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Research Article

Weak Convergence Theorems for a System of Mixed Equilibrium Problems and Nonspreading Mappings in a Hilbert Space

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We introduce an iterative sequence and prove a weak convergence theorem for finding a solution of a system of mixed equilibrium problems and the set of fixed points of a quasi-nonexpansive mapping in Hilbert spaces. Moreover, we apply our result to obtain a weak convergence theorem for finding a solution of a system of mixed equilibrium problems and the set of fixed points of a nonspreading mapping. The result obtained in this paper improves and extends the recent ones announced by Moudafi (2009), Iemoto and Takahashi (2009), and many others. Using this result, we improve and unify several results in fixed point problems and equilibrium problems.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping T of C into itself is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and a mapping F is said to be *firmly nonexpansive* if $\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle$ for all $x, y \in C$. Let E be a smooth, strictly convex and reflexive Banach space; let J be the duality mapping of E and C a nonempty closed convex subset of E. A mapping $S : C \to C$ is said to be *nonspreading* if

$$\phi(Sx, Sy) + \phi(Sy, Sx) \leq \phi(Sx, y) + \phi(Sy, x) \tag{1.1}$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$; see, for instance, Kohsaka and Takahashi [1]. In the case when E is a Hilbert space, we know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$. Then a nonspreading mapping $S : C \to C$ in a Hilbert space H is defined as follows:

$$2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|x - Sy\|^2 \tag{1.2}$$

for all $x, y \in C$. Let $F(Q)$ be the set of fixed points of Q, and $F(Q)$ be nonempty; a mapping $Q : C \to C$ is said to be *quasi-nonexpansive* if $\|Qx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(Q)$.

Remark 1.1. In a Hilbert space, we know that every firmly nonexpansive mapping is nonspreading and if the set of fixed points of a nonspreading mapping is nonempty, the nonspreading mapping is quasi-nonexpansive; see [1].

Fixed point iterations process for nonexpansive mappings and asymptotically nonexpansive mappings in Banach spaces including Mann and Ishikawa iterations process have been studied extensively by many authors to solve the nonlinear operator equations. In 1953, Mann [2] introduced Mann iterative process defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \tag{1.3}$$

where $\alpha_n \in [0, 1]$ and satisfies the assumptions $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ and proved that in case E is a Banach space, and C is closed, and T is continuous, then the convergence of $\{x_n\}$ to a point y implies that $Ty = y$. Recently, Dotson [3] proved that a Mann iteration process was applied to the approximation of fixed points of quasi-nonexpansive mappings in Hilbert space and in uniformly convex and strictly convex Banach spaces.

On the other hand, Kohsaka and Takahashi [1] proved an existence theorem of fixed points for nonspreading mappings in a Banach space. Very recently, Iemoto and Takahashi [4] studied the approximation theorem of common fixed points for a nonexpansive mapping T of C into itself and a nonspreading mapping S of C into itself in a Hilbert space. In particular, this result reduces to approximation fixed points of a nonspreading mapping S of C into itself in a Hilbert space by using iterative scheme

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Sx_n. \tag{1.4}$$

Let $\varphi : C \to \mathbb{R}$ be a real-valued function and $F : C \times C \to \mathbb{R}$ an equilibrium bifunction, that is, $F(u, u) = 0$ for each $u \in C$. The mixed equilibrium problem is to find $x^* \in C$ such that

$$F(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0 \quad \text{for all } y \in C. \tag{1.5}$$

Denote the set of solutions of (1.5) by $\mathrm{MEP}(F, \varphi)$. The mixed equilibrium problems include fixed point problems, optimization problems, variational inequality problems, Nash equiliubrium problems, and the equilibrium problems as special cases (see, e.g., Blum and Oettli [5]). In particular, if $\varphi = 0$, this problem reduces to the equilibrium problem, which is to find $x^* \in C$ such that

$$F(x^*, y) \geq 0 \quad \text{for all } y \in C. \tag{1.6}$$

The set of solutions of (1.6) is denoted by $\text{EP}(F)$. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.6). Let $F_1, F_2 : C \times C \to \mathbb{R}$ be two monotone bifunctions and $\lambda > 0$ is constant. In 2009, Moudafi [6] introduced an alternating algorithm for approximating a solution of the system of equilibrium problems: finding $(x^*, y^*) \in C \times C$ such that

$$F_1(x^*, z) + \frac{1}{\lambda}\langle y^* - x^*, x^* - z\rangle \geq 0, \quad \forall z \in C,$$
$$F_2(y^*, z) + \frac{1}{\lambda}\langle x^* - y^*, y^* - z\rangle \geq 0, \quad \forall z \in C. \tag{1.7}$$

