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Research Article

Hardy-Littlewood and Caccioppoli-Type Inequalities for A-Harmonic Tensors

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We prove the new versions of the weighted Hardy-Littlewood inequality and Caccioppoli-type inequality for A-harmonic tensors. We also explore applications of our results to K-quasiregular mappings and p-harmonic functions in $\mathbf{R}^n$.

1. Introduction

The purpose of this paper is to prove the new versions of the weighted Hardy-Littlewood and Caccioppoli-type inequalities for the A-harmonic tensors. Our results may have applications in different fields, particularly, in the study of the integrability of solutions to the A-harmonic equation in some domains. Roughly speaking, the A-harmonic tensors are solutions of the A-harmonic equation, which is intimately connected to the fields, including potential theory, quasiconformal mappings, and the theory of elasticity. The investigation of the A-harmonic equation has developed rapidly in the recent years see [1–11].

In this paper, we still keep using the standard notations and symbols. All notations and definitions involved in this paper can be found in [1] cited in the paper. We always assume that M is a bounded and convex domain in $\mathbf{R}^n$, $n \geq 2$. We write $\mathbf{R} = \mathbf{R}^1$. Let $e_1, e_2, \ldots, e_n$ be the standard unit basis of $\mathbf{R}^n$ and $\wedge^l = \wedge^l(\mathbf{R}^n)$ the linear space of l-vectors, generated by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots e_{i_l}$, corresponding to all ordered l-tuples $I = (i_1, i_2, \ldots, i_l)$, $1 \leq i_1 < i_2 < \cdots < i_l \leq n$, $l = 0, 1, \ldots, n$. The Grassman algebra $\wedge = \oplus \wedge^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \wedge$ and $\beta = \sum \beta^I e_I \in \wedge$, the inner product in $\wedge$ is given by $\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$, with summation over all l-tuples $I = (i_1, i_2, \ldots, i_l)$ and all integers $l = 0, 1, \ldots, n$. We define the Hodge star operator $\star\colon \wedge \to \wedge$ by the rule $\star 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$

and $\alpha \wedge \star\beta = \beta \wedge \star\alpha = \langle \alpha, \beta \rangle (\star 1)$ for all $\alpha, \beta \in \wedge$. The norm of $\alpha \in \wedge$ is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star\alpha) \in \wedge^0 = \mathbf{R}$. The Hodge star is an isometric isomorphism on $\wedge$ with $\star : \wedge^l \to \wedge^{n-l}$ and $\star\star\, (-1)^{l(n-l)} : \wedge^l \to \wedge^l$.

It is well known that a differential l-form ω on M is a de Rham current (see [12, Chapter III]) on M with values in $\wedge^l(\mathbf{R}^n)$. Let $\Lambda^l M$ be the lth exterior power of the cotangent bundle. We use $D'(M, \Lambda^l)$ to denote the space of all differential l-forms and $L^p(\Lambda^l M)$ to denote the l-forms

$$\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \cdots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l} \tag{1.1}$$

on M satisfying $\int_M |\omega_I|^p < \infty$ for all ordered l-tuples I, where $I = (i_1, i_2, \ldots, i_l)$, $1 \le i_1 < i_2 < \cdots < i_l \le n$, and $\omega_{i_1 i_2 \cdots i_l}(x)$ are differentiable functions. Thus, $L^p(\Lambda^l M)$ is a Banach space with norm $\|\omega\|_{p,M} = \left(\int_M |\omega(x)|^p dx\right)^{1/p} = \left(\int_M \left(\sum_I |\omega_I(x)|^2\right)^{p/2} dx\right)^{1/p}$. Here, $|u(x)| = \left(\sum_I |\omega_I(x)|^2\right)^{1/2} = \left(\sum_I |\omega_{i_1 i_2 \cdots i_l}(x)|^2\right)^{1/2}$. We denote the exterior derivative by $d : D'(M, \Lambda^l) \to D'(M, \Lambda^{l+1})$ for $l = 0, 1, \ldots, n$. The Hodge codifferential operator $d^\star : D'(M, \Lambda^{l+1}) \to D'(M, \Lambda^l)$ is given by $d^\star = (-1)^{nl+1} \star d \star$ on $D'(M, \wedge^{l+1})$, $l = 0, 1, \ldots, n$. We use B to denote a ball and σB, $\sigma > 0$, is the ball with the same center as B and with $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$. We do not distinguish the balls from cubes in this paper. For any measurable set $E \subset \mathbf{R}^n$, we write $|E|$ for the n-dimensional Lebesgue measure of E. We call w a weight if $w \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $w > 0$ a.e.. For $0 < p < \infty$, we write $f \in L^p(\Lambda^l E, w^\alpha)$ if the weighted L^p-norm of f over E satisfies $\|f\|_{p,E,w^\alpha} = \left(\int_E |f(x)|^p w(x)^\alpha dx\right)^{1/p} < \infty$, where α is a real number. See [1] or [13] for more properties of differential forms.

