$$\left. + \int_{\delta_i(x_0)}^{\delta_i(x)} \int_{\gamma_i(y_0)}^{\gamma_i(y)} u^q(s,t)\tilde{f}_i(X,Y,s,t)w_i(u(s,t))ds\,dt \right\}, \quad (3.2)$$

for all $(x,y) \in \Lambda_{(X,\,Y)}$, where $x_0 \leq X \leq X_1$ and $y_0 \leq Y \leq Y_1$ are chosen arbitrarily. Let $z_1(x,y)$ denote the function on the right-hand side of (3.2), which is a nonnegative and nondecreasing function on $\Lambda_{(X,\,Y)}$ and $z_1(x_0,y) = a(X,Y)$. Then, we get the equivalent form of (3.2)

$$u(x,y) \leq \psi^{-1}(z_1(x,y)), \quad \forall (x,y) \in \Lambda_{(X,\,Y)}. \quad (3.3)$$

Since w_i is nondecreasing and satisfies $w_i(u) > 0$ for $u > 0$. By the definition of z_1, hypothesis (H_4), the monotonicity of ψ^{-1} and z_1, and (3.3), we have

$$\frac{\partial z_1(x,y)}{\partial x} = \sum_{i=1}^{n} \left\{ \alpha_i'(x) \int_{\beta_i(y_0)}^{\beta_i(y)} u^q(\alpha_i(x),t)\tilde{g}_i(X,Y,\alpha_i(x),t)dt \right.$$
$$\left. + \delta_i'(x) \int_{\gamma_i(y_0)}^{\gamma_i(y)} u^q(\delta_i(x),t)\tilde{f}_i(X,Y,\delta_i(x),t)w_i(u(\delta_i(x),t))dt \right\}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{n}\Bigg\{\alpha_i'(x)\int_{\beta_i(y_0)}^{\beta_i(y)}\left(\varphi^{-1}(z_1(\alpha_i(x),t))\right)^q\widetilde{g}_i(X,Y,\alpha_i(x),t)dt \\
&\qquad + \delta_i'(x)\int_{\gamma_i(y_0)}^{\gamma_i(y)}\left(\varphi^{-1}(z_1(\delta_i(x),t))\right)^q\widetilde{f}_i(X,Y,\delta_i(x),t) \\
&\qquad\qquad \times w_i\left(\varphi^{-1}(z_1(\delta_i(x),t))\right)dt\Bigg\} \\
&\leq \left(\varphi^{-1}(z_1(x,y))\right)^q\sum_{i=1}^{n}\Bigg\{\alpha_i'(x)\int_{\beta_i(y_0)}^{\beta_i(y)}\widetilde{g}_i(X,Y,\alpha_i(x),t)dt \\
&\qquad + \delta_i'(x)\int_{\gamma_i(y_0)}^{\gamma_i(y)}\widetilde{f}_i(X,Y,\delta_i(x),t)w_i\left(\varphi^{-1}(z_1(\delta_i(x),t))\right)dt\Bigg\}.
\end{aligned}
\tag{3.4}
$$

From (3.4), we have

$$
\begin{aligned}
\frac{\partial z_1(x,y)/\partial x}{\left(\varphi^{-1}(z_1(x,y))\right)^q} &\leq \sum_{i=1}^{n}\Bigg\{\alpha_i'(x)\int_{\beta_i(y_0)}^{\beta_i(y)}\widetilde{g}_i(X,Y,\alpha_i(x),t)dt \\
&\qquad + \delta_i'(x)\int_{\gamma_i(y_0)}^{\gamma_i(y)}\widetilde{f}_i(X,Y,\delta_i(x),t) \\
&\qquad\qquad \times w_i\left(\varphi^{-1}(z_1(\delta_i(x),t))\right)dt\Bigg\}.
\end{aligned}
\tag{3.5}
$$

Keeping y fixed in (3.5), setting $s = x$, integrating both sides of (3.5) with respect to s from x_0 to x, and using the definition of Ψ in (2.3), we have

$$
\begin{aligned}
&\Psi(z_1(x,y)) \\
&\quad\leq \Psi(z_1(x_0,y)) + \sum_{i=1}^{n}\Bigg\{\int_{x_0}^{x}\left(\alpha_i{}'(s)\int_{\beta_i(y_0)}^{\beta_i(y)}\widetilde{g}_i(X,Y,\alpha_i(s),t)dt\right)ds \\
&\qquad + \int_{x_0}^{x}\left(\delta_i{}'(s)\int_{\gamma_i(y_0)}^{\gamma_i(y)}\widetilde{f}_i(X,Y,\delta_i(s),t)w_i\left(\varphi^{-1}(z_1(\delta_i(s),t))\right)dt\right)ds\Bigg\} \\
&\quad\leq \Psi(a(X,Y)) + \sum_{i=1}^{n}\Bigg\{\int_{\alpha_i(x_0)}^{\alpha_i(X)}\int_{\beta_i(y_0)}^{\beta_i(Y)}\widetilde{g}_i(X,Y,s,t)ds\,dt \\
&\qquad + \int_{\delta_i(x_0)}^{\delta_i(x)}\int_{\gamma_i(y_0)}^{\gamma_i(y)}\widetilde{f}_i(X,Y,s,t)w_i\left(\varphi^{-1}(z_1(s,t))\right)ds\,dt\Bigg\} \\
&\quad\leq A_n(X,Y) + \sum_{i=1}^{n}\int_{\delta_i(x_0)}^{\delta_i(x)}\int_{\gamma_i(y_0)}^{\gamma_i(y)}\widetilde{f}_i(X,Y,s,t)w_i\left(\varphi^{-1}(z_1(s,t))\right)ds\,dt,
\end{aligned}
\tag{3.6}
$$