Let $F_1, F_2 : C \times C \to \mathbb{R}$ be two monotone bifunctions and $\lambda, \mu > 0$ are two constants. In this paper, we consider the following problem for finding $(x^*, y^*) \in C \times C$ such that

$$F_1(x^*, z) + \varphi(z) - \varphi(x^*) + \frac{1}{\lambda}\langle y^* - x^*, x^* - z\rangle \geq 0, \quad \forall z \in C,$$
$$F_2(y^*, z) + \varphi(z) - \varphi(y^*) + \frac{1}{\mu}\langle x^* - y^*, y^* - z\rangle \geq 0, \quad \forall z \in C, \tag{1.8}$$

which is called a system of mixed equilibrium problems. In particular, if $\lambda = \mu$, then problem (1.8) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$F_1(x^*, z) + \varphi(z) - \varphi(x^*) + \frac{1}{\lambda}\langle y^* - x^*, x^* - z\rangle \geq 0, \quad \forall z \in C,$$
$$F_2(y^*, z) + \varphi(z) - \varphi(y^*) + \frac{1}{\lambda}\langle x^* - y^*, y^* - z\rangle \geq 0, \quad \forall z \in C. \tag{1.9}$$

The system of nonlinear variational inequalities close to these introduced by Verma [7] is also a special case: by taking $\varphi = 0$, $F_1(x, y) = \langle A(x), y - x\rangle$, and $F_2(x, y) = \langle B(x), y - x\rangle$, where $A, B : C \to H$ are two nonlinear mappings. In this case, we can reformulate problem (1.7) to finding $(x^*, y^*) \in C \times C$ such that

$$\langle \lambda A(x^*) + x^* - y^*, z - x^*\rangle \geq 0, \quad \forall z \in C,$$
$$\langle \mu B(y^*) + y^* - x^*, z - y^*\rangle \geq 0, \quad \forall z \in C, \tag{1.10}$$

which is called a general system of variational inequalities where $\lambda > 0$ and $\mu > 0$ are two constants. Moreover, if we add up the requirement that $x^* = y^*$, then problem (1.10) reduces to the classical variational inequality $\text{VI}(A, C)$.

In 2008,Ceng and Yao [8] considered a new iterative scheme for finding a common element of the set of solutions of MEP and the set of common fixed points of finitely many nonexpansive mappings. They also proved a strong convergence theorem for the iterative scheme. In the same year, Yao et al. [9] introduced a new hybrid iterative algorithm for finding a common element of the set of fixed points of an infinite family of nonexpansive mappings, the set of solutions of the variational inequality of a monotone mapping, and the set of solutions of a mixed equilibrium problem. Very recently, Ceng et al. [10] introduced and studied a relaxed extragradient method for finding a common of the set of solution (1.10) for the α and β-inverse strongly monotones and the set of fixed points of a nonexpansive mapping T of C into a real Hilbert space H. Let $x_1 = u \in C$, and $\{x_n\}$ are given by

$$\begin{aligned} y_n &= P_C(x_n - \mu B x_n), \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n T P_C(y_n - \lambda A y_n), \quad n \in \mathbf{N}. \end{aligned} \tag{1.11}$$

Then, they proved that the iterative sequence $\{x_n\}$ converges strongly to a common element of the set of fixed points of a nonexpansive mapping and a general system of variational inequalities with inverse-strongly monotone mappings under some parameters controlling conditions.