For any differential k-form $u(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \cdots i_k}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$, $k = 1, 2, \ldots, n$, the vector-valued differential form ∇u is defined by

$$\begin{gathered}
\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right) = \left(\sum_I \frac{\partial u_I}{\partial x_1} dx_I, \sum_I \frac{\partial u_I}{\partial x_2} dx_I, \ldots, \sum_I \frac{\partial u_I}{\partial x_n} dx_I\right), \\
|\nabla u| = \left(\sum_{j=1}^n \left|\frac{\partial u}{\partial x_j}\right|^2\right)^{1/2} = \left(\sum_{j=1}^n \sum_I \left|\frac{\partial u_I}{\partial x_j}\right|^2\right)^{1/2}.
\end{gathered} \tag{1.2}$$

Also, we all know that

$$\begin{gathered}
du(x) = \sum_{k=1}^n \sum_{1 \le i_1 < i_2 < \cdots < i_k} \frac{\partial \omega_{i_1 i_2 \cdots i_k}(x)}{\partial x_k} dx_k \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}, \quad k = 0, 1, \ldots, n-1, \\
|du(x)| = \left(\sum_{k=1}^n \sum_{1 \le i_1 < i_2 < \cdots < i_k} \left|\frac{\partial \omega_{i_1 i_2 \cdots i_k}(x)}{\partial x_k}\right|^2\right)^{1/2}.
\end{gathered} \tag{1.3}$$

There has been remarkable work in the study of the A-harmonic equation

$$d^\star A(x, d\omega) = 0 \tag{1.4}$$

for differential forms, where $A : M \times \wedge^l(\mathbf{R}^n) \rightarrow \wedge^1(\mathbf{R}^n)$ satisfies the following conditions:

$$|A(x,\xi)| \leq a|\xi|^{p-1}, \qquad \langle A(x,\xi),\xi\rangle \geq |\xi|^p \tag{1.5}$$

for almost every $x \in M$ and all $\xi \in \wedge^l(\mathbf{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.4). A solution to (1.4) is an element of the Sobolev space $W^1_{p,\mathrm{loc}}(\Omega, \wedge^{l-1})$ such that $\int_\Omega \langle A(x,d\omega), d\varphi\rangle = 0$ for all $\varphi \in W^1_p(M,\wedge^{l-1})$ with compact support.

Definition 1.1. We call u an A-harmonic tensor on M if u satisfies the A-harmonic equation (1.4) on M.

A differential l-form $u \in D'(M,\wedge^l)$ is called a closed form if $du = 0$ on M. Similarly, a differential $l+1$-form $v \in D'(M,\wedge^{l+1})$ is called a coclosed form if $d^\star v = 0$. The equation

$$A(x,du) = d^\star v \tag{1.6}$$

is called the conjugate A-harmonic equation. Suppose that u is a solution to (1.4) in Ω. Then, at least locally in a ball B, there exists a form $v \in W^1_q(B,\wedge^{l+1})$, $1/p + 1/q = 1$, such that (1.6) holds.

Definition 1.2. When u and v satisfy (1.6) on M, and A^{-1} exists on M, we call u and v conjugate A-harmonic tensors on M.

Let $Q \subset \mathbf{R}^n$ be a cube or a ball. To each $y \in Q$ there corresponds a linear operator $K_y : C^\infty(Q,\wedge^l) \rightarrow C^\infty(Q,\wedge^{l-1})$ defined by $(K_y\omega)(x;\xi_1,\ldots,\xi_l) = \int_0^1 t^{l-1}\omega(tx + y - ty; x - y, \xi_1,\ldots,\xi_{l-1})dt$ and the decomposition $\omega = d(K_y\omega) + K_y(d\omega)$. The linear operator $T_Q : C^\infty(Q,\wedge^l) \rightarrow C^\infty(Q,\wedge^{l-1})$ is defined by averaging K_y over all points y in $Q T_Q\omega = \int_Q \varphi(y)K_y\omega\, dy$, where $\varphi \in C_0^\infty(Q)$ is normalized by $\int_Q \varphi(y)dy = 1$. See [1] for more property for the operator T_Q. We define the l-form $\omega_Q \in D'(Q,\wedge^l)$ by $\omega_Q = |Q|^{-1}\int_Q \omega(y)dy,\ l = 0$, and $\omega_Q = d(T_Q\omega),\ l = 1,2,\ldots,n$, for all $\omega \in L^p(Q,\wedge^l)$, $1 \leq p < \infty$.

2. The Local Hardy-Littlewood Inequality

We first introduce the following two-weight class which is an extension of A_r-weight and $A_r(\lambda)$-weights.

Definition 2.1. We say the weight $(w_1(x), w_2(x))$ satisfies the $A_r(\lambda, M)$ condition for $r > 1$ and $0 < \lambda < \infty$, write $(w_1, w_2) \in A_r(\lambda, M)$, if $w_1(x) > 0$, $w_2(x) > 0$ a.e., and

$$\sup_B \left(\frac{1}{|B|}\int_B w_1^\lambda dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w_2}\right)^{1/(r-1)} dx\right)^{(r-1)} < \infty \tag{2.1}$$

for any ball $B \subset M$.

