

Research Article

On Jordan Type Inequalities for Hyperbolic Functions

R. Klén, M. Visuri, and M. Vuorinen

Department of Mathematics, University of Turku, 20014 Turku, Finland

Correspondence should be addressed to R. Klén, riku.klen@utu.fi

Received 28 January 2010; Accepted 29 April 2010

Academic Editor: Andrea Laforgia

Copyright © 2010 R. Klén et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper deals with some inequalities for trigonometric and hyperbolic functions such as the Jordan inequality and its generalizations. In particular, lower and upper bounds for functions such as $(\sin x)/x$ and $x/\sinh x$ are proved.

1. Introduction

During the past several years there has been a great deal of interest in trigonometric inequalities [1–7]. The classical Jordan inequality [8, page 31]

$$\frac{2}{\pi}x \leq \sin x \leq x, \quad 0 < x < \frac{\pi}{2} \quad (1.1)$$

has been in the focus of these studies and many refinements have been proved for it by Wu and Srivastava [9, 10], Zhang et al. [11], J.-L. Li and Y.-L. Li [5, 12], Wu and Debnath [13–15], Özban [16], Qi et al. [17], Zhu [18–29], Sándor [30, 31], Baricz and Wu [32, 33], Neuman and Sándor [34], Agarwal et al. [35], Niu et al. [36], Pan and Zhu [37], and Qi and Guo [38]. For a long list of recent papers on this topic see [7] and for an extensive survey see [17]. The proofs are based on familiar methods of calculus. In particular, a method based on a l'Hospital type criterion for monotonicity of the quotient of two functions from Anderson et al. [39] is a key tool in these studies. Some other applications of this criterion are reviewed in [40, 41]. Pinelis has found several applications of this criterion in [42] and in several other papers.

The inequality

$$\frac{1 + \cos x}{2} \leq \frac{\sin x}{x} \leq \frac{2 + \cos x}{3}, \quad (1.2)$$

where $x \in (-\pi/2, \pi/2)$ is well-known and it was studied recently by Baricz in [43, page 111].

The second inequality of (1.2) is given in [8, page 354, 3.9.32] for $0 \leq x \leq \pi$. For a refinement of the first inequality in (1.2) see Remark 1.3(1) and of the second inequality see Theorem 2.4.

This paper is motivated by these studies and it is based on the Master Thesis of Visuri [44]. Some of our main results are the following theorems.

Theorem 1.1. For $x \in (0, \pi/2)$

$$\frac{x^2}{\sinh^2 x} < \frac{\sin x}{x} < \frac{x}{\sinh x}. \quad (1.3)$$

Theorem 1.2. For $x \in (0, 1)$

$$\left(\frac{1}{\cosh x}\right)^{1/2} < \frac{x}{\sinh x} < \left(\frac{1}{\cosh x}\right)^{1/4}. \quad (1.4)$$

We will consider quotients $\sin x/x$ and $x/\sinh x$ at origin as limiting values $\lim_{x \rightarrow 0} \sin x/x = 1$ and $\lim_{x \rightarrow 0} x/\sinh x = 1$.

Remark 1.3. (1) Let

$$g_1(x) = \frac{1 + 2 \cos x}{3} + \frac{x \sin x}{6}, \quad g_2(x) = \frac{1 + \cos x}{2}. \quad (1.5)$$

Then $g_1(x) - g_2(x) > 0$ on $(0, \pi/2)$ because

$$\frac{d}{dx}(g_1(x) - g_2(x)) = \frac{x \cos x}{6} > 0. \quad (1.6)$$

In [45, (27)] it is proved that for $x \in (0, \pi/2)$

$$\frac{\sin x}{x} \geq g_1(x). \quad (1.7)$$

Hence $(1 + 2 \cos x)/3 + (x \sin x)/6$ is a better lower bound for $(\sin x)/x$ than (1.2) for $x \in (0, \pi/2)$.

(2) Observe that

$$\frac{2 + \cos x}{3} = \frac{2 + 2\cos^2(x/2) - 1}{3} \leq \cos \frac{x}{2}, \quad (1.8)$$

which holds true as equality if and only if $\cos(x/2) = (3 \pm 1)/4$. In conclusion, (1.8) holds for all $x \in (-2\pi/3, 2\pi/3)$. Together with (1.2) we now have

$$\cos^2 \frac{x}{2} = \frac{1 + \cos x}{2} > \cos x, \quad (1.9)$$

and by (1.8)

$$\cos^2 \frac{x}{2} < \frac{\sin x}{x} < \cos \frac{x}{2}. \quad (1.10)$$

2. Jordan's Inequality

In this section we will find upper and lower bounds for $(\sin x)/x$ by using hyperbolic trigonometric functions.