for all $(x, y) \in \Lambda_{(X, Y)}$, where

$$A_n(X, Y) = \Psi(a(X, Y)) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(X)} \int_{\beta_i(y_0)}^{\beta_i(Y)} \widetilde{g}_i(X, Y, s, t) ds\, dt. \tag{3.7}$$

Let

$$v(x, y) = \Psi(z_1(x, y)). \tag{3.8}$$

From (3.6), we have

$$v(x, y) \leq A_n(X, Y) + \sum_{i=1}^{n} \int_{\delta_i(x_0)}^{\delta_i(x)} \int_{\gamma_i(y_0)}^{\gamma_i(y)} \widetilde{f}_i(X, Y, s, t) w_i\left(\varphi^{-1}\left(\Psi^{-1}(v(s, t))\right)\right) ds\, dt, \tag{3.9}$$

for all $(x, y) \in \Lambda_{(X, Y)}$. We claim that the unknown function v in (3.9) satisfies

$$v(x, y) \leq W_n^{-1}\left(W_n(E_n(X, Y)) + \int_{\delta_n(x_0)}^{\delta_n(X)} \int_{\gamma_n(y_0)}^{\gamma_n(Y)} \widetilde{f}_n(X, Y, s, t) ds\, dt \right), \tag{3.10}$$

for all $(x, y) \in \Lambda_{(X, Y)}$, where

$$E_1(X, Y) := A_n(X, Y), \tag{3.11}$$

$$E_i(X, Y) := W_{i-1}^{-1}\left(W_{i-1}(E_{i-1}(X, Y)) + \int_{\delta_{i-1}(x_0)}^{\delta_{i-1}(X)} \int_{\gamma_{i-1}(y_0)}^{\gamma_{i-1}(Y)} \widetilde{f}_{i-1}(X, Y, s, t) ds dt \right), \; i = 2,\ 3, \ldots, n \tag{3.12}$$

Now, we prove (3.10) by induction. For $n = 1$, let $z_2(x, y)$ denote the function on the right-hand side of (3.9), which is a nonnegative and nondecreasing function on $\Lambda_{(X, Y)}$, $z_2(x_0, y) = A_1(X, Y)$ and $v(x, y) \leq z_2(x, y)$. Then we have

$$\begin{aligned} \frac{\partial z_2(x, y)}{\partial x} &= \delta_1'(x) \int_{\gamma_1(y_0)}^{\gamma_1(y)} \widetilde{f}_1(X, Y, \delta_1(x), t) w_1\left(\varphi^{-1}\left(\Psi^{-1}(v(\delta_1(x), t))\right)\right) dt \\ &\leq w_1\left(\varphi^{-1}\left(\Psi^{-1}(z_2(x, y))\right)\right) \left[\delta_1'(x) \int_{\gamma_1(y_0)}^{\gamma_1(y)} \widetilde{f}_1(X, Y, \delta_1(x), t) dt \right], \end{aligned} \tag{3.13}$$

for all $(x, y) \in \Lambda_{(X, Y)}$. From (3.13), we have

$$\frac{\partial z_2(x, y)/\partial x}{w_1(\varphi^{-1}(\Psi^{-1}(z_2(x, y))))} \leq \delta_1'(x) \int_{\gamma_1(y_0)}^{\gamma_1(y)} \widetilde{f}_1(X, Y, \delta_1(x), t) dt, \quad \forall (x, y) \in \Lambda_{(X, Y)}. \tag{3.14}$$

Keeping y fixed in (3.14), setting $s = x$, integrating both sides of (3.14) with respect to s from x_0 to x, and using the definition of W_i in (2.4), we have

$$\begin{aligned}
W_1(z_2(x,y)) &\le W_1(z_2(x_0,y)) + \int_{x_0}^{x}\left(\delta_1'(s)\int_{\gamma_1(y_0)}^{\gamma_1(y)} \tilde{f}_1(X,Y,\delta_1(s),t)dt\right)ds \\
&= W_1(A_1(X,Y)) + \int_{\delta_1(x_0)}^{\delta_1(x)}\int_{\gamma_1(y_0)}^{\gamma_1(y)} \tilde{f}_1(X,Y,s,t)ds\,dt \\
&\le W_1(A_1(X,Y)) + \int_{\delta_1(x_0)}^{\delta_1(X)}\int_{\gamma_1(y_0)}^{\gamma_1(Y)} \tilde{f}_1(X,Y,s,t)ds\,dt,
\end{aligned} \tag{3.15}$$

for all $(x,y) \in \Lambda_{(X,\,Y)}$. Using $v(x,y) \le z_2(x,y)$, from (3.15), we obtain

$$v(x,y) \le z_2(x,y) \le W_1^{-1}\left(W_1(A_1(X,Y)) + \int_{\delta_1(x_0)}^{\delta_1(X)}\int_{\gamma_1(y_0)}^{\gamma_1(Y)} \tilde{f}_1(X,Y,s,t)ds\,dt\right), \tag{3.16}$$

for all $(x,y) \in \Lambda_{(X,\,Y)}$. This proves that (3.10) is true for $n = 1$.