If we choose $w_1 = w_2$ in Definition 2.1, we obtain the usual $A_r(\lambda)$-weights introduced in [7]. Also, if $\lambda = 1$ and $w_1 = w_2$, the above weight reduces to the well-known A_r-weight.

See [1, 14, 15] for more properties of weights. We will also need the following generalized Hölder inequality.

Lemma 2.2. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$, and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If f and g are measurable functions on $\mathbf{R}^n$, then*

$$\|fg\|_{s,M} \leq \|f\|_{\alpha,M} \cdot \|g\|_{\beta,M} \tag{2.2}$$

for any $M \subset \mathbf{R}^n$.

The following two versions of the Hardy-Littlewood integral inequality (Theorem A and Theorem B) appear in [16] and [9], respectively.

Theorem A. *For each $p > 0$, there is a constant C such that*

$$\int_D |u - u(0)|^p dx\, dy \leq C \int_D |v - v(0)|^p dx\, dy \tag{2.3}$$

for all analytic functions $f = u + iv$ in the unit disk D.

Theorem B. *Let u and v be conjugate A-harmonic tensors in $M \subset \mathbf{R}^n$, $\sigma > 1$, and $0 < s, t < \infty$. Then there exists a constant C, independent of u and v, such that*

$$\|u - u_B\|_{s,B} \leq C|B|^{\beta} \|v - c\|_{t,\sigma B}^{q/p} \tag{2.4}$$

for all balls B with $\sigma B \subset M$. Here c is any form in $W^1_{p,\text{loc}}(M, \Lambda)$ with $d^\star c = 0$ and $\beta = 1/s + 1/n - (1/t + 1/n)q/p$.

Now we prove the following local two-weight Hardy-Littlewood integral inequality.

Theorem 2.3. *Let u and v be conjugate A-harmonic tensors on $M \subset \mathbf{R}^n$ and $(w_1, w_2) \in A_r(\lambda, M)$ for some $r > 1$ and $\lambda > 0$. Let $0 < s, t < \infty$. Then there exists a constant C, independent of u and v, such that*

$$\left(\int_B |u - u_B|^s w_1^{\lambda/\alpha} dx \right)^{1/s} \leq C|B|^{\gamma} \left(\int_{\sigma B} |v - c|^t w_2^{pt/\alpha qs} dx \right)^{q/pt} \tag{2.5}$$

for all balls B with $\sigma B \subset M \subset \mathbf{R}^n$, $\sigma > 1$ and $\alpha > 1$. Here c is any form in $W^1_{q,\text{loc}}(M, \Lambda)$ with $d^* c = 0$ and $\gamma = 1/s + 1/n - (1/t + 1/n)q/p$.

Note that (2.5) can be written as the following symmetric form:

$$\left(\frac{1}{|B|} \int_B |u - u_B|^s w_1^{\lambda/\alpha} dx \right)^{1/qs} \leq C|B|^{(1/q - 1/p)/n} \left(\frac{1}{|B|} \int_{\sigma B} |v - c|^t w_2^{pt/\alpha qs} dx \right)^{1/pt}. \tag{2.6'}$$

Proof. Let $k = \alpha s/(\alpha-1)$. Since $\alpha > 1$, then $k > 0$ and $k > s$. Applying the Hölder inequality, we have

$$\begin{aligned}\left(\int_B |u-u_B|^s w_1^{\lambda/\alpha}dx\right)^{1/s} &= \left(\int_B \left(|u-u_B| w_1^{\lambda/\alpha s}\right)^s dx\right)^{1/s}\\ &\le \|u-u_B\|_{k,B}\left(\int_B w_1^{k\lambda/\alpha(k-s)}dx\right)^{(k-s)/ks}\\ &= \|u-u_B\|_{k,B}\left(\int_B w_1^{\lambda}dx\right)^{1/\alpha s}.\end{aligned} \tag{2.6}$$

Choose $m = \alpha qst/(\alpha qs + pt(r-1))$, then $m < t$. By Theorem B we have

$$\|u-u_B\|_{k,B} \le C_1|B|^{\beta}\|v-c\|_{m,\sigma B}^{q/p}, \tag{2.7}$$

where $\beta = 1/k + 1/n - (1/m+1/n)q/p$. Since $1/m = 1/t + (t-m)/mt$, by the Hölder inequality again, we obtain

$$\begin{aligned}\|v-c\|_{m,\sigma B} &= \left(\int_{\sigma B}\left(|v-c| w_2^{p/\alpha qs} w_2^{-p/\alpha qs}\right)^m dx\right)^{1/m}\\ &\le \left(\int_{\sigma B}|v-c|^t w_2^{pt/\alpha qs}dx\right)^{1/t}\left(\int_{\sigma B}\left(\frac{1}{w_2}\right)^{pmt/\alpha qs(t-m)}dx\right)^{(t-m)/mt}\\ &= \left(\int_{\sigma B}|v-c|^t w_2^{pt/\alpha qs}dx\right)^{1/t}\left(\int_{\sigma B}\left(\frac{1}{w_2}\right)^{1/(r-1)}dx\right)^{p(r-1)/\alpha qs}.\end{aligned} \tag{2.8}$$