In this paper, we introduce an iterative sequence and prove a weak convergence theorem for finding a solution of a system of mixed equilibrium problems and the set of fixed points of a quasi-nonexpansive mapping in Hilbert spaces. Moreover, we apply our result to obtain a weak convergence theorem for finding a solution of a system of mixed equilibrium problems and the set of fixed points of a nonspreading mapping.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\| \quad \text{for all } y \in C. \tag{2.1}$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y \rangle \ge \|P_C x - P_C y\|^2 \tag{2.2}$$

for all $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\begin{aligned} &\langle x - P_C x, y - P_C y \rangle \le 0, \\ &\|x - y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C y\|^2 \end{aligned} \tag{2.3}$$

for all $x \in H, y \in C$. Further, for all $x \in H$ and $y \in C$, $y = P_C x$ if and only if $\langle x - y, y - z \rangle \ge 0$, for all $z \in C$.

A space X is said to satisfy Opial's condition if for each sequence $\{x_n\}_{n=1}^{\infty}$ in X which converges weakly to point $x \in X$, we have

$$\begin{aligned} \liminf_{n \to \infty} \|x_n - x\| &< \liminf_{n \to \infty} \|x_n - y\|, \quad \forall y \in X,\ y \ne x, \\ \limsup_{n \to \infty} \|x_n - x\| &< \limsup_{n \to \infty} \|x_n - y\|, \quad \forall y \in X,\ y \ne x. \end{aligned} \tag{2.4}$$

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 2.1 (see [11]). *Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha+\beta+\gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2. \quad (2.5)$$

Lemma 2.2 (see [12]). *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$.*

Lemma 2.3 (see [1]). *Let H be a Hilbert space and C a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself. Then the following are equivalent:*

(1) *there exists $x \in C$ such that $\{S^n x\}$ is bounded;*

(2) *$F(S)$ is nonempty.*

Lemma 2.4 (see [1]). *Let H be a Hilbert space and C a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself. Then $F(S)$ is closed and convex.*

Lemma 2.5 (see [4]). *Let H be a Hilbert space, C a closed convex subset of H, and $S : C \to C$ a nonspreading mapping with $F(S) \neq \emptyset$. Then S is demiclosed, that is, $x_n \rightharpoonup u$ and $x_n - Sx_n \to 0$ imply $u \in F(S)$.*

Lemma 2.6 (see [13]). *Let C be a closed convex subset of a real Hilbert space H and let $\{x_n\}$ be a sequence in H. Suppose that for all $u \in C$,*

$$\|x_{n+1} - u\| \leq \|x_n - u\| \quad (2.6)$$

for every $n = 0, 1, 2, \ldots$. Then, $\{P_C x_n\}$ converges strongly to some $z \in C$.

Lemma 2.7 (see [14]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n, \quad n \geq 0, \quad (2.7)$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbf{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty}(\delta_n/\alpha_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

For solving the mixed equilibrium problems for an equilibrium bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $y \in C$, $x \mapsto F(x, y)$ is weakly upper semicontinuous;

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex, semicontinuous.

The following lemma appears implicitly in [5, 15].

Lemma 2.8 (see [5]). *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbf{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \text{for all } y \in C. \tag{2.8}$$

The following lemma was also given in [15].

Lemma 2.9 (see [15]). *Assume that $F : C \times C \to \mathbf{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\} \tag{2.9}$$

for all $z \in H$. Then, the following hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, that is, for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*

(3) $F(T_r) = \mathrm{EP}(F)$;

(4) *$\mathrm{EP}(F)$ is closed and convex.*

We note that Lemma 2.9 is equivalent to the following lemma.

Lemma 2.10. *Let C a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be an equilibrium bifunction satisfying (A1)–(A4) and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional. For $r > 0$ and $x \in H$, define a mapping $S_r(x) : H \to C$ as follows.*

$$S_r(x) = \left\{ y \in C : F(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y - x, z - y\rangle \geq 0,\ \forall z \in C \right\}, \quad \forall x \in H. \tag{2.10}$$

Then, the following results hold:

(i) *for each $x \in H$, $S_r(x) \neq \emptyset$;*

(ii) *S_r is single-valued;*

(iii) *S_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\left\| S_r(x) - S_r(y) \right\|^2 \leq \langle S_r(x) - S_r(y), x - y\rangle; \tag{2.11}$$

(iv) $F(S_r) = \mathrm{MEF}(F, \varphi)$;