Next, we make the inductive assumption that (3.10) is true for $n = k$. Now, we consider

$$v(x,y) \le A_{k+1}(X,Y) + \sum_{i=1}^{k+1}\int_{\delta_i(x_0)}^{\delta_i(x)}\int_{\gamma_i(y_0)}^{\gamma_i(y)} \tilde{f}_i(X,Y,s,t)w_i\left(\varphi^{-1}\left(\Psi^{-1}(v(s,t))\right)\right)ds\,dt, \tag{3.17}$$

for all $(x,y) \in \Lambda_{(X,\,Y)}$. Let $z_3(x,y)$ denote the nonnegative and nondecreasing function on the right-hand side of (3.17). Then $z_3(x_0,y) = A_{k+1}(X,Y)$ and

$$v(x,y) \le z_3(x,y). \tag{3.18}$$

Let

$$\phi_i(u) := \frac{w_{i+1}(u)}{w_1(u)}, \quad i = 1,2,\ldots,k. \tag{3.19}$$

By (2.2), we see that each ϕ_i, $i = 1, 2, \ldots, k$, is a nondecreasing function. Then, we have

$$\begin{aligned}\frac{\partial z_3(x,y)/\partial x}{w_1(\varphi^{-1}(\Psi^{-1}(z_3(x,y))))} &= \frac{\sum_{i=1}^{k+1}\delta_i'(x)\int_{\gamma_i(y_0)}^{\gamma_i(y)}\tilde{f}_i(X,Y,\delta_i(x),t)w_i(\varphi^{-1}(\Psi^{-1}(v(\delta_i(x),t))))dt}{w_1(\varphi^{-1}(\Psi^{-1}(z_3(x,y))))}\\ &\le \frac{\sum_{i=1}^{k+1}\delta_i'(x)\int_{\gamma_i(y_0)}^{\gamma_i(y)}\tilde{f}_i(X,Y,\delta_i(x),t)w_i(\varphi^{-1}(\Psi^{-1}(z_3(\delta_i(x),t))))dt}{w_1(\varphi^{-1}(\Psi^{-1}(z_3(x,y))))}\\ &\le \delta_1'(x)\int_{\gamma_1(y_0)}^{\gamma_1(y)}\tilde{f}_1(X,Y,\delta_1(x),t)dt\\ &\quad+\sum_{i=1}^{k}\delta_{i+1}'(x)\int_{\gamma_{i+1}(y_0)}^{\gamma_{i+1}(y)}\tilde{f}_{i+1}(X,Y,\delta_{i+1}(x),t)\\ &\qquad\times\phi_i\Big(\varphi^{-1}\Big(\Psi^{-1}(z_3(\delta_{i+1}(x),t))\Big)\Big)dt,\end{aligned} \tag{3.20}$$

for all $(x,y)\in\Lambda_{(X,\,Y)}$. Keeping y fixed in (3.20), setting $s = x$, integrating both sides of (3.20) with respect to s from x_0 to x, and using the definition of W_i in (2.4), we have

$$\begin{aligned}W_1(z_3(x,y)) \le{}& W_1(A_{k+1}(X,Y)) + \int_{\delta_1(x_0)}^{\delta_1(X)}\int_{\gamma_1(y_0)}^{\gamma_1(Y)}\tilde{f}_1(X,Y,s,t)\,ds\,dt\\ &+\sum_{i=1}^{k}\int_{\delta_{i+1}(x_0)}^{\delta_{i+1}(x)}\int_{\gamma_{i+1}(y_0)}^{\gamma_{i+1}(y)}\tilde{f}_{i+1}(X,Y,s,t)\phi_i\Big(\varphi^{-1}\Big(\Psi^{-1}(z_3(s,t))\Big)\Big)ds\,dt,\end{aligned} \tag{3.21}$$

for all $(x,y)\in\Lambda_{(X,\,Y)}$. Let

$$\eta(x,y) := W_1(z_3(x,y)), \tag{3.22}$$

$$\theta_1(X,Y) := W_1(A_{k+1}(X,Y)) + \int_{\delta_1(x_0)}^{\delta_1(X)}\int_{\gamma_1(y_0)}^{\gamma_1(Y)}\tilde{f}_1(X,Y,s,t)\,ds\,dt. \tag{3.23}$$