Hence

$$\|v-c\|_{m,\sigma B}^{q/p} \le \left(\int_{\sigma B}\left(\frac{1}{w_2}\right)^{1/(r-1)}dx\right)^{(r-1)/\alpha s}\left(\int_{\sigma B}|v-c|^t w_2^{pt/\alpha qs}dx\right)^{q/pt}. \tag{2.9}$$

Combining (2.6), (2.7), and (2.9) yields

$$\begin{aligned}&\left(\int_B |u-u_B|^s w_1^{\lambda/\alpha}dx\right)^{1/s}\\ &\quad\le C_1|B|^{\beta}\left(\int_B w_1^{\lambda}dx\right)^{1/\alpha s}\left(\int_{\sigma B}\left(\frac{1}{w_2}\right)^{1/(r-1)}dx\right)^{(r-1)/\alpha s}\left(\int_{\sigma B}|v-c|^t w_2^{pt/\alpha qs}dx\right)^{q/pt}.\end{aligned} \tag{2.10}$$

Using the condition that $(w_1, w_2) \in A_r(\lambda, M)$, we obtain

$$\begin{aligned}
&\left(\int_B w_1^\lambda dx\right)^{1/\alpha s}\left(\int_{\sigma B}\left(\frac{1}{w_2}\right)^{1/(r-1)} dx\right)^{(r-1)/\alpha s} \\
&\leq |\sigma B|^{r/\alpha s}\left(\left(\frac{1}{|\sigma B|}\int_B w_1^\lambda dx\right)\left(\frac{1}{|\sigma B|}\int_{\sigma B}\left(\frac{1}{w_2}\right)^{1/(r-1)} dx\right)\right)^{1/\alpha s} \\
&\leq C_2|\sigma B|^{r/\alpha s} \\
&= C_3|B|^{r/\alpha s}.
\end{aligned} \tag{2.11}$$

Putting (2.11) into (2.10) and noting that $\beta + r/\alpha s = 1/k + 1/n - (1/m + 1/n)q/p + r/\alpha s = 1/s + 1/n - (1/t + 1/n)q/p$, we have

$$\left(\int_B |u - u_B|^s w_1^{\lambda/\alpha} dx\right)^{1/s} \leq C|B|^\gamma \left(\int_{\sigma B} |v - c|^t w_2^{pt/\alpha q s} dx\right)^{q/pt}, \tag{2.12}$$

where $\gamma = 1/s + 1/n - (1/t + 1/n)q/p$. We have completed the proof of Theorem 2.3. □

Note that in Theorem 2.3, $\alpha > 1$ is arbitrary. Hence, if we choose α to be some special values, we will have some different versions of the Hardy-Littlewood inequality. For example, if we let $\alpha = \lambda$, $\lambda > 1$. By Theorem 2.3, we have

$$\left(\int_B |u - u_B|^s w_1 dx\right)^{1/s} \leq C|B|^\gamma \left(\int_{\sigma B} |v - c|^t w_2^{pt/\lambda q s} dx\right)^{q/pt} \tag{2.13}$$

for all balls B with $\sigma B \subset M \subset \mathbf{R}^n$, $\sigma > 1$, and $\gamma = 1/s + 1/n - (1/t + 1/n)q/p$.

If we choose $\alpha = p$ in Theorem 2.3, we obtain the following result:

$$\left(\int_B |u - u_B|^s w_1^{\lambda/p} dx\right)^{1/s} \leq C|B|^\gamma \left(\int_{\sigma B} |v - c|^t w_2^{t/q s} dx\right)^{q/pt} \tag{2.14}$$

for all balls B with $\sigma B \subset M \subset \mathbf{R}^n$, $\sigma > 1$, and $\gamma = 1/s + 1/n - (1/t + 1/n)q/p$.

As an application of Theorem 2.3, we have the following example.

Example 2.4. Let $f(x) = (f^1, f^2, \ldots, f^n)$ be K-quasiregular in $\mathbf{R}^n$, then

$$u = f^l df^1 \wedge df^2 \wedge \cdots \wedge df^{l-1}, \qquad v = * f^{l+1} df^{l+2} \wedge \cdots \wedge df^n, \tag{2.15}$$

$l = 1, 2, \ldots, n-1$, are conjugate A-harmonic tensors with $p = n/l$ and $q = n/(n-l)$, where A is some operator satisfying (1.5). Then by Theorem 2.3, we obtain

$$\begin{aligned}&\left(\int_B \left| f^l df^1 \wedge df^2 \wedge \cdots \wedge df^{l-1} - \left(f^l df^1 \wedge df^2 \wedge \cdots \wedge df^{l-1}\right)_B \right|^s w_1^{\lambda/\alpha} dx\right)^{1/s} \\ &\quad \le C|B|^\gamma \left(\int_{\sigma B} |* f^{l+1} df^{l+2} \wedge \cdots \wedge df^n - c|^t w_2^{pt/\alpha q s} dx\right)^{q/pt},\end{aligned} \tag{2.16}$$

where C is independent of f, $\gamma = 1/s + 1/n - (1/t + 1/n)q/p$ and $d^* c = 0$.