(v) *$\mathrm{MEF}(F, \varphi)$ is closed and convex.*

Proof. Define $G(x,y) := F(x,y)+\varphi(y)+\varphi(x)$, for all $x,y \in H$. Thus, the bifunction $F : C\times C \to \mathbb{R}$satisfies, (A1)–(A4). Hence, by Lemmas 2.8 and 2.9, we have (i)–(v). □

Lemma 2.11 (see [6]). *Let C be a closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two mappings from $C \times C \to \mathbf{R}$ satisfying (A1)–(A4) and let $S_{1,\lambda}$ and $S_{2,\mu}$ be defined as in Lemma 2.10 associated to F_1 and F_2, respectively. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem* (1.8) *if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = S_{1,\lambda}(S_{2,\mu}x), \quad \forall x \in C, \tag{2.12}$$

where $y^* = S_{2,\mu}x^*$.

Proof. By a similar argument as in the proof of Proposition 2.1 in [6], we obtain the desired result. □

We note from Lemma 2.11 that the mapping G is nonexpansive. Moreover, if C is a closed bounded convex subset of H, then the solution of problem (1.8) always exists. Throughout this paper, we denote the set of solutions of (1.8) by Ω.

3. Main Result

In this section, we prove a weak convergence theorem for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of the system of mixed equilibrium problems.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two bifunctions from $C \times C \to \mathbb{R}$ satisfying (A1)–(A4). Let $\lambda, \mu > 0$ and let $S_{1,\lambda}$ and $S_{2,\mu}$ be defined as in Lemma 2.10 associated to F_1 and F_2, respectively. Let T be a quasi-nonexpansive mapping of C into itself such that $F(T) \cap \Omega \neq \emptyset$. Suppose $x_0 = x \in C$ and $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ are given by*

$$\begin{gathered} z_n \in C, \quad F_2(z_n, z) + \varphi(z) - \varphi(z_n) + \frac{1}{\mu}\langle z - z_n, z_n - x_n\rangle \geq 0, \quad \forall z \in C, \\ y_n \in C, \quad F_1(y_n, z) + \varphi(z) - \varphi(y_n) + \frac{1}{\lambda}\langle z - y_n, y_n - z_n\rangle \geq 0, \quad \forall z \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n, \end{gathered} \tag{3.1}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a,b]$ for some $a, b \in (0,1)$ and satisfy $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$. Then $\{x_n\}$ converges weakly to $\overline{x} = \lim_{n\to\infty} P_{F(T)\cap\Omega}x_n$ and $(\overline{x}, \overline{y})$ is a solution of problem (1.8)*, where $\overline{y} = S_{2,\mu}\overline{x}$.*

Proof. Let $x^* \in F(T) \cap \Omega$. Then $x^* = Tx^*$ and $x^* = S_{1,\lambda}(S_{2,\mu}x^*)$. Put $y^* = S_{2,\mu}x^*$, $y_n = S_{1,\lambda}z_n$ and $z_n = S_{2,\mu}x_n$. Since

$$\begin{aligned}\left\|y_n - x^*\right\| &= \left\|S_{1,\lambda}z_n - S_{1,\lambda}y^*\right\| \\ &\le \left\|z_n - y^*\right\| \\ &= \left\|S_{2,\mu}x_n - S_{2,\mu}x^*\right\| \\ &\le \|x_n - x^*\|,\end{aligned} \tag{3.2}$$

it follows by Lemma 2.1 that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \left\|\alpha_n x_n + (1-\alpha_n)Ty_n\right\|^2 \\ &= \left\|\alpha_n(x_n - x^*) + (1-\alpha_n)(Ty_n - x^*)\right\|^2 \\ &= \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\left\|Ty_n - x^*\right\|^2 - \alpha_n(1-\alpha_n)\left\|Ty_n - x_n\right\|^2 \\ &\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\left\|y_n - x^*\right\|^2 - \alpha_n(1-\alpha_n)\left\|Ty_n - x_n\right\|^2 \\ &\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 - \alpha_n(1-\alpha_n)\left\|Ty_n - x_n\right\|^2 \\ &= \|x_n - x^*\|^2 - \alpha_n(1-\alpha_n)\left\|Ty_n - x_n\right\| \\ &\le \|x_n - x^*\|^2.\end{aligned} \tag{3.3}$$