Using (3.22) and (3.23), from (3.21), we have

$$\eta(x,y) \le \theta_1(X,Y) + \sum_{i=1}^{k}\int_{\delta_{i+1}(x_0)}^{\delta_{i+1}(x)}\int_{\gamma_{i+1}(y_0)}^{\gamma_{i+1}(y)}\tilde{f}_{i+1}(X,Y,s,t)\phi_i\Big(\varphi^{-1}\Big(\Psi^{-1}\Big(W_1^{-1}(\eta(s,t))\Big)\Big)\Big)ds\,dt,\quad \forall (x,y)\in\Lambda_{(X,\,Y)}. \tag{3.24}$$

It has the same form as (3.9). Let $\rho(s) := \varphi^{-1}(\Psi^{-1}(W_1^{-1}(s)))$. Since $\varphi^{-1}, \Psi^{-1}, W_1^{-1}$, and ϕ_i are continuous, nondecreasing, and positive on $(0,\infty)$, each $\phi_i(\rho(s))$ is continuous, nondecreasing, and positive on $(0,\infty)$. Moreover,

$$\frac{\phi_{i+1}(\rho(s))}{\phi_i(\rho(s))} = \frac{w_{i+1}(\rho(s))}{w_i(\rho(s))} = \max_{\tau\in[0,\,\rho(s)]}\left\{\frac{\varphi_{i+1}(\tau)}{w_i(\tau)}\right\}, \quad i = 1,2,\ldots,k-1, \tag{3.25}$$

which are also continuous, nondecreasing, and positive on $(0,\infty)$. Therefore, the inductive assumption for (3.9) can be used to (3.24), and then we have

$$\eta(x,y) \le \Phi_k^{-1}\left(\Phi_k(\theta_k(X,Y)) + \int_{\delta_{k+1}(x_0)}^{\delta_{k+1}(X)}\int_{\gamma_{k+1}(y_0)}^{\gamma_{k+1}(Y)} \tilde{f}_{k+1}(X,Y,s,t)\,ds\,dt\right), \tag{3.26}$$

for all $(x,y) \in \Lambda_{(X,\,Y)}$, where

$$\Phi_i(u) := \int_0^u \frac{ds}{\phi_i(\varphi^{-1}(\Psi^{-1}(W_1^{-1}(s))))}, \quad u > 0,\ i = 1,2,\ldots,k, \tag{3.27}$$

$$\theta_{i+1}(X,Y) := \Phi_i^{-1}\left(\Phi_i(\theta_i(X,Y)) + \int_{\delta_{i+1}(x_0)}^{\delta_{i+1}(X)}\int_{\gamma_{i+1}(y_0)}^{\gamma_{i+1}(Y)} \tilde{f}_{i+1}(X,Y,s,t)\,ds\,dt\right), \tag{3.28}$$

$i = 1,2,\ldots,k-1$

We note that

$$\begin{aligned}\Phi_i(u) &= \int_0^u \frac{w_1(\varphi^{-1}(\Psi^{-1}(W_1^{-1}(s))))ds}{w_{i+1}(\varphi^{-1}(\Psi^{-1}(W_1^{-1}(s))))}\\ &= \int_0^{W_1^{-1}(u)} \frac{ds}{w_{i+1}(\varphi^{-1}(\Psi^{-1}(s)))}\\ &= W_{i+1}\left(W_1^{-1}(u)\right), \quad i = 1,2,\ldots,k.\end{aligned} \tag{3.29}$$

Thus, from (3.18), (3.22), (3.26), and (3.29), we have

$$\begin{aligned}v(x,y) &\le z_3(x,y) = W_1^{-1}(\eta(x,y))\\ &\le W_1^{-1}\left(\Phi_k^{-1}\left(\Phi_k(\theta_k(X,Y)) + \int_{\delta_{k+1}(x_0)}^{\delta_{k+1}(X)}\int_{\gamma_{k+1}(y_0)}^{\gamma_{k+1}(Y)} \tilde{f}_{k+1}(X,Y,s,t)\,ds\,dt\right)\right)\\ &= W_{k+1}^{-1}\left(W_{k+1}\left(W_1^{-1}(\theta_k(X,Y))\right) + \int_{\delta_{k+1}(x_0)}^{\delta_{k+1}(X)}\int_{\gamma_{k+1}(y_0)}^{\gamma_{k+1}(Y)} \tilde{f}_{k+1}(X,Y,s,t)\,ds\,dt\right),\end{aligned} \tag{3.30}$$

for all $(x, y) \in \Lambda_{(X,Y)}$. We can prove that the term of $W_1^{-1}(\theta_k(X,Y))$ in (3.30) is just the same as $E_{k+1}(X,Y)$ defined in (3.12). Let $\tilde{\theta}_i(X,Y) := W_1^{-1}(\theta_i(X,Y))$. By (3.23), we have

$$\begin{aligned}\tilde{\theta}_1(X,Y) &= W_1^{-1}(\theta_1(X,Y))\\ &= W_1^{-1}\left(W_1(A_{k+1}(X,Y)) + \int_{\delta_1(x_0)}^{\delta_1(X)}\int_{\gamma_1(y_0)}^{\gamma_1(Y)} \tilde{f}_1(X,Y,s,t)ds\,dt\right)\\ &= E_2(X,Y).\end{aligned} \tag{3.31}$$