For more examples of conjugate harmonic tensors, see [3]. We will have different versions of the global two-weight Hardy-Littlewood inequality if we choose α and λ to be some special values as we did in the local case. Recently, Xing and Ding introduced the following $A(\alpha, \beta, \gamma; E)$-weights in [17].

Definition 2.5. We say that a measurable function $g(x)$ defined on a subset $E \subset \mathbf{R}^n$ satisfies the $A(\alpha, \beta, \gamma; E)$-condition for some positive constants α, β, γ, write $g(x) \in A(\alpha, \beta, \gamma; E)$ if $g(x) > 0$ a.e., and

$$\sup_B \left(\frac{1}{|B|}\int_B g^\alpha dx\right)\left(\frac{1}{|B|}\int_B g^{-\beta} dx\right)^{\gamma/\beta} < \infty, \tag{2.17}$$

where the supremum is over all balls $B \subset E$. We say $g(x)$ satisfies the $A(\alpha, \beta; E)$-condition if (2.17) holds for $\gamma = 1$ and write $g(x) \in A(\alpha, \beta; E) = A(\alpha, \beta, 1; E)$.

We should notice that there are three parameters in the definition of the $A(\alpha, \beta, \gamma; E)$-weights. If we choose some special values for these parameters, we may obtain some existing weighted classes. For example, it is easy to see that the $A(\alpha, \beta, \gamma; E)$-class reduces to the usual $A_r(E)$-class if $\alpha = \gamma = 1$ and $\beta = 1/(r-1)$. Moreover, it has been proved in [17] that the $A_r(E)$-weight is a proper subset of the $A(\alpha, \beta, \gamma; E)$-weight. Using the similar method to the proof of Theorem 1.5.5 in [1], we can prove the following version of the Hardy-Littlewood inequality. Considering the length of the paper, we do not include the proof here.

Theorem 2.6. *Let u and v be conjugate A-harmonic tensors on $M \subset \mathbf{R}^n$ and $g(x) \in A(\alpha, \beta, \alpha; M)$ with $\alpha > 1$ and $\beta > 0$. Let $0 < s, t < \infty$. Then, there exists a constant C, independent of u and v, such that*

$$\left(\int_B |u - u_B|^s g\, dx\right)^{1/s} \le C|B|^\gamma \left(\int_{\sigma B} |v - c|^t g^{pt/qs} dx\right)^{q/pt} \tag{2.18}$$

for all balls B with $\sigma B \subset M \subset \mathbf{R}^n$ and $\sigma > 1$. Here c is any form in $W^1_{q,\mathrm{loc}}(M, \Lambda)$ with $d^* c = 0$ and $\gamma = 1/s + 1/n - (1/t + 1/n)q/p$.

Example 2.7. Let

$$u(x) = \frac{3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \tag{2.19}$$

be a harmonic function in $\mathbf{R}^3$ and v a 2-form in $\mathbf{R}^3$ defined by

$$v = v_3 dx_1 \wedge dx_2 + v_2 dx_1 \wedge dx_3 + v_1 dx_2 \wedge dx_3, \tag{2.20}$$

where v_1, v_2, and v_3 are defined as follows:

$$\begin{aligned}
v_1 &= \frac{x_2 x_3}{\sqrt{\sum x_i^2}} \frac{x_2^4 - x_3^4}{\prod_{i<j}\left(x_i^2 + x_j^2\right)} = \frac{x_2 x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \frac{x_2^2 - x_3^2}{\left(x_1^2 + x_2^2\right)\left(x_1^2 + x_3^2\right)}, \\
v_2 &= \frac{x_1 x_3}{\sqrt{\sum x_i^2}} \frac{x_1^4 - x_3^4}{\prod_{i<j}\left(x_i^2 + x_j^2\right)} = \frac{x_1 x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \frac{x_1^2 - x_3^2}{\left(x_1^2 + x_2^2\right)\left(x_2^2 + x_3^2\right)}, \\
v_3 &= \frac{x_1 x_2}{\sqrt{\sum x_i^2}} \frac{x_1^4 - x_2^4}{\prod_{i<j}\left(x_i^2 + x_j^2\right)} = \frac{x_1 x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \frac{x_1^2 - x_2^2}{\left(x_1^2 + x_3^2\right)\left(x_2^2 + x_3^2\right)}.
\end{aligned} \tag{2.21}$$

Then u and v are a pair of conjugate harmonic tensors; see [3]. Hence, the Hardy-Littlewood inequality is applicable. Using inequality (2.5) with $w_1 = w_2 = 1$ and $c = 0$ over any ball B, we can obtain the norm comparison inequality for u and v defined by (2.19) and (2.20), respectively.