Hence $\{\|x_{n+1} - x^*\|\}$ is a decreasing sequence and therefore $\lim_{n\to\infty}\|x_n - x^*\|$ exists. This implies that $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{Ty_n\}$ are bounded. From (3.3), we note that

$$\alpha_n(1-\alpha_n)\left\|Ty_n - x_n\right\| \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{3.4}$$

Since $0 < a \le \alpha_n \le b < 1$ and $\lim_{n\to\infty}\|x_n - x^*\|^2 = \lim_{n\to\infty}\|x_{n+1} - x^*\|^2$, we obtain

$$a(1-b)\left\|Ty_n - x_n\right\| \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \longrightarrow 0. \tag{3.5}$$

This implies that $\lim_{n\to\infty}\|Ty_n - x_n\| = 0$. Since $S_{1,\lambda}$ and $S_{2,\mu}$ are firmly nonexpansive, it follows that

$$\begin{aligned}\left\|z_n - y^*\right\|^2 &= \left\|S_{2,\mu}x_n - S_{2,\mu}x^*\right\|^2 \\ &\le \left\langle S_{2,\mu}x_n - S_{2,\mu}x^*, x_n - x^*\right\rangle \\ &= \frac{1}{2}\left(\left\|z_n - y^*\right\|^2 + \|x_n - x^*\|^2 - \left\|z_n - y^* - x_n + x^*\right\|^2\right)\end{aligned} \tag{3.6}$$

and so $\|z_n - y^*\|^2 \leq \|x_n - x^*\|^2 - \|z_n - x_n + x^* - y^*\|^2$. By the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|\alpha_n x_n + (1-\alpha_n) T y_n\right\|^2 \\
&= \left\|\alpha_n (x_n - x^*) + (1-\alpha_n)(T y_n - x^*)\right\|^2 \\
&\leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\left\|T y_n - x^*\right\|^2 \\
&\leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\left\|y_n - x^*\right\|^2 \\
&\leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\left\|z_n - y^*\right\|^2 \\
&\leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\left[\|x_n - x^*\|^2 - \left\|z_n - x_n + x^* - y^*\right\|^2\right].
\end{aligned} \tag{3.7}$$

This implies that

$$(1-\alpha_n)\left\|z_n - x_n + x^* - y^*\right\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{3.8}$$

Since $0 < a \leq \alpha_n \leq b < 1$ and $\lim_{n\to\infty}\|x_n - x^*\|^2 = \lim_{n\to\infty}\|x_{n+1} - x^*\|^2$, we obtain $\lim_{n\to\infty}\|z_n - x_n + x^* - y^*\| = 0$. Similarly, we note that

$$\begin{aligned}
\left\|y_n - x^*\right\|^2 &= \left\|S_{1,\lambda} z_n - S_{1,\lambda} y^*\right\|^2 \\
&\leq \left\langle S_{1,\lambda} z_n - S_{1,\lambda} y^*, z_n - y^*\right\rangle \\
&= \frac{1}{2}\left(\left\|y_n - x^*\right\|^2 + \left\|z_n - y^*\right\|^2 - \left\|y_n - x^* - z_n + y^*\right\|^2\right) \\
&\leq \frac{1}{2}\left(\left\|y_n - x^*\right\|^2 + \|x_n - x^*\|^2 - \left\|y_n - z_n - x^* + y^*\right\|^2\right)
\end{aligned} \tag{3.9}$$

and so $\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|y_n - z_n - x^* + y^*\|^2$. Thus, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\left\|y_n - x^*\right\|^2 \\
&\leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\left[\|x_n - x^*\|^2 - \left\|y_n - z_n - x^* + y^*\right\|^2\right]
\end{aligned} \tag{3.10}$$

and hence

$$(1-\alpha_n)\left\|y_n - z_n - x^* + y^*\right\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{3.11}$$

It follows from $0 < a \leq \alpha_n \leq b < 1$ and $\lim_{n\to\infty}\|x_n - x^*\|^2 = \lim_{n\to\infty}\|x_{n+1} - x^*\|^2$ that $\lim_{n\to\infty}\|y_n - z_n - x^* + y^*\| = 0$. Hence

$$\left\|T y_n - y_n\right\| \leq \left\|T y_n - x_n\right\| + \left\|x_n - z_n - x^* + y^*\right\| + \left\|z_n - y_n + x^* - y^*\right\| \longrightarrow 0, \tag{3.12}$$

and therefore

$$\|y_n - x_n\| \le \|y_n - Ty_n\| + \|Ty_n - x_n\| \longrightarrow 0. \tag{3.13}$$