Theorem 2.1. For $x \in (0, \pi/2)$

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \frac{x}{\sinh x}. \quad (2.1)$$

Proof. The lower bound of $\sin x/x$ holds true if the function $f(x) = \sin x \cosh x - x$ is positive on $(0, \pi/2)$. Since

$$f''(x) = 2 \cos x \sinh x, \quad (2.2)$$

we have $f''(x) > 0$ for $x \in (0, \pi/2)$ and $f'(x)$ is increasing on $(0, \pi/2)$. Therefore

$$f'(x) = \cos x \cosh x + \sin x \sinh x - 1 > f'(0) = 0, \quad (2.3)$$

and the function $f(x)$ is increasing on $(0, \pi/2)$. Now $f(x) > f(0) = 0$ for $x \in (0, \pi/2)$.

The upper bound of $\sin x/x$ holds true if the function $g(x) = x^2 - \sin x \sinh x$ is positive on $(0, \pi/2)$. Let us denote $h(x) = \tan x - \tanh x$. Since $\cos x < 1 < \cosh x$ for $x \in (0, \pi/2)$ we have $h'(x) = \cosh^{-2} x - \cos^{-2} x > 0$ and $h(x) > h(0) = 0$ for $x \in (0, \pi/2)$. Now

$$g'''(x) = 2(\cos x \cosh x)h(x), \quad (2.4)$$

which is positive on $(0, \pi/2)$, because $\cos x \cosh x > 0$ and $h(x) > 0$ for $x \in (0, \pi/2)$. Therefore

$$\begin{aligned} g''(x) &= 2(1 - \cos x \cosh x) > g''(0) = 0, \\ g'(x) &= 2x - \cos x \sinh x - \sin x \cosh x > g'(0) = 0 \end{aligned} \quad (2.5)$$

for $x \in (0, \pi/2)$. Now $g(x) > g(0) = 0$ for $x \in (0, \pi/2)$. □

Proof of Theorem 1.1. The upper bound of $\sin x/x$ is clear by Theorem 2.1. The lower bound of $\sin x/x$ holds true if the function $f(x) = \sin x \sinh^2 x - x^3$ is positive on $(0, \pi/2)$.

Let us assume $x \in (0, \pi/2)$. Since $\sin x > x - x^3/6 = (6x - x^3)/6$ we have $f(x) > ((6x - x^3)\sinh^2 x)/6 - x^3$. We will show that

$$g(x) = \frac{6 - x^2}{6} \sinh^2 x - x^2 \quad (2.6)$$

is positive which implies the assertion.

Now $g(x) > 0$ is equivalent to

$$\frac{\sinh x}{x} > \frac{\sqrt{6}}{\sqrt{6-x^2}}. \quad (2.7)$$

Since $x^{-1} \sinh x > 1 + x^2/6$ it is sufficient to show that $1 + x^2/6 > \sqrt{6}/\sqrt{6-x^2}$, which is equivalent to

$$x^2(-x^4 - 6x^2 + 36) > 0. \quad (2.8)$$

Let us denote $h(x) = -x^4 - 6x^2 + 36$. Now $h'(x) = -4x(x^2 + 3)$ and therefore $h'(x) \neq 0$ and $h(x) > \min\{h(0), h(\pi/2)\} > 0$. Therefore inequality (2.8) holds for $x \in (0, \pi/2)$ and the assertion follows. \square

We next show that for $x \in (0, 1)$ the upper and lower bounds of (1.2) are better than the upper and lower bounds in Theorem 2.1.

Theorem 2.2. (i) For $x \in (-1, 1)$

$$\frac{2 + \cos x}{3} \leq \frac{x}{\sinh x}. \quad (2.9)$$

(ii) For $x \in (-\pi/2, \pi/2)$

$$\frac{1}{\cosh x} \leq \frac{1 + \cos x}{2} = \cos^2 \frac{x}{2}. \quad (2.10)$$

(iii) For $x \in (-\pi/2, \pi/2)$

$$\frac{1}{1 + \sin^2 x} \leq \frac{1 + \cos^2 x}{2} \leq \frac{1 + \cos x}{2}. \quad (2.11)$$