Then using (3.28) and (3.29), we get

$$\begin{aligned}\tilde{\theta}_i(X,Y) &= W_1^{-1}\left(\Phi_{i-1}^{-1}\left(\Phi_{i-1}(\theta_{i-1}(X,Y)) + \int_{\delta_i(x_0)}^{\delta_i(X)}\int_{\gamma_i(y_0)}^{\gamma_i(Y)} \tilde{f}_i(X,Y,s,t)ds\,dt\right)\right)\\ &= W_i^{-1}\left(W_i\left(W_1^{-1}(\theta_{i-1}(X,Y))\right) + \int_{\delta_i(x_0)}^{\delta_i(X)}\int_{\gamma_i(y_0)}^{\gamma_i(Y)} \tilde{f}_i(X,Y,s,t)ds\,dt\right)\\ &= W_i^{-1}\left(W_i\left(\tilde{\theta}_{i-1}(X,Y)\right) + \int_{\delta_i(x_0)}^{\delta_i(X)}\int_{\gamma_i(y_0)}^{\gamma_i(Y)} \tilde{f}_i(X,Y,s,t)ds\,dt\right)\\ &= W_i^{-1}\left(W_i(E_i(X,Y)) + \int_{\delta_i(x_0)}^{\delta_i(X)}\int_{\gamma_i(y_0)}^{\gamma_i(Y)} \tilde{f}_i(X,Y,s,t)ds\,dt\right)\\ &= E_{i+1}(X,Y), \quad i = 2,3,\ldots,k.\end{aligned} \tag{3.32}$$

This proves that $W_1^{-1}(\theta_k(X,Y))$ in (3.30) is just the same as $E_{k+1}(X,Y)$ defined in (3.12). Therefore, from (3.29), (3.30), and (3.32), we obtain

$$\begin{aligned}&\Phi_i(\theta_i(X,Y)) + \int_{\delta_{i+1}(x_0)}^{\delta_{i+1}(X)}\int_{\gamma_{i+1}(y_0)}^{\gamma_{i+1}(Y)} \tilde{f}_{i+1}(X,Y,s,t)ds\,dt\\ &\quad = W_{i+1}(E_{i+1}(X,Y)) + \int_{\delta_{i+1}(x_0)}^{\delta_{i+1}(X)}\int_{\gamma_{i+1}(y_0)}^{\gamma_{i+1}(Y)} \tilde{f}_{i+1}(X,Y,s,t)ds\,dt\\ &\quad \le \int_0^{\infty} \frac{ds}{w_{i+1}\left(\varphi^{-1}\left(\Psi^{-1}(s)\right)\right)} = \int_0^{W_1(\infty)} \frac{ds}{\phi_i\left(\varphi^{-1}\left(\Psi^{-1}\left(W_1^{-1}(s)\right)\right)\right)},\end{aligned} \tag{3.33}$$

$i = 1,2,\ldots,k$. The relations of (3.33) imply that in (3.26) and (3.28)

$$\Phi_i(\theta_i(X,Y)) + \int_{\delta_{i+1}(x_0)}^{\delta_{i+1}(X)}\int_{\gamma_{i+1}(y_0)}^{\gamma_{i+1}(Y)} \tilde{f}_{i+1}(X,Y,s,t)ds\,dt \in \operatorname{Dom}\left(\Phi_i^{-1}\right), \tag{3.34}$$

$i = 1, 2, \ldots, k$. Hence, (3.30) can be equivalently written as

$$v(x,y) \le W_{k+1}^{-1}\left(W_{k+1}(E_{k+1}(X,Y)) + \int_{\delta_{k+1}(x_0)}^{\delta_{k+1}(X)} \int_{\gamma_{k+1}(y_0)}^{\gamma_{k+1}(Y)} \tilde{f}_{i+1}(X,Y,s,t)ds\,dt, \right. \tag{3.35}$$

for all $(x,y) \in \Lambda_{(X,\,Y)}$. The claim in (3.10) is proved by induction.

Therefore, by (3.3), (3.8), and (3.10), we have

$$\begin{aligned} u(x,y) &\le \varphi^{-1}(z_1(x,y)) \le \varphi^{-1}\left(\Psi^{-1}(v(x,y))\right) \\ &\le \varphi^{-1}\left(\Psi^{-1}\left(W_n^{-1}\left(W_n(E_n(X,Y)) + \int_{\delta_n(x_0)}^{\delta_n(X)} \int_{\gamma_n(y_0)}^{\gamma_n(Y)} \tilde{f}_n(X,Y,s,t)ds\,dt\right)\right)\right), \end{aligned} \tag{3.36}$$

for all $(x,y) \in \Lambda_{(X,\,Y)}$. Hence, we obtain the estimation of the unknown function u in the auxiliary inequality (3.2).

