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*Research Article*

# $\epsilon$-Duality Theorems for Convex Semidefinite Optimization Problems with Conic Constraints

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A convex semidefinite optimization problem with a conic constraint is considered. We formulate a Wolfe-type dual problem for the problem for its $\epsilon$-approximate solutions, and then we prove $\epsilon$-weak duality theorem and $\epsilon$-strong duality theorem which hold between the problem and its Wolfe type dual problem. Moreover, we give an example illustrating the duality theorems.

## 1. Introduction

Convex semidefinite optimization problem is to optimize an objective convex function over a linear matrix inequality. When the objective function is linear and the corresponding matrices are diagonal, this problem becomes a linear optimization problem.

For convex semidefinite optimization problem, Lagrangean duality without constraint qualification [1, 2], complete dual characterization conditions of solutions [1, 3, 4], saddle point theorems [5], and characterizations of optimal solution sets [6, 7] have been investigated.

To get the $\epsilon$-approximate solution, many authors have established $\epsilon$-optimality conditions, $\epsilon$-saddle point theorems and $\epsilon$-duality theorems for several kinds of optimization problems [1, 8–16].

Recently, Jeyakumar and Glover [11] gave $\epsilon$-optimality conditions for convex optimization problems, which hold without any constraint qualification. Yokoyama and Shiraishi [16] gave a special case of convex optimization problem which satisfies $\epsilon$-optimality conditions. Kim and Lee [12] proved sequential $\epsilon$-saddle point theorems and $\epsilon$-duality theorems for convex semidefinite optimization problems which have not conic constraints.

The purpose of this paper is to extend the $\epsilon$-duality theorems by Kim and Lee [12] to convex semidefinite optimization problems with conic constraints. We formulate a Wolfe type dual problem for the problem for its $\epsilon$-approximate solutions, and then prove

$\epsilon$-weak duality theorem and $\epsilon$-strong duality theorem for the problem and its Wolfe type dual problem, which hold under a weakened constraint qualification. Moreover, we give an example illustrating the duality theorems.

## 2. Preliminaries

Consider the following convex semidefinite optimization problem:

$$\begin{aligned} \text{(SDP)}\quad & \text{Minimize} \quad f(x), \\ & \text{subject to} \quad F_0 + \sum_{i=1}^{m} x_i F_i \succeq 0, \quad (x_1, x_2, \ldots, x_m) \in C, \end{aligned} \tag{2.1}$$

where $f : \mathbb{R}^m \to \mathbb{R}$ is a convex function, $C$ is a closed convex cone of $\mathbb{R}^m$, and for $i = 0, 1, \ldots, m, F_i \in S_n$, where $S_n$ is the space of $n \times n$ real symmetric matrices. The space $S_n$ is partially ordered by the Löwner order, that is, for $M, N \in S_n, M \succeq N$ if and only if $M - N$ is positive semidefinite. The inner product in $S_n$ is defined by $(M, N) = \mathrm{Tr}[MN]$, where $\mathrm{Tr}[\cdot]$ is the trace operation.

Let $S := \{M \in S_n \mid M \succeq 0\}$. Then $S$ is self-dual, that is,

$$S^+ := \left\{\theta \in S_n \mid (\theta, Z) \geqq 0, \text{ for any } Z \in S\right\} = S. \tag{2.2}$$

Let $F(x) := F_0 + \sum_{i=1}^{m} x_i F_i$, $\widehat{F}(x) := \sum_{i=1}^{m} x_i F_i$, $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$. Then $\widehat{F}$ is a linear operator from $\mathbb{R}^m$ to $S_n$ and its dual is defined by

$$\widehat{F}^*(Z) = (\mathrm{Tr}[F_1 Z], \ldots, \mathrm{Tr}[F_m Z]), \tag{2.3}$$

for any $Z \in S_n$. Clearly, $A := \{x \in C \mid F(x) \in S\}$ is the feasible set of SDP.

*Definition 2.1.* Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function.

(1) The subdifferential of $g$ at $a \in \operatorname{dom} g$, where $\operatorname{dom} g = \{x \in \mathbb{R}^n \mid g(x) < +\infty\}$, is given by

$$\partial g(a) = \left\{v \in \mathbb{R}^n \mid g(x) \geqq g(a) + \langle v, x - a\rangle, \ \forall x \in \mathbb{R}^n\right\}, \tag{2.4}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product on $\mathbb{R}^n$.