Proof. (i) The claim holds true if the function $f(x) = 3x - 2 \sinh x - \sinh x \cos x$ is nonnegative on $[0, 1)$. By a simple computation we obtain $f''(x) = 2(\cosh x \sin x - \sinh x)$. Inequality $f''(x) \geq 0$ is equivalent to $\sin x \geq \tanh x$. By the series expansions of $\sin x$ and $\tanh x$ we obtain

$$\begin{aligned} \sin x - \tanh x &= \sum_{n \geq 3, n \equiv 1 \pmod{2}} \frac{(-1)^{(n-1)/2} (n+1) - 2^{n+1} (2^{n+1} - 1) B_{n+1}}{(n+1)!} x^n \\ &= \sum_{n \geq 3, n \equiv 1 \pmod{2}} c_n x^n, \end{aligned} \quad (2.12)$$

where B_j is the j th Bernoulli number. By the properties of the Bernoulli numbers $c_1 = 1/6$, $c_3 = -1/8$, coefficients c_n , for $n \equiv 1(2)$, form an alternating sequence, $|c_n x^n| \rightarrow 0$ as $n \rightarrow \infty$ and $|c_{2m+1}| > |c_{2m+3}|$ for $m \geq 1$. Therefore by Leibniz Criterion

$$\sin x - \tanh x \geq \frac{x^3}{6} - \frac{x^5}{8} = \frac{x^3}{24}(4 - 3x^2) \quad (2.13)$$

and $\sin x \geq \tanh x$ for all $x \in [0, 1)$. Now $f(x)$ is a convex function on $[0, 1)$ and $f'(x)$ is nondecreasing on $[0, 1)$ with $f'(0) = 0$. Therefore $f(x)$ is nondecreasing and $f(x) \geq f(0) = 0$.

- (ii) The claim holds true if the function $g(x) = \cosh x(1 + \cos x) - 2$ is nonnegative on $[0, \pi/2)$. By the series expansion of $\cos x$ we have $\cos x - 1 + x^2/2 \geq 0$ and therefore by the series expansion of $\cosh x$

$$\begin{aligned} g(x) &\geq \left(1 + \frac{x^2}{2}\right)(1 + \cos x) - 2 \\ &= \cos x - 1 + \frac{x^2}{2} + \frac{x^2 \cos x}{2} \\ &\geq \cos x - 1 + \frac{x^2}{2} \geq 0, \end{aligned} \quad (2.14)$$

and the assertion follows.

- (iii) Clearly we have

$$(1 + \cos^2 x)(1 + \sin^2 x) = 2 + \sin^2 x \cos^2 x \geq 2, \quad (2.15)$$

which implies the first inequality of the claim. The second inequality is trivial since $\cos x \in (0, 1)$. \square

Theorem 2.3. Let $x \in (0, \pi/2)$. Then

- (i) the function

$$f(t) = \cos^t \frac{x}{t} \quad (2.16)$$

is increasing on $(1, \infty)$,

- (ii) the function

$$g(t) = \sin^t \frac{x}{t} \quad (2.17)$$

is decreasing on $(1, \infty)$,

- (iii) the functions $\bar{f}(t) = \cosh^t(x/t)$ and $\bar{g}(t) = \sinh^t(x/t)$ are decreasing on $(0, \infty)$.

Proof. (i) Let us consider instead of $f(x)$ the function

$$f_1(y) = \frac{x}{y} \log \cos y \quad (2.18)$$

for $y \in (0, x)$. Note that $f(t) = \exp(f_1(x/t))$ and therefore the claim is equivalent to the function $f_1(y)$ being decreasing on $(0, x)$. We have

$$f_1'(y) = -\frac{x}{y^2} (\log \cos y + y \tan y), \quad (2.19)$$

and $f_1'(y) \leq 0$ is equivalent to $f_2(y) = \log \cos y + y \tan y \geq 0$. Since $f_2'(y) = y/\cos^2 y \geq 0$ we have $f_2(y) \geq f_2(0) = 0$. Therefore $f(t)$ is increasing on $(1, \infty)$.

(ii) We will consider instead of $g(x)$ the function

$$g_1(y) = \frac{x}{y} \log \sin y \quad (2.20)$$

for $y \in (0, x)$. Note that $g(t) = \exp(g_1(x/t))$ and therefore the claim is equivalent to the function $g_1(y)$ being increasing on $(0, x)$. We have

$$g_1'(y) = \frac{x}{y^2} \left(\frac{y}{\tan y} - \log \cos y \right), \quad (2.21)$$

and $g_1'(y) \geq 0$ is equivalent to $g_2(y) = y/\tan y - \log \cos y \geq 0$. Since $g_2'(y) = ((1/\cos y) - (y/\sin y))/\sin y \geq 0$ we have $g_2(y) \geq g_2(0) = 1$. Therefore $g_1'(y) \geq 0$ and the assertion follows.