3. The Local Caccioppoli-Type Inequality

The purpose of this section is to obtain some estimates which give upper bounds for the L^p-norm of ∇u or du in terms of the corresponding norm u or $u - c$, where u is a differential form satisfying the A-harmonic equation (1.4) and c is any closed form. These kinds of estimates are called the Caccioppoli-type estimates or the Caccioppoli inequalities. From [9], we can obtain the following Caccioppoli-type inequality.

Theorem C. *Let u be an A-harmonic tensor on M and let $\sigma > 1$. Then there exists a constant C, independent of u, such that*

$$\|du\|_{s,B} \leq C \operatorname{diam}(B)^{-1} \|u - c\|_{s,\sigma B} \tag{3.1}$$

for all balls or cubes B with $\sigma B \subset M$ and all closed forms c. Here $1 < s < \infty$.

The following weak reverse Hölder inequality appears in [9].

Theorem D. *Let u be an A-harmonic tensor in Ω, $\sigma > 1$ and $0 < s,t < \infty$. Then there exists a constant C, independent of u, such that*

$$\|u\|_{s,B} \le C|B|^{(t-s)/st}\|u\|_{t,\sigma B} \tag{3.2}$$

for all balls or cubes B with $\sigma B \subset \Omega$.

Now, we prove the following local two-weight Caccioppoli-type inequality for A-harmonic tensors.

Theorem 3.1. *Let $u \in D'(M, \wedge^l)$, $l = 0,1,\ldots,n$, be an A-harmonic tensor on $M \subset \mathbf{R}^n$, $\rho > 1$ and $0 < \alpha < 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation and $(w_1, w_2) \in A_r(\lambda, M)$ for some $r > 1$ and $\lambda > 0$. Then there exists a constant C, independent of u, such that*

$$\left(\int_B |du|^s w_1^{\alpha\lambda} dx\right)^{1/s} \le \frac{C}{\operatorname{diam}(B)}\left(\int_{\rho B} |u-c|^s w_2^{\alpha} dx\right)^{1/s} \tag{3.3}$$

for all balls B with $\rho B \subset M$ and all closed forms c.

Proof. Choose $t = s/(1-\alpha)$, then $1 < s < t$. Since $1/s = 1/t + (t-s)/st$, by Hölder inequality and Theorem C, we have

$$\begin{aligned}\left(\int_B |du|^s w_1^{\alpha\lambda} dx\right)^{1/s} &= \left(\int_B \left(|du| w_1^{\alpha\lambda/s}\right)^s dx\right)^{1/s} \\ &\le \left(\int_B |du|^t dx\right)^{1/t}\left(\int_B \left(w_1^{\alpha\lambda/s}\right)^{st/(t-s)} dx\right)^{(t-s)/st} \\ &\le \|du\|_{t,B}\cdot\left(\int_B w_1^{\lambda} dx\right)^{\alpha/s} \\ &= C_1 \operatorname{diam}(B)^{-1}\|u-c\|_{t,\sigma B}\left(\int_B w_1^{\lambda} dx\right)^{\alpha/s}\end{aligned} \tag{3.4}$$

for all balls B with $\sigma B \subset \Omega$ and all closed forms c. Since c is a closed form and u is an A-harmonic tensor, then $u - c$ is still an A-harmonic tensor. Taking $m = s/(1+\alpha(r-1))$, we find that $m < s < t$. Applying Theorem D yields

$$\begin{aligned}\|u-c\|_{t,\sigma B} &\le C_2|B|^{(m-t)/mt}\|u-c\|_{m,\sigma^2 B} \\ &= C_2|B|^{(m-t)/mt}\|u-c\|_{m,\rho B},\end{aligned} \tag{3.5}$$

where $\rho = \sigma^2$. Substituting (3.5) in (3.4), we have

$$\left(\int_B |du|^s w_1^{\alpha\lambda} dx\right)^{1/s} \le C_3 \operatorname{diam}(B)^{-1}|B|^{(m-t)/mt}\|u-c\|_{m,\rho B}\left(\int_B w_1^{\lambda} dx\right)^{\alpha/s}. \tag{3.6}$$

Now $1/m = 1/s + (s-m)/sm$, by the Hölder inequality again, we obtain

$$\begin{aligned}\|u-c\|_{m,\rho B} &= \left(\int_{\rho B} |u-c|^m dx\right)^{1/m} \\ &= \left(\int_{\rho B} \left(|u-c| w_2^{\alpha/s} w_2^{-\alpha/s}\right)^m dx\right)^{1/m} \\ &\le \left(\int_{\rho B} |u-c|^s w_2^{\alpha} dx\right)^{1/s} \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{1/(r-1)} dx\right)^{\alpha(r-1)/s}\end{aligned} \tag{3.7}$$

for all balls B with $\rho B \subset \Omega$ and all closed forms c. Combining (3.6) and (3.7), we obtain

$$\left(\int_B |du|^s w_1^{\alpha\lambda} dx\right)^{1/s} \le C_3 \operatorname{diam}(B)^{-1} |B|^{(m-t)/mt} \|w_1\|_{\lambda,B}^{\alpha\lambda/s} \left\|\frac{1}{w_2}\right\|_{1/(r-1),\rho B}^{\alpha/s} \left(\int_{\rho B} |u-c|^s w_2^{\alpha} dx\right)^{1/s}. \tag{3.8}$$