Letting $x = X$, $y = Y$, from (3.36), we have

$$u(X,Y) \le \varphi^{-1}\left(\Psi^{-1}\left(W_n^{-1}\left(W_n(E_n(X,Y)) + \int_{\delta_n(x_0)}^{\delta_n(X)} \int_{\gamma_n(y_0)}^{\gamma_n(Y)} \tilde{f}_n(X,Y,s,t)ds\,dt\right)\right)\right), \tag{3.37}$$

for all $x_0 \le X \le X_1$, $y_0 \le Y \le Y_1$. Since $\Xi_i(X,Y) = E_i(X,Y)$ and X and Y are arbitrarily chosen, this proves (2.7).

The remainder case is that $a(x,y) = 0$ for some $(x,y) \in \Lambda$. Let

$$a_\varepsilon(x,y) = a(x,y) + \varepsilon, \tag{3.38}$$

where $\varepsilon > 0$ is an arbitrary small number. Obviously, $a_\varepsilon(x,y) > 0$, for all $(x,y) \in \Lambda$. Using the same arguments as above, where $a(x,y)$ is replaced with $a_\varepsilon(x,y)$, we get

$$u(x,y) \le \varphi^{-1}\left(\Psi^{-1}\left(W_n^{-1}\left(W_n(E_{n,\,\varepsilon}(x,y)) + \int_{\delta_n(x_0)}^{\delta_n(x)} \int_{\gamma_n(y_0)}^{\gamma_n(y)} \tilde{f}_n(X,Y,s,t)ds\,dt\right)\right)\right), \tag{3.39}$$

for all $(x,y) \in \Lambda_{(X_1,\,Y_1)}$. Letting $\varepsilon \to 0_+$, we obtain (2.7) because of continuity of a_ε in ε and continuity of $\varphi^{-1}, \Psi^{-1}, W_i$, and W_i^{-1} for $i = 1, 2, \ldots, n$. This completes the proof. □

The proofs of Corollary 2.2, 2.3, 2.5 and Theorem 2.4 are similar to the argument in the proofs of Theorem 2.1 with appropriate modification. We omit the details here.

Remark 3.1. When $a(x,y) = a, n = 1$ and $g(x,y,s,t) = (p/(p-q))g_1(s,t), f(x,y,s,t) = (p/(p-q))g_2(s,t)$, Corollary 2.2 reduces to Theorem 2.4 in [9].

Remark 3.2. When $\varphi(u) = u^p$ and $q = 0$, Corollary 2.3 reduces to Theorem 1 of Wang [16].

Remark 3.3. When $f_i(x,y,s,t) = c(x,y)f_i(s,t)$, $g_i(x,y,s,t) = c(c,y)g_i(s,t)$, Theorem 2.4 reduces to Theorem 2.1 of Kim [12].

4. Applications

In this section, we apply our results to study the boundedness and uniqueness of the solutions of boundary value problem to a partial differential equation. We consider the partial differential equation with the initial boundary conditions:

$$\frac{\partial^2 \psi(z(x,y))}{\partial x \partial y} = F(x,y,\varphi_1(z(\delta_1(x),\gamma_1(y))),\dots,\varphi_n(z(\delta_n(x),\gamma_n(y)))), \tag{4.1}$$

$$\psi(z(x,y_0)) = a_1(x), \quad \psi(z(x_0,y)) = a_2(y), \quad a_1(x_0) = a_2(y_0) = 0, \tag{4.2}$$

for all $(x,y) \in \Lambda$, where $\Lambda = I \times J$ is defined as in Section 2, δ_i, γ_i are defined in (H_4), ψ is a continuous and strictly increasing odd function on $\mathbb{R}$, satisfying $\psi(0) = 0$, $\psi(u) > 0$ for $u > 0$, $F : \Lambda \times \mathbb{R}^n \to \mathbb{R}$, $a_1 : I \to \mathbb{R}$, $a_2 : J \to \mathbb{R}$, and $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing continuous functions, and the ratio φ_{i+1}/φ_i is also nondecreasing, and $\varphi_i(u) > 0$ for $u > 0$ $i = 1,2,\dots,n$.

In the following corollary, we firstly apply our result to discuss boundedness on the solution of problem (4.1).

Corollary 4.1. *Assume that $F : \Lambda\times\mathbb{R}^n \to \mathbb{R}$ is a continuous function for which there exist a constant $q > 0$, nonnegative functions $f_i(x,y) \in C(\Lambda,\mathbb{R}_+), \varphi_i \in C(\mathbb{R}_+,\mathbb{R}_+)$, $i = 1,2,\dots,n$, such that*

$$\left|F(x,y,\varphi_1(u_1),\dots,\varphi_n(u_n))\right| \le \sum_{i=1}^{n} |u_i|^q f_i(x,y)\varphi_i(|u_i|), \tag{4.3}$$

$$\left|a_1(x) + a_2(y)\right| \le a(x,y), \tag{4.4}$$

for all $(x,y) \in \Lambda$, and $a : \Lambda \to \mathbb{R}_+$ is nondecreasing in each variable. If $z(x,y)$ is any solution of problem (4.1) *with condition* (4.2), *then*