(iii) We will show that $h_1(y) = (x/y) \log \cosh y$ is increasing on $(0, \infty)$. Now $h_1'(y) = (x(y \tanh y - \log \cosh y))/y^2$,

$$\frac{d(y \tanh y - \log \cosh y)}{dy} = \frac{y}{\cosh^2 y} > 0, \quad (2.22)$$

and $y \tanh y - \log \cosh y \geq 0$. Therefore the function $h_1(y)$ is increasing on $(0, \infty)$ and $\bar{f}(t)$ is decreasing on $(0, \infty)$.

We will show that $h_2(y) = (x/y) \log \sinh y$ is increasing on $(0, \infty)$. Now $h_2'(y) = (x(y \coth y - \log \sinh y))/y^2$,

$$\frac{d(y \coth y - \log \sinh y)}{dy} = -\frac{y}{\sinh^2 y} < 0, \quad (2.23)$$

and $y \coth y - \log \sinh y \geq \lim_{y \rightarrow \infty} y \coth y - \log \sinh y = 0$. Therefore the function $h_2(y)$ is increasing on $(0, \infty)$ and $\bar{g}(t)$ is decreasing on $(0, \infty)$. \square

We next will improve the upper bound of (1.2).

Theorem 2.4. For $x \in (-\sqrt{27/5}, \sqrt{27/5})$

$$\cos^2 \frac{x}{2} \leq \frac{\sin x}{x} \leq \cos^3 \left(\frac{x}{3} \right) \leq \frac{2 + \cos x}{3}. \quad (2.24)$$

Proof. The first inequality of (2.24) follows from (1.2).

By the series expansions of $\sin x$ and $\cos x$

$$\frac{\sin x}{x} \leq 1 - \frac{x^2}{6} + \frac{x^4}{120} \leq \left(1 - \frac{x^2}{18} \right)^3 \leq \cos^3 \left(\frac{x}{3} \right), \quad (2.25)$$

where the second inequality is equivalent to $x^4(27-5x^2)/29160 \geq 0$ and the second inequality of (2.24) follows.

By the identity $\cos^3 x = (\cos 3x + 3 \cos x)/4$ the upper bound of (2.24) is equivalent to $0 \leq 8 + \cos x - 9 \cos(x/3)$. By the series expansion of $\cos x$

$$8 + \cos x - 9 \cos \left(\frac{x}{3} \right) = \sum_{n=2}^{\infty} (-1)^n \frac{3^{2n} - 9}{3^{2n} (2n)!} x^{2n}, \quad (2.26)$$

and by the Leibniz Criterion the assertion follows. \square

3. Hyperbolic Jordan's Inequality

In this section we will find upper and lower bounds for the functions $x/\sinh x$ and $\cosh x$.

Theorem 3.1. For $x \in (-\pi/2, \pi/2)$

$$1 - \frac{x^2}{6} \leq \frac{\sin x}{x} \leq 1 - \frac{2x^2}{3\pi^2}. \quad (3.1)$$

Proof. We obtain from the series expansion of $\sin x$

$$\frac{\sin x}{x} \geq 1 - \frac{x^2}{6}, \quad (3.2)$$

which proves the lower bound.

By using the identity $1 - \cos x = 2\sin^2(x/2)$ the chain of inequalities (1.2) gives

$$\frac{\sin x}{x} \leq 1 - \frac{2\sin^2(x/2)}{3} \quad (3.3)$$

and the assertion follows from inequality $\sin^2(x/2) \geq (x/\pi)^2$. \square

Remark 3.2. J.-L. Li and Y.-L. Li have proved [12, (4.9)] that

$$\frac{\sin x}{x} < p(x) \left(1 - \frac{x^2}{\pi^2} \right) < 1 - \frac{x^2}{\pi^2}, \quad 0 < x < \pi, \quad (3.4)$$

where $p(x) = 1/\sqrt{1 + 3(x/\pi)^4} < 1$. This result improves Theorem 3.1.

Lemma 3.3. For $x \in (0, 1)$

- (i) $\sinh x < x + x^3/5$,
- (ii) $\cosh x < 1 + 5x^2/9$,
- (iii) $1/\cosh x < 1 - x^2/3$.