Since $(w_1, w_2) \in A_r(\lambda, M)$, then we have

$$\begin{aligned}\|w_1\|_{\lambda,B}^{\alpha\lambda/s} \cdot \left\|\frac{1}{w_2}\right\|_{1/(r-1),\rho B}^{\alpha/s} &\le \left(\left(\left(\int_{\rho B} w_1^{\lambda} dx\right)\left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{1/(r-1)} dx\right)^{r-1}\right)\right)^{\alpha/s} \\ &= \left(|\rho B|^r \left(\frac{1}{|\rho B|}\int_{\rho B} w_1^{\lambda} dx\right)\left(\frac{1}{|\rho B|}\int_{\rho B} \left(\frac{1}{w_2}\right)^{1/(r-1)} dx\right)^{r-1}\right)^{\alpha/s} \\ &\le C_4 |B|^{\alpha r/s}.\end{aligned} \tag{3.9}$$

Substituting (3.9) in (3.8), we find that

$$\left(\int_B |du|^s w_1^{\alpha\lambda} dx\right)^{1/s} \le \frac{C}{\operatorname{diam}(B)} \left(\int_{\rho B} |u-c|^s w_2^{\alpha} dx\right)^{1/s} \tag{3.10}$$

for all balls B with $\rho B \subset M$ and all closed forms c. This ends the proof of Theorem 3.1. □

Note that if $\lambda = 1$, then $A_r(\lambda, M) = A_r(1, M)$ becomes the usual $A_r(M)$ weight. See [14] for the properties of $A_r(M)$ weights. Thus, choosing $\lambda = 1$ and $w_1 = w_2$ in Theorem 3.1, we have the following $A_r(M)$-weighted Caccioppoli-type inequality.

Theorem 3.2. *Let $u \in D'(M, \wedge^l)$, $l = 0, 1, \ldots, n$, be an A-harmonic tensor in a domain $M \subset \mathbf{R}^n$, $\rho > 1$ and $0 < \alpha < 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation and $w \in A_r(M)$ for some $r > 1$. Then there exists a constant C, independent of u, such that*

$$\left(\int_B |du|^s w^\alpha dx\right)^{1/s} \le \frac{C}{\operatorname{diam}(B)}\left(\int_{\rho B} |u - c|^s w^\alpha dx\right)^{1/s} \tag{3.11}$$

for all balls B with $\rho B \subset M$ and all closed forms c.

We also need to note that in Theorem 3.1α is a parameter with $0 < \alpha < 1$. Thus, we will obtain different versions of the Caccioppoli-type inequality if we let α be some particular values. For example, putting $\alpha = 1/s$, we have the following result.

Theorem 3.3. *Let $u \in D'(M, \wedge^l)$, $l = 0, 1, \ldots, n$, be an A-harmonic tensor in a domain $M \subset \mathbf{R}^n$ and $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation and $(w_1, w_2) \in A_r(\lambda, M)$ for some $r > 1$ and $\lambda > 0$. Then there exists a constant C, independent of u, such that*

$$\left(\int_B |du|^s w_1^{\lambda/s} dx\right)^{1/s} \le \frac{C}{\operatorname{diam}(B)}\left(\int_{\rho B} |u - c|^s w_2^{1/s} dx\right)^{1/s} \tag{3.12}$$

for all balls B with $\rho B \subset M$ and all closed forms c.

If we choose $\alpha = 1/s$ in Theorem 3.2, then $0 < \alpha < 1$ since $1 < s < \infty$. Thus, Theorem 3.2 reduces to the following version.

Theorem 3.4. *Let $u \in D'(M, \wedge^l)$, $l = 0, 1, \ldots, n$, be an A-harmonic tensor in a domain $M \subset \mathbf{R}^n$ and $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation and $w \in A_r(M)$ for some $r > 1$. Then there exists a constant C, independent of u, such that*

$$\left(\int_B |du|^s w^{1/s} dx\right)^{1/s} \le \frac{C}{\operatorname{diam}(B)}\left(\int_{\rho B} |u - c|^s w^{1/s} dx\right)^{1/s} \tag{3.13}$$

for all balls B with $\rho B \subset M$ and all closed forms c.

Example 3.5. Let $A : M \times \wedge^l(\mathbf{R}^n) \to \wedge^l(\mathbf{R}^n)$ be an operator defined by $A(x, \xi) = \xi|\xi|^{p-2}$. Then A satisfies the condition (1.5). Equation (1.4) reduces to the p-harmonic equation

$$d^\star\left(du|u|^{p-2}\right) = 0 \tag{3.14}$$

and (1.6) reduces to the conjugate p-harmonic equation

$$du|u|^{p-2} = d^{\star}v \tag{3.15}$$

for differential forms, respectively. If u is a function (0-form), (3.14) reduces to the usual p-harmonic equation

$$\operatorname{div}\left(\nabla u|\nabla u|^{p-2}\right) = 0. \tag{3.16}$$

Also, (3.16) becomes the usual Laplace equation if we let $p = 2$ in (3.16). Now assume that u is a solution to (3.14). By theorems obtained above, we know that u satisfies (3.3), (3.11), (3.12), and (3.13), respectively.