(2) The $\epsilon$-subdifferential of $g$ at $a \in \operatorname{dom} g$ is given by

$$\partial_\epsilon g(a) = \left\{v \in \mathbb{R}^n \mid g(x) \geqq g(a) + \langle v, x - a\rangle - \epsilon, \ \forall x \in \mathbb{R}^n\right\}. \tag{2.5}$$

*Definition 2.2.* Let $\epsilon \geqq 0$. Then $\overline{x} \in A$ is called an $\epsilon$-approximate solution of SDP, if, for any $x \in A$,

$$f(x) \geqq f(\overline{x}) - \epsilon. \tag{2.6}$$

*Definition 2.3.* The conjugate function of a function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$g^*(v) = \sup\{\langle v, x\rangle - g(x) \mid x \in \mathbb{R}^n\}. \tag{2.7}$$

*Definition 2.4.* The epigraph of a function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $epi\ g$, is defined by

$$epig = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid g(x) \leqq r\}. \tag{2.8}$$

If $g$ is sublinear (i.e., convex and positively homogeneous of degree one), then $\partial_\epsilon g(0) = \partial g(0)$, for all $\epsilon \geqq 0$. If $\tilde{g}(x) = g(x) - k$, $x \in \mathbb{R}^n$, $k \in \mathbb{R}$, then $epi\tilde{g}^* = epig^* + (0, k)$. It is worth nothing that if $g$ is sublinear, then

$$epig^* = \partial g(0) \times \mathbb{R}_+. \tag{2.9}$$

Moreover, if $g$ is sublinear and if $\tilde{g}(x) = g(x) - k$, $x \in \mathbb{R}^n$, and $k \in \mathbb{R}$, then

$$epi\tilde{g}^* = \partial g(0) \times [k, \infty). \tag{2.10}$$

*Definition 2.5.* Let $C$ be a closed convex set in $\mathbb{R}^n$ and $x \in C$.

(1) Let $N_C(x) = \{v \in \mathbb{R}^n \mid \langle v, y - x\rangle \leqq 0, \text{for all } y \in C\}$. Then $N_C(x)$ is called the normal cone to $C$ at $x$.

(2) Let $\epsilon \geqq 0$. Let $N_C^\epsilon(x) = \{v \in \mathbb{R}^n \mid \langle v, y - x\rangle \leqq \epsilon, \text{for all } y \in C\}$. Then $N_C^\epsilon(x)$ is called the $\epsilon$-normal set to $C$ at $x$.

(3) When $C$ is a closed convex cone in $R^n$, $N_C(0)$ we denoted by $C^*$ and called the negative dual cone of $C$.

**Proposition 2.6** (see [17, 18]). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $\delta_C$ be the indicator function with respect to a closed convex subset $C$ of $\mathbb{R}^n$, that is, $\delta_C(x) = 0$ if $x \in C$, and $\delta_C(x) = +\infty$ if $x \notin C$. Let $\epsilon \geqq 0$. Then*

$$\partial_\epsilon(f + \delta_C)(\overline{x}) = \bigcup_{\substack{\epsilon_0 \geqq 0, \epsilon_1 \geqq 0 \\ \epsilon_0 + \epsilon_1 = \epsilon}} \{\partial_{\epsilon_0} f(\overline{x}) + \partial_{\epsilon_1}\delta_C(\overline{x})\}. \tag{2.11}$$

**Proposition 2.7** (see [7]). *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous convex function and let $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then*

$$epi(g + h)^* = epig^* + epih^*. \tag{2.12}$$

Following the proof of Lemma 2.2 in [1], we can prove the following lemma.