Proof. (i) For $x \in (0, 1)$ we have $x^2(1 - x^2) > 0$ which is equivalent to

$$\frac{1}{1 - x^2/6} < 1 + \frac{x^2}{5}. \quad (3.5)$$

By Theorems 2.1, 3.1, and (3.5)

$$\sinh x \leq \frac{x^2}{\sin x} \leq \frac{x}{1 - x^2/6} < x + \frac{x^3}{5}. \quad (3.6)$$

(ii) Since $(2n)! > 6^n$ for $n \geq 3$ we have

$$\begin{aligned} 1 + \frac{5x^2}{9} - \cosh x &= \frac{x^2}{18} - \sum_{n=2}^{\infty} \frac{x^{2n}}{(2n)!} \\ &\geq \frac{x^2}{18} - \sum_{n=2}^{\infty} \frac{x^2}{(2n)!} \\ &= x^2 \left(\frac{1}{72} - \sum_{n=3}^{\infty} \frac{1}{(2n)!} \right) \\ &\geq x^2 \left(\frac{1}{72} - \sum_{n=3}^{\infty} \frac{1}{6^n} \right) \\ &> 0. \end{aligned} \quad (3.7)$$

(iii) By the series expansion of $\cosh x$ we have

$$\cosh x \left(1 - \frac{x^2}{3} \right) \geq \left(1 + \frac{x^2}{2} \right) \left(1 - \frac{x^2}{3} \right) = 1 + \frac{x^2}{6} - \frac{x^4}{6} > 1. \quad (3.8)$$

□

Proof of Theorem 1.2. The lower bound of $x/\sinh x$ follows from Lemma 3.3 and Theorem 3.1 since

$$\frac{1}{\cosh x} < 1 - \frac{x^2}{3} < \left(1 - \frac{x^2}{6}\right)^2 \leq \left(\frac{x}{\sinh x}\right)^2. \quad (3.9)$$

The upper bound of $x/\sinh x$ holds true if the function $g(x) = \sinh^4 x - x^4 \cosh x$ is positive on $(0, 1)$. By the series expansion it is clear that

$$\sinh x > x + \frac{x^3}{6}. \quad (3.10)$$

By Lemma 3.3 and (3.10)

$$g(x) > \left(x + \frac{x^3}{6}\right)^4 - x^4 \left(1 + \frac{2x^2}{3}\right) = \frac{x^6}{1296} (x^6 + 24x^4 + 216x^2 + 144) > 0, \quad (3.11)$$

and the assertion follows. \square

Theorem 3.4. For $x \in (0, \pi/4)$

$$\cosh x < \frac{\cos x}{\sqrt{(\cos x)^2 - (\sin x)^2}}. \quad (3.12)$$

Proof. The upper bound of $\cosh x$ holds true if the function $f(x) = \cos^2 x - \cosh^2 x (\cos^2 x - \sin^2 x)$ is positive on $(0, \pi/4)$. Since

$$f''(x) = 4 \sin(2x) \sinh(2x) > 0, \quad (3.13)$$

we have

$$f'(x) = \sin(2x) \sinh(2x) - \cos(2x) \cosh(2x) > f'(0) = 0. \quad (3.14)$$

Therefore $f(x) > f(0) = 0$ and the assertion follows. \square

Theorem 3.5. For $x \in (0, \pi/4)$

$$\frac{1}{(\cos x)^{2/3}} < \cosh x < \frac{1}{\cos x}. \quad (3.15)$$

Proof. The upper bound of $\cosh x$ holds true if the function $f(x) = 1 - \cos x \cosh x$ is positive on $(0, \pi/4)$. Since $f''(x) = 2 \sin x \sinh x > 0$ the function $f'(x) = \cosh x \sin x - \cos x \sinh x$ is increasing. Therefore $f'(x) > f'(0) = 0$ and $f(x) > f(0) = 0$.

The lower bound of $\cosh x$ holds true if the function $g(x) = \cos^2 x \cosh^3 x - 1$ is positive on $(0, \pi/4)$. By the series expansions we have

$$g(x) > \left(1 - \frac{x^2}{2}\right)^2 \left(1 + \frac{x^2}{2}\right)^3 - 1 = \frac{x^2}{32} (x^8 + 2x^6 - 8x^4 - 16x^2 + 16). \quad (3.16)$$

By a straightforward computation we see that the polynomial $h(x) = x^8 + 2x^6 - 8x^4 - 16x^2 + 16$ is strictly decreasing on $(0, \pi/4)$. Therefore

$$\begin{aligned} h(x) &> h(\pi/4) \\ &= 16 - \pi^2 - \frac{\pi^4}{32} + \frac{\pi^6}{2048} + \frac{\pi^8}{65536} \\ &> 16 - \frac{16^2}{5} - 32^{-1} \frac{16^4}{5} + \frac{3^6}{2048} + \frac{3^8}{65536} \\ &= \frac{120392497}{40960000} \\ &> 0, \end{aligned} \quad (3.17)$$

and the assertion follows. \square

Remark 3.6. Baricz and Wu have shown in [33, page 276-277] that the right hand side of Theorem 2.1 is true for $x \in (0, \pi)$ and the right hand side of Theorem 3.5 is true for $x \in (0, \pi/2)$. Their proof is based on the infinite product representations.