$$\left|z(x,y)\right| \le \psi^{-1}\left\{\Psi^{-1}\left[W_n^{-1}\left(W_n\left(\widetilde{K}_n(x,y)\right) + \int_{x_0}^{x}\int_{y_0}^{y} f_n(s,t)ds\,dt\right)\right]\right\}, \tag{4.5}$$

for all $(x,y) \in \Lambda_{(X_6,Y_6)}$, where

$$\widetilde{K}_1(x,y) := \Psi(a(x,y)),$$

$$\widetilde{K}_i(x,y) := W_{i-1}^{-1}\left(W_{i-1}\left(\widetilde{K}_{i-1}(x,y)\right) + \int_{x_0}^{x}\int_{y_0}^{y} f_{i-1}(s,t)ds\,dt\right), \tag{4.6}$$

$$i = 2,3,\dots,n,$$

and $\Psi, \Psi^{-1}, W, W^{-1}$ *are as defined in Theorem 2.1.* $(X_6, Y_6) \in \Lambda$ *lies on the boundary of the planar region*

$$\mathcal{R}_6 := \left\{ (x,y) \in \Lambda : W_i\left(\widetilde{K}_i(x,y)\right) + \int_{x_0}^{x}\int_{y_0}^{y} f_i(s,t)ds\,dt \right.$$
$$\leq \int_0^{\infty} \frac{ds}{\varphi_i(\psi^{-1}(\Psi^{-1}(s)))},\ i = 1,2,\ldots,n, \tag{4.7}$$
$$\left. W_n^{-1}\left(W_n\left(\widetilde{K}_n(x,y)\right) + \int_{x_0}^{x}\int_{y_0}^{y} f_n(s,t)ds\,dt\right) \leq \int_0^{\infty} \frac{ds}{(\psi^{-1}(s))^q} \right\}.$$

Proof. It is easy to see that the solution $z(x,y)$ of (4.1) satisfies the following equivalent integral equation:

$$\psi(z(x,y)) = a_1(x) + a_2(y) + \int_{x_0}^{x}\int_{y_0}^{y} F(s,t,\varphi_1(z(\delta_1(s),\gamma_1(t))),\ldots,\varphi_n(z(\delta_n(s),\gamma_n(t))))ds\,dt. \tag{4.8}$$

By (4.3),(4.4), and (4.8), we have

$$\begin{aligned}
|\psi(z(x,y))| &= |a_1(x) + a_2(y)| \\
&\quad + \int_{x_0}^{x}\int_{y_0}^{y} |F(s,t,\varphi_1(z(\delta_1(s),\gamma_1(t))),\ldots,\varphi_n(z(\delta_n(s),\gamma_n(t))))ds\,dt| \\
&\leq a(x,y) + \sum_{i=1}^{n}\int_{x_0}^{x}\int_{y_0}^{y} |z(\delta_i(s),\gamma_i(t))|^q f_i(s,t)\varphi_i(|z(\delta_i(s),\gamma_i(t))|)ds\,dt \\
&\leq a(x,y) + \sum_{i=1}^{n}\int_{\delta_i(x_0)}^{\delta_i(x)}\int_{\gamma_i(y_0)}^{\gamma_i(y)} \frac{f_i(\delta_i^{-1}(s),\gamma_i^{-1}(t))}{\delta_i'(\delta_i^{-1}(s))\gamma_i'(\gamma_i^{-1}(t))}|z(s,t)|^q\varphi_i(|z(s,t)|)ds\,dt.
\end{aligned} \tag{4.9}$$

Since $|\psi(z(x,y))| = \psi(|z(x,y)|)$, (4.9) is the form of (2.14). Applying Corollary 2.3 to inequality (4.9), using the relation

$$\int_{\delta_i(x_0)}^{\delta_i(x)}\int_{\gamma_i(y_0)}^{\gamma_i(y)} \frac{f_i(\delta_i^{-1}(s),\gamma_i^{-1}(t))}{\delta_i'(\delta_i^{-1}(s))\gamma_i'(\gamma_i^{-1}(t))}ds\,dt = \int_{x_0}^{x}\int_{y_0}^{y} f_i(s,t)ds\,dt, \tag{4.10}$$

we obtain the estimation of $z(x,y)$ as given in (4.5).

Corollary 4.1 gives a condition of boundedness for solutions, concretely. If there is a $M > 0$,

$$\widetilde{K}_i(x,y) < M, \qquad \int_{x_0}^{x}\int_{y_0}^{y} f_i(s,t)ds\,dt < M, \quad i = 1,2,\ldots,n, \tag{4.11}$$

for all $(x,y) \in \Lambda$. Then every solution $z(x,y)$ of (4.1) is bounded on Λ.