The following example appeared in [18] which shows us how to use the Caccioppoli inequality to estimate the norm of the harmonic function u in $\mathbf{R}^2$.

Example 3.6. Let $u(x, y)$ be a function (0-form) defined in $\mathbf{R}^2$ by

$$u(x,y) = \frac{1}{\pi}\left(\arctan\frac{y}{x-1} - \arctan\frac{y}{x+1}\right). \tag{3.17}$$

It is easy to check that $u(x, y)$ satisfies the Laplace equation $u_{xx}(x, y) + u_{yy}(x, y) = 0$ in the upper half-plane; that is, $u(x, y)$ is a harmonic function in the upper half-plane. Let $r > 0$ be a constant, (x_0, y_0) be a fixed point with $y_0 > r$, and $B = \{(x,y) : \ (x-x_0)^2 + (y-y_0)^2 \le r^2\}$. To obtain the upper bound for the L^s-norm $\|du(x, y)\|_{s,B}$ with $s > 1$, it would be very complicated if we evaluate the integral $(\int_B |du(x,y)|^s dx \wedge dy)^{1/s}$ directly. However, using Caccioppoli inequality (3.11) with $w(x) = 1$ and $n = 2$, we can easily obtain the upper bound of the norm $\|du(x, y)\|_{s,B}$ as follows. First, we know that $|B| = \pi r^2$ and

$$\begin{aligned} |u(x,y)| &\le \frac{1}{\pi}\left|\arctan\frac{y}{x-1} - \arctan\frac{y}{x+1}\right| \\ &\le \frac{1}{\pi}\left|\arctan\frac{y}{x-1}\right| + \left|\arctan\frac{y}{x+1}\right| \\ &\le \frac{1}{\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = 1. \end{aligned} \tag{3.18}$$

Applying (3.11) and (3.18), we have

$$\begin{aligned}
\|du(x,y)\|_{s,B} &= \left(\int_B |du(x,y)|^s dx \wedge dy\right)^{1/s} \\
&\le C|B|^{-1/2}\left(\int_{\sigma B} |u(x,y)|^s dx \wedge dy\right)^{1/s} \\
&\le C\pi^{-1/2}r^{-1}\left(\int_{\sigma B} dx \wedge dy\right)^{1/s} \\
&= C\pi^{-1/2}r^{-1}\left(\pi(\sigma r)^2\right)^{1/s} \\
&= C\pi^{1/s-1/2}r^{2/s-1}\sigma^{2/s} \\
&= C\left(\pi^{2-s}r^{4-2s}\sigma^4\right)^{1/2s}.
\end{aligned} \tag{3.19}$$

4. The Global Hardy-Littlewood Inequality

Finally, we should notice that the local Hardy-Littlewood inequality can be extended into the global case in the John domain. A proper subdomain $\Omega \subset \mathbf{R}^n$ is called a δ-John domain, $\delta > 0$, if there exists a point $x_0 \in \Omega$ which can be joined with any other point $x \in \Omega$ by a continuous curve $\gamma \subset \Omega$ so that

$$d(\xi, \partial\Omega) \ge \delta|x - \xi| \tag{4.1}$$

for each $\xi \in \gamma$. Here $d(\xi, \partial\Omega)$ is the Euclidean distance between ξ and $\partial\Omega$.

Using the properties of John domain and the well-known Covering Lemma, we can prove the following global two-weight Hardy-Littlewood inequality.

Theorem 4.1. *Let $u \in D'(\Omega, \Lambda^0)$ and $v \in D'(\Omega, \Lambda^2)$ be conjugate A-harmonic tensors in a John domain Ω. Assume that $q \le p$, $v - c \in L^t(\Omega, \Lambda^2)$, $(w_1, w_2) \in A_r(\lambda, \Omega)$, and $w_1 \in A_r(\Omega)$ for some $r > 1$ and $\lambda > 0$. If s is defined by $s = npt/(nq + t(q-p))$, $0 < t < \infty$, then there exists a constant C, independent of u and v, such that*

$$\left(\int_\Omega |u - u_{Q_0}|^s w_1^{\lambda/\alpha} dx\right)^{1/s} \le C\left(\int_\Omega |v - c|^t w_2^{pt/\alpha qs} dx\right)^{q/pt} \tag{4.2}$$

for any real number $\alpha > 1$. Here c is any form in $W^1_{q,\mathrm{loc}}(\Omega, \Lambda)$ with $d^\star c = 0$ and $Q_0 \subset \Omega$ is a fixed cube.

It is easy to see that our global results can also be used to study K-quasiregular mappings and p-harmonic functions in $\mathbf{R}^n$ as we did in the local cases. Similar to the local case, some global versions of the two-weight inequalities will be obtained if we choose λ and α to be some special values in Theorem 4.1. Considering the length of the paper, we do not list these similar results here.

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