**Lemma 2.8.** *Let $F_i \in S_n, i = 0, 1, \ldots, m$. Suppose that $A \neq \emptyset$. Let $u \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$. Then the following are equivalent:*

$$\text{(i)} \quad \left\{ x \in C \mid F_0 + \sum_{i=1}^{m} F_i x_i \succeq 0 \right\} \subset \{ x \in \mathbb{R}^m \mid \langle u, x \rangle \geqq \alpha \},$$
$$\text{(ii)} \quad \begin{pmatrix} u \\ \alpha \end{pmatrix} \in \mathrm{cl}\left( \bigcup_{(Z,\delta) \in S \times \mathbb{R}_+} \left\{ \begin{pmatrix} \widehat{F}^*(Z) \\ -\operatorname{Tr}[ZF_0] - \delta \end{pmatrix} \right\} - C^* \times \mathbb{R}_+ \right). \tag{2.13}$$

## 3. $\epsilon$-Duality Theorem

Now we give $\epsilon$-duality theorems for SDP. Using Lemma 2.8, we can obtain the following lemma which is useful in proving our $\epsilon$-strong duality theorems for SDP.

**Lemma 3.1.** *Let $\overline{x} \in A$. Suppose that*

$$\bigcup_{(Z,\delta) \in S \times \mathbb{R}_+} \left\{ \begin{pmatrix} \widehat{F}^*(Z) \\ -\operatorname{Tr}[ZF_0] - \delta \end{pmatrix} \right\} - C^* \times \mathbb{R}_+ \tag{3.1}$$

*is closed. Then $\overline{x}$ is an $\epsilon$-approximate solution of SDP if and only if there exists $Z \in S$ such that for any $x \in C$,*

$$f(x) - \operatorname{Tr}[ZF(x)] \geqq f(\overline{x}) - \epsilon. \tag{3.2}$$

*Proof.* ($\Longrightarrow$) Let $\overline{x}$ be an $\epsilon$-approximate solution of SDP. Then $f(x) \geqq f(\overline{x}) - \epsilon$, for any $x \in A$. Let $h(x) = f(x) - f(\overline{x}) + \epsilon$. Then $h(x) + \delta_A(x) \geqq 0$, for any $x \in \mathbb{R}^n$. Thus we have, from Proposition 2.7,

$$\begin{aligned} 0 \in epi(h + \delta_A)^* &= epih^* + epi\delta_A{}^* \\ &= epif^* + (0, f(\overline{x}) - \epsilon) + epi\delta_A^*, \end{aligned} \tag{3.3}$$

and hence, $(0, \epsilon - f(\overline{x})) \in epif^* + epi\delta_A^*$. So there exists $(u, r) \in epif^*$ such that $(-u, \epsilon - f(\overline{x}) - r) \in epi\delta_A{}^*$ and hence there exists $(u, r) \in epif^*$ such that $\langle -u, x \rangle \leqq \epsilon - f(\overline{x}) - r$ for any $x \in A$. Since $f^*(u) \leqq r$, $\langle -u, x \rangle \leqq \epsilon - f(\overline{x}) - f^*(u)$ for any $x \in A$; and hence it follows from Lemma 2.8 that

$$\begin{pmatrix} u \\ -\epsilon + f(\overline{x}) + f^*(u) \end{pmatrix} \in \bigcup_{(Z,\delta) \in S \times \mathbb{R}_+} \left\{ \begin{pmatrix} \widehat{F}^*(Z) \\ -\operatorname{Tr}[ZF_0] - \delta \end{pmatrix} \right\} - C^* \times \mathbb{R}_+. \tag{3.4}$$

Thus there exist $(Z, \delta) \in S \times \mathbb{R}_+, c^* \in C^*$, and $\gamma \in \mathbb{R}_+$ such that

$$\begin{aligned} u &= \widehat{F}^*(Z) - c^*, \\ -\epsilon + f(\overline{x}) + f^*(u) &= -\operatorname{Tr}[ZF_0] - \delta - \gamma. \end{aligned} \tag{3.5}$$

This gives

$$\left\langle \widehat{F}^*(Z), x \right\rangle - \langle c^*, x\rangle - f(x) = \langle u, x\rangle - f(x) \leqq f^*(u) \\ = -\operatorname{Tr}[ZF_0] - \delta - \gamma - f(\overline{x}) + \epsilon, \tag{3.6}$$

for any $x \in \mathbb{R}^n$. Thus we have

$$\begin{aligned} f(\overline{x}) - \epsilon &\leqq -\langle u, x\rangle + f(x) - \operatorname{Tr}[ZF_0] - \delta - \gamma \\ &= f(x) - \left\langle \widehat{F}^*(Z), x \right\rangle + \langle c^*, x\rangle - \operatorname{Tr}[ZF_0] - \delta - \gamma \\ &= f(x) - \operatorname{Tr}[ZF(x)] + \langle c^*, x\rangle - \delta - \gamma \\ &\leqq f(x) - \operatorname{Tr}[ZF(x)] \end{aligned} \tag{3.7}$$

for any $x \in C$.