Next, we discuss the uniqueness of the solutions of (4.1). □

Corollary 4.2. *Additionally, assume that*

$$\begin{aligned}
&\left|F(s,t,\varphi_1(u_{11}),\ldots,\varphi_n(u_{1n})) - F(s,t,\varphi_1(u_{21}),\ldots,\varphi_n(u_{2n}))\right| \\
&\quad \le \sum_{i=1}^{n} f_i(s,t)\left|\psi(u_{1i}) - \psi(u_{2i})\right|^q \varphi_i\left(\left|\psi(u_{1i}) - \psi(u_{2i})\right|\right),
\end{aligned} \tag{4.12}$$

for $u_{1i}, u_{2i} \in \mathbb{R}, i = 1,2,\ldots,n$, and $(x,y) \in \Lambda$, where Λ is defined as in Section 2, $0 < q < 1$ is a constant, $f_i \in C(\Lambda,\mathbb{R}_+)$, $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,2,\ldots,n$ are continuous nondecreasing with the nondecreasing ratio φ_{i+1}/φ_i such that $\varphi_i(u) > 0$ for all $u > 0$, and $\int_0^{\infty}(ds/\varphi_i(s)) = \infty$, $i = 1,2,\ldots,n$, and $\psi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing odd function satisfying $\psi(u) > 0$, for all $u > 0$. Then, (4.1) *has at most one solution on Λ*

Proof. Let $z(x,y)$ and $\tilde{z}(x,y)$ be two solutions of (4.1). By (4.8) and (4.12), we have

$$\begin{aligned}
&\left|\psi(z(x,y)) - \psi(\tilde{z}(x,y))\right| \\
&\quad \le \varepsilon + \sum_{i=1}^{n}\int_{x_0}^{x}\int_{y_0}^{y} f_i(s,t)\left|\psi(z(\delta_i(s),\gamma_i(t))) - \psi(\tilde{z}(\delta_i(s),\gamma_i(t)))\right|^q \\
&\qquad\qquad \times \varphi_i\left(\left|\psi(z(\delta_i(s),\gamma_i(t))) - \psi(\tilde{z}(\delta_i(s),\gamma_i(t)))\right|\right)ds\,dt \\
&\quad = \varepsilon + \sum_{i=1}^{n}\int_{\delta_i(x_0)}^{\delta_i(x)}\int_{\gamma_i(y_0)}^{\gamma_i(y)} \frac{f_i(\delta_i^{-1}(s),\gamma_i^{-1}(t))}{\delta_{i'}(\delta_i^{-1}(s))\gamma_{i'}(\gamma_i^{-1}(t))}\left|\psi(z(s,t)) - \psi(\tilde{z}(s,t))\right|^q \\
&\qquad\qquad \times \varphi_i\left(\left|\psi(z(s,t)) - \psi(\tilde{z}(s,t))\right|\right)ds\,dt
\end{aligned} \tag{4.13}$$

for all $(x,y) \in \Lambda$, which is an inequality of the form (2.14), where $\varepsilon > 0$ is an arbitrary small number. Applying Corollary 2.2, we obtain an estimation of the difference $|\psi(z(x,y)) - \psi(\tilde{z}(x,y))|$ in the form (4.5). Namely,

$$\left|\psi(z(x,y)) - \psi(\tilde{z}(x,y))\right| \le \left[\Omega_n^{-1}\left(\Omega_n(P_n(x,y)) + (1-q)\int_{x_0}^{x}\int_{y_0}^{y} f_n(s,t)ds\,dt\right)\right]^{1/(1-q)}, \tag{4.14}$$

for all $(x,y) \in \Lambda$, where

$$P_1(x,y) := \varepsilon^{1-q},$$

$$P_i(x,y) := \Omega_{i-1}^{-1}\left(\Omega_{i-1}(P_{i-1}(x,y)) + (1-q)\int_{x_0}^{x}\int_{y_0}^{y} f_{i-1}(s,t)ds dt\right),\ i = 2,3,\ldots,n, \tag{4.15}$$

$$\Omega_i(u) := \int_1^u \frac{ds}{\varphi_i(s^{1/1-q})}, \quad u \geq 0, \quad i = 1,2,\ldots,n,$$

and Ω_i^{-1} denotes the inverse function of Ω_i.

Furthermore, by the definition of Ω_i, we conclude that

$$\lim_{u\to 0}\Omega_i(u) = -\infty, \quad \lim_{u\to -\infty}\Omega_i^{-1}(u) = 0, \quad i = 1,2,\ldots,n. \tag{4.16}$$

Letting $\varepsilon \to 0$, it follows that

$$\Omega_i(P_i(x,y)) + (1-q)\int_{x_0}^{x}\int_{y_0}^{y} f_i(s,t)ds\,dt = -\infty, \quad i = 1,2,\ldots,n,$$
$$\Omega_i^{-1}\left[\Omega_i(P_i(x,y)) + (1-q)\int_{x_0}^{x}\int_{y_0}^{y} f_i(s,t)ds\,dt\right] = 0, \quad i = 2,\ldots,n. \tag{4.17}$$

Thus, from (4.5), we deduce that $|\psi(z(x,y)) - \psi(\tilde{z}(x,y))| \leq 0$, implying that $z(x,y) = \tilde{z}(x,y)$, for all $(x,y) \in \Lambda$, since ψ is strictly increasing. The uniqueness is proved. □

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