Since $\{y_n\}$ is a bounded sequence, there exists a subsequence $\{y_{n_i}\} \subset \{y_n\}$ such that $\{y_{n_i}\}$ converges weakly to $\overline{x}$. From Lemma 2.5, we have $\overline{x} \in F(T)$. Let G be a mapping which is defined as in Lemma 2.11. Thus, we have

$$\|y_n - G(y_n)\| = \|S_{1,\lambda}S_{2,\mu}x_n - G(y_n)\| = \|G(x_n) - G(y_n)\| \le \|x_n - y_n\|, \tag{3.14}$$

and hence

$$\|x_n - G(x_n)\| \le \|x_n - y_n\| + \|y_n - G(y_n)\| + \|G(y_n) - G(x_n)\| \le 3\|x_n - y_n\| \longrightarrow 0. \tag{3.15}$$

From $\lim_{n\to\infty}\|x_n - y_n\| = 0$ and $y_{n_i} \rightharpoonup \overline{x}$, we get $x_{n_i} \rightharpoonup \overline{x}$. According to Lemmas 2.2 and 2.11, we have $\overline{x} \in \Omega$. Hence $\overline{x} \in F(T) \cap \Omega$. Since $y_{n_i} \rightharpoonup \overline{x}$ and $\|y_n - x_n\| \to 0$, we obtain $x_{n_i} \rightharpoonup \overline{x}$. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $\widehat{x}$. We may show that $\overline{x} = \widehat{x}$, suppose not. Since $\lim_{n\to\infty}\|x_n - x^*\|$ exists for all $x^* \in F(T) \cap \Omega$, it follows by the Opial's condition that

$$\begin{aligned}\lim_{n\to\infty}\|x_n - \overline{x}\| &= \liminf_{i\to\infty}\|x_{n_i} - \overline{x}\| < \liminf_{i\to\infty}\|x_{n_i} - \widehat{x}\| = \lim_{n\to\infty}\|x_n - \widehat{x}\| \\ &= \liminf_{j\to\infty}\left\|x_{n_j} - \widehat{x}\right\| < \liminf_{j\to\infty}\left\|x_{n_j} - \overline{x}\right\| = \lim_{n\to\infty}\|x_n - \overline{x}\|.\end{aligned} \tag{3.16}$$

This is a contradiction. Thus, we have $\overline{x} = \widehat{x}$. This implies that $\{x_n\}$ converges weakly to $\overline{x} \in F(T) \cap \Omega$. Put $u_n = P_{F(T)\cap\Omega}x_n$. Finally, we show that $\overline{x} = \lim_{n\to\infty}u_n$. Now from (2.2) and $\overline{x} \in F(T) \cap \Omega$, we have

$$\langle \overline{x} - u_n, u_n - x_n \rangle \ge 0. \tag{3.17}$$

Since $\{\|x_n - x^*\|\}$ is nonnegative and decreasing for any $x^* \in F(S) \cap \Omega$, it follows by Lemma 2.6 that $\{u_n\}$ converges strongly to some $\widehat{x} \in F(T) \cap \Omega$ and hence

$$\langle \overline{x} - \widehat{x}, \widehat{x} - \overline{x} \rangle \ge 0. \tag{3.18}$$

Therefore, $\overline{x} = \widehat{x}$. □

Corollary 3.2. *Let C be a closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two bifunctions from $C \times C \to \mathbb{R}$ satisfying (A1)–(A4). Let $\lambda, \mu > 0$ and let $S_{1,\lambda}$ and $S_{2,\mu}$ be defined as* *in Lemma 2.10 associated to* F_1 *and* F_2*, respectively. Let* T *be a nonspreading mapping of* C *into itself such that* $F(T)\cap\Omega\neq\emptyset$*. Suppose* $x_0 = x \in C$ *and* $\{x_n\}$*,* $\{y_n\}$*,* $\{z_n\}$ *are given by*