Note that for $x \in (0, \pi/4)$

$$\frac{1}{\cos x} \leq \frac{\cos x}{\sqrt{(\cos x)^2 - (\sin x)^2}}. \quad (3.18)$$

Hence, the upper bound in Theorem 3.5 is better than in Theorem 3.4.

4. Trigonometric Inequalities

Theorem 4.1. For $x \in (0, 1)$ the following inequalities hold

- (i) $x / \arcsin x \leq \sin x / x$,
- (ii) $x / \operatorname{arcsinh} x \leq \sinh x / x$,
- (iii) $x / \arctan x \leq \tan x / x$,
- (iv) $x / \operatorname{arctanh} x \leq \tanh x / x$.

Proof. (i) By setting $x = \sin t$ the assertion is equivalent to

$$\operatorname{sinc} t \leq \operatorname{sinc}(\sin t), \quad (4.1)$$

which is true because $\operatorname{sinc} t = (\sin t)/t$ is decreasing on $(0, \pi/2)$ and $\sin t \leq t$.

(ii) By the series expansions of $\sinh x$ and $\operatorname{arcsinh} x$ we have by Leibniz Criterion

$$\begin{aligned} (\sinh x)\operatorname{arcsinh} x - x^2 &\geq \left(x + \frac{x^3}{6}\right)\left(x - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{5x^7}{112}\right) - x^2 \\ &= \frac{x^6}{10080}(-75x^4 - 324x^2 + 476), \end{aligned} \quad (4.2)$$

and since $-75x^4 - 324x^2 + 476 > 77$ on $(0, 1)$ the assertion follows.

(iii) By the series expansions of $\tan x$ and $\arctan x$ we have by Leibniz Criterion

$$\begin{aligned} (\tan x)\arctan x - x^2 &\geq \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315}\right)\left(x - \frac{x^3}{3}\right) - x^2 \\ &= \frac{x^6}{945}(21 + 9x^2 - 17x^4), \end{aligned} \quad (4.3)$$

and since $21 + 9x^2 - 17x^4 > 4$ on $(0, 1)$ the assertion follows.

(iv) By the series expansions of $\tanh x$ and $\operatorname{arctanh} x$ we have by Leibniz Criterion

$$\begin{aligned} (\tanh x)\operatorname{arctanh} x - x^2 &\geq \left(x - \frac{x^3}{3}\right)\left(x + \frac{x^3}{3} + \frac{x^5}{5}\right) - x^2 \\ &= \frac{x^6}{45}(4 - 3x^2), \end{aligned} \quad (4.4)$$

and since $4 - 3x^2 > 1$ on $(0, 1)$ the assertion follows. \square

Remark 4.2. Similar inequalities to Theorem 4.1 have been considered by Neuman in [46, page 34-35].

Theorem 4.3. *Let $k \in (0, 1)$. Then*

(i) *for $x \in (0, \pi)$*

$$\frac{\sin x}{x} \leq \frac{\sin(kx)}{kx}, \quad (4.5)$$

(ii) *for $x > 0$*

$$\frac{\sinh x}{x} \geq \frac{\sinh(kx)}{kx}, \quad (4.6)$$

(iii) *for $x \in (0, 1)$*

$$\frac{\tanh x}{x} \leq \frac{\tanh(kx)}{kx}. \quad (4.7)$$

Proof. (i) The claim follows from the fact that sinc is decreasing on $(0, \pi)$.

(ii) The claim is equivalent to saying that the function $f(x) = (\sinh x)/x$ is increasing for $x > 0$. Since $f'(x) = (\cosh x)/x - (\sinh x)/x^2 \geq 0$ and $f'(x) \geq 0$ is equivalent to $\tanh x \leq x$ the assertion follows.

(iii) The claim is equivalent to $\tanh(kx) - k \tanh x \geq 0$. By the series expansion of $\tanh x$ we have

$$\tanh(kx) - k \tanh x = k \sum_{n=1}^{\infty} \frac{4^{n+1}(4^{n+1} - 1)B_{2(n+1)}x^{2n+1}}{(2n+2)!} (k^{2n} - 1), \quad (4.8)$$

where B_j is the j th Bernoulli number ($B_0 = 1, B_1 = -1/2, B_2 = 1/6, \dots$). The assertion follows from the Leibniz Criterion, if

$$\frac{k - k^3}{3}x^3 - \frac{2(k - k^5)}{15}x^5 > 0 \quad (4.9)$$

for all $x \in (0, 1)$. Since (4.9) is equivalent to

$$x^2 < \frac{5}{2(1 + k^2)}, \quad (4.10)$$

the assertion follows from the assumptions $k \in (0, 1)$ and $x \in (0, 1)$. \square

References

- [1] Á. Baricz, "Redheffer type inequality for Bessel functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 8, no. 1, article 11, pp. 1–6, 2007.
- [2] Á. Baricz, "Jordan-type inequalities for generalized Bessel functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 9, no. 2, article 39, pp. 1–6, 2008.
- [3] L. Debnath and C.-J. Zhao, "New strengthened Jordan's inequality and its applications," *Applied Mathematics Letters*, vol. 16, no. 4, pp. 557–560, 2003.
- [4] W. D. Jiang and H. Yun, "Sharpening of Jordan's inequality and its applications," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 3, article 102, pp. 1–4, 2006.
- [5] J.-L. Li, "An identity related to Jordan's inequality," *International Journal of Mathematics and Mathematical Sciences*, vol. 2006, Article ID 76782, 6 pages, 2006.
- [6] S.-H. Wu, H. M. Srivastava, and L. Debnath, "Some refined families of Jordan-type inequalities and their applications," *Integral Transforms and Special Functions*, vol. 19, no. 3-4, pp. 183–193, 2008.
- [7] L. Zhu and J. Sun, "Six new Redheffer-type inequalities for circular and hyperbolic functions," *Computers & Mathematics with Applications*, vol. 56, no. 2, pp. 522–529, 2008.
- [8] D. S. Mitrinović, *Analytic Inequalities*, vol. 16 of *Die Grundlehren der mathematischen Wissenschaften*, Springer, New York, NY, USA, 1970, in cooperation with P. M. Vasić.
- [9] S.-H. Wu and H. M. Srivastava, "A weighted and exponential generalization of Wilker's inequality and its applications," *Integral Transforms and Special Functions*, vol. 18, no. 7-8, pp. 529–535, 2007.
- [10] S.-H. Wu and H. M. Srivastava, "A further refinement of a Jordan type inequality and its application," *Applied Mathematics and Computation*, vol. 197, no. 2, pp. 914–923, 2008.
- [11] X. Zhang, G. Wang, and Y. Chu, "Extensions and sharpenings of Jordan's and Kober's inequalities," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 2, article 63, pp. 1–3, 2006.
- [12] J.-L. Li and Y.-L. Li, "On the strengthened Jordan's inequality," *Journal of Inequalities and Applications*, vol. 2007, Article ID 74328, 8 pages, 2007.