$$\begin{gathered}
z_n \in C, \quad F_2(z_n, z) + \varphi(z) - \varphi(z_n) + \frac{1}{\mu}\langle z - z_n, z_n - x_n\rangle \geq 0, \quad \forall z \in C, \\
y_n \in C, \quad F_1(y_n, z) + \varphi(z) - \varphi(y_n) + \frac{1}{\lambda}\langle z - y_n, y_n - z_n\rangle \geq 0, \quad \forall z \in C, \\
x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n,
\end{gathered} \tag{3.19}$$

for all $n \in \mathbb{N}$*, where* $\{\alpha_n\}\subset[a,b]$ *for some* $a, b \in (0,1)$ *and satisfy* $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$*. Then* $\{x_n\}$ *converges weakly to* $\overline{x} = \lim_{n\to\infty}P_{F(T)\cap\Omega}x_n$ *and* $(\overline{x}, \overline{y})$ *is a solution of problem* (1.8)*, where* $\overline{y} = S_{2,\mu}\overline{x}$*.*

Setting $\lambda = \mu$ and $T = I$ in Theorem 3.1, we have following result.

Corollary 3.3. *Let* C *be a closed convex subset of a real Hilbert space* H*. Let* F_1 *and* F_2 *be two bifunctions from* $C\times C \to \mathbb{R}$ *satisfying (A1)–(A4). Let* $\mu > 0$ *and let* $S_{1,\mu}$ *and* $S_{2,\mu}$ *be defined as in Lemma 2.10 associated to* F_1 *and* F_2*, respectively, such that* $\Omega\neq\emptyset$*. Suppose* $x_0 = x \in C$ *and* $\{x_n\}$*,* $\{y_n\}$*,* $\{z_n\}$ *are given by*

$$\begin{gathered}
z_n \in C, \quad F_2(z_n, z) + \varphi(z) - \varphi(z_n) + \frac{1}{\mu}\langle z - z_n, z_n - x_n\rangle \geq 0, \quad \forall z \in C, \\
y_n \in C, \quad F_1(y_n, z) + \varphi(z) - \varphi(y_n) + \frac{1}{\mu}\langle z - y_n, y_n - z_n\rangle \geq 0, \quad \forall z \in C, \\
x_{n+1} = \alpha_n x_n + (1-\alpha_n)y_n,
\end{gathered} \tag{3.20}$$

for all $n \in \mathbb{N}$*, where* $\{\alpha_n\}\subset[a,b]$ *for some* $a, b \in (0,1)$ *and satisfy* $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$*. Then* $\{x_n\}$ *converges weakly to* $\overline{x} = \lim_{n\to\infty}P_{\Omega}x_n$ *and* $(\overline{x}, \overline{y})$ *is a solution of problem* (1.9)*, where* $\overline{y} = S_{2,\mu}\overline{x}$*.*

Setting $F_2 = \varphi = 0$ in Theorem 3.1, we have the following result.

Corollary 3.4. *Let* C *be a closed convex subset of a real Hilbert space* H*. Let* F_1 *be a bifunction from* $C\times C \to \mathbb{R}$ *satisfying (A1)–(A4). Let* $\lambda > 0$ *and let* $T_{1,\lambda}$ *be defined as in Lemma 2.9 associated to* F_1*. Let* T *be a quasi-nonexpansive mapping of* C *into itself such that* $F(T)\cap \mathrm{EP}(F_1)\neq\emptyset$*. Suppose* $x_0 = x \in C$ *and* $\{x_n\}$ *and* $\{y_n\}$ *are given by*

$$\begin{gathered}
y_n \in C; \quad F_1(y_n, z) + \frac{1}{\lambda}\langle z - y_n, y_n - x_n\rangle \geq 0, \quad \forall z \in C, \\
x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n,
\end{gathered} \tag{3.21}$$

for all $n \in \mathbb{N}$*, where* $\{\alpha_n\}\subset[a,b]$ *for some* $a, b \in (0,1)$ *and satisfy* $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$*. Then* $\{x_n\}$ *converges weakly to* $\overline{x} = \lim_{n\to\infty}P_{F(T)\cap \mathrm{EP}(F)}x_n$ *and* $(\overline{x}, \overline{y})$ *is a solution of problem* (1.7)*, where* $\overline{y} = T_{1,\lambda}\overline{x}$*.*

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