- [13] S. Wu and L. Debnath, "A new generalized and sharp version of Jordan's inequality and its applications to the improvement of the Yang Le inequality," *Applied Mathematics Letters*, vol. 19, no. 12, pp. 1378–1384, 2006.
- [14] S. Wu and L. Debnath, "A new generalized and sharp version of Jordan's inequality and its applications to the improvement of the Yang Le inequality. II," *Applied Mathematics Letters*, vol. 20, no. 5, pp. 532–538, 2007.
- [15] S. Wu and L. Debnath, "Jordan-type inequalities for differentiable functions and their applications," *Applied Mathematics Letters*, vol. 21, no. 8, pp. 803–809, 2008.
- [16] A. Y. Özban, "A new refined form of Jordan's inequality and its applications," *Applied Mathematics Letters*, vol. 19, no. 2, pp. 155–160, 2006.
- [17] F. Qi, D.-W. Niu, and B.-N. Guo, "Refinements, generalizations, and applications of Jordan's inequality and related problems," *Journal of Inequalities and Applications*, vol. 2009, Article ID 271923, 52 pages, 2009.
- [18] L. Zhu, "Sharpening of Jordan's inequalities and its applications," *Mathematical Inequalities & Applications*, vol. 9, no. 1, pp. 103–106, 2006.
- [19] L. Zhu, "Sharpening Jordan's inequality and the Yang Le inequality," *Applied Mathematics Letters*, vol. 19, no. 3, pp. 240–243, 2006.
- [20] L. Zhu, "Sharpening Jordan's inequality and Yang Le inequality. II," *Applied Mathematics Letters*, vol. 19, no. 9, pp. 990–994, 2006.
- [21] L. Zhu, "A general refinement of Jordan-type inequality," *Computers & Mathematics with Applications*, vol. 55, no. 11, pp. 2498–2505, 2008.
- [22] L. Zhu, "General forms of Jordan and Yang Le inequalities," *Applied Mathematics Letters*, vol. 22, no. 2, pp. 236–241, 2009.
- [23] L. Zhu, "Sharpening Redheffer-type inequalities for circular functions," *Applied Mathematics Letters*, vol. 22, no. 5, pp. 743–748, 2009.
- [24] L. Zhu, "Some new inequalities of the Huygens type," *Computers & Mathematics with Applications*, vol. 58, no. 6, pp. 1180–1182, 2009.
- [25] L. Zhu, "Some new Wilker-type inequalities for circular and hyperbolic functions," *Abstract and Applied Analysis*, vol. 2009, Article ID 485842, 9 pages, 2009.
- [26] L. Zhu, "A source of inequalities for circular functions," *Computers & Mathematics with Applications*, vol. 58, no. 10, pp. 1998–2004, 2009.
- [27] L. Zhu, "Generalized lazarevic's inequality and its applications—part II," *Journal of Inequalities and Applications*, vol. 2009, Article ID 379142, 4 pages, 2009.
- [28] L. Zhu, "Jordan type inequalities involving the Bessel and modified Bessel functions," *Computers & Mathematics with Applications*, vol. 59, no. 2, pp. 724–736, 2010.
- [29] L. Zhu, "Inequalities for hyperbolic functions and their applications," *Journal of Inequalities and Applications*. In press.
- [30] J. Sándor, "On the concavity of $\sin x/x$," *Octagon Mathematical Magazine*, vol. 13, no. 1, pp. 406–407, 2005.
- [31] J. Sándor, "A note on certain Jordan type inequalities," *RGMI A Research Report Collection*, vol. 10, no. 1, article 1, 2007.
- [32] Á. Baricz and S. Wu, "Sharp Jordan-type inequalities for Bessel functions," *Publicationes Mathematicae Debrecen*, vol. 74, no. 1-2, pp. 107–126, 2009.
- [33] Á. Baricz and S. Wu, "Sharp exponential Redheffer-type inequalities for Bessel functions," *Publicationes Mathematicae Debrecen*, vol. 74, no. 3-4, pp. 257–278, 2009.
- [34] E. Neuman and J. Sándor, "On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities," *Mathematical Inequalities & Applications*, vol. 1973, pp. 1–9, 2010.
- [35] R. P. Agarwal, Y.-H. Kim, and S. K. Sen, "A new refined Jordan's inequality and its application," *Mathematical Inequalities & Applications*, vol. 12, no. 2, pp. 255–264, 2009.
- [36] D.-W. Niu, Z.-H. Huo, J. Cao, and F. Qi, "A general refinement of Jordan's inequality and a refinement of L. Yang's inequality," *Integral Transforms and Special Functions*, vol. 19, no. 3-4, pp. 157–164, 2008.
- [37] W. Pan and L. Zhu, "Generalizations of Shafer-Fink-type inequalities for the arc sine function," *Journal of Inequalities and Applications*, vol. 2009, Article ID 705317, 6 pages, 2009.
- [38] F. Qi and B.-N. Guo, "A concise proof of Oppenheim's double inequality relating to the cosine and sine functions," <http://arxiv.org/abs/arxiv:0902.2511>.

- [39] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorine, *Conformal Invariants, Inequalities and Quasiconformal Mappings*, John Wiley & Sons, New York, NY, USA, 1997.
- [40] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, "Topics in special functions," in *Papers on Analysis: A Volume Dedicated to Olli Martio on the Occasion of His 60th Birthday*, J. Heinonen, T. Kilpeläinen, and P. Koskela, Eds., vol. 83 of *Rep. Univ. Jyväskylä Dep. Math. Stat.*, pp. 5–26, Univ. Jyväskylä, Jyväskylä, Finland, 2001.
- [41] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, "Topics in special functions. II," *Conformal Geometry and Dynamics*, vol. 11, pp. 250–270, 2007.
- [42] I. Pinelis, "Hospital rules for monotonicity and the Wilker-Anglesio inequality," *The American Mathematical Monthly*, vol. 111, no. 10, pp. 905–909, 2004.
- [43] Á. Baricz, *Generalized Bessel functions of the first kind*, Ph.D. thesis, Babes-Bolyai University, Cluj-Napoca, Romania, 2008.
- [44] M. Lehtonen, *Yleistetty konveksisuus*, M.S. thesis, University of Turku, March 2008, written under the supervision of Prof. M. Vuorinen.
- [45] F. Qi, L.-H. Cui, and S.-L. Xu, "Some inequalities constructed by Tchebysheff's integral inequality," *Mathematical Inequalities & Applications*, vol. 2, no. 4, pp. 517–528, 1999.
- [46] E. Neuman, "Inequalities involving inverse circular and inverse hyperbolic functions," *Publikacije Elektrotehničkog Fakulteta Univerzitet u Beogradu. Serija Matematika*, vol. 18, pp. 32–37, 2007.