($\Longleftarrow$) Suppose that there exists $Z \in S$ such that

$$f(x) - \operatorname{Tr}[ZF(x)] \geqq f(\overline{x}) - \epsilon, \tag{3.8}$$

for any $x \in C$. Then we have

$$f(x) \geqq f(x) - \operatorname{Tr}[ZF(x)] \geqq f(\overline{x}) - \epsilon, \tag{3.9}$$

for any $x \in A$. Thus $f(x) \geqq f(\overline{x}) - \epsilon$, for any $x \in A$. Hence $\overline{x}$ is an $\epsilon$-approximate solution of SDP. □

Now we formulate the dual problem SDD of SDP as follows:

$$\begin{aligned} \text{(SDD)} \quad \text{maximize} \quad & f(x) - \operatorname{Tr}[ZF(x)], \\ \text{subject to} \quad & 0 \in \partial_{\epsilon_0} f(x) - \widehat{F}^*(Z) + N_C^{\epsilon_1}(x), \\ & Z \succeq 0, \\ & \epsilon_0 + \epsilon_1 \in [0, \epsilon]. \end{aligned} \tag{3.10}$$

We prove $\epsilon$-weak and $\epsilon$-strong duality theorems which hold between SDP and SDD.

**Theorem 3.2** ($\epsilon$-weak duality). *For any feasible solution $x$ of SDP and any feasible solution $(y, Z)$ of SDD,*

$$f(x) \geqq f(y) - \operatorname{Tr}[ZF(y)] - \epsilon. \tag{3.11}$$

*Proof.* Let $x$ and $(y, Z)$ be feasible solutions of SDP and SDD respectively. Then $\operatorname{Tr}[ZF(x)] \geqq 0$ and there exist $v \in \partial_{\epsilon_0} f(y)$ and $\omega \in N_C^{\epsilon_1}(y)$ such that $v = -\omega + \widehat{F}^*(Z)$. Thus, we have

$$
\begin{aligned}
f(x) - \{f(y) - \operatorname{Tr}[ZF(y)]\} &\geqq \langle v, x-y\rangle - \epsilon_0 + \operatorname{Tr}[ZF(y)] \\
&= \left\langle -\omega + \widehat{F}^*(Z), x-y \right\rangle - \epsilon_0 + \operatorname{Tr}[ZF(y)] \\
&\geqq \left\langle \widehat{F}^*(Z), x-y \right\rangle - \epsilon_0 - \epsilon_1 + \operatorname{Tr}[ZF(y)] \\
&= \left\langle \widehat{F}^*(Z), x \right\rangle - \left\langle \widehat{F}^*(Z), y \right\rangle - \epsilon_0 - \epsilon_1 \\
&\quad + \operatorname{Tr}[ZF(y)] \\
&= \operatorname{Tr}\left[Z\sum_{i=1}^m x_i F_i\right] - \operatorname{Tr}\left[Z\sum_{i=1}^m y_i F_i\right] - \epsilon_0 - \epsilon_1 \\
&\quad + \operatorname{Tr}[ZF_0] + \operatorname{Tr}\left[Z\sum_{i=1}^m y_i F_i\right] \\
&= \operatorname{Tr}[ZF(x)] - \epsilon_0 - \epsilon_1 \\
&\geqq -\epsilon_0 - \epsilon_1 \\
&\geqq -\epsilon.
\end{aligned}
\tag{3.12}
$$

Hence $f(x) \geqq f(y) - \operatorname{Tr}[ZF(y)] - \epsilon$. □

**Theorem 3.3** ($\epsilon$-strong duality). *Suppose that*

$$
\bigcup_{(Z,\delta)\in S\times\mathbb{R}_+} \left\{ \begin{pmatrix} \widehat{F}^*(Z) \\ -\operatorname{Tr}[ZF_0] - \delta \end{pmatrix} \right\} - C^* \times \mathbb{R}_+
\tag{3.13}
$$

*is closed. If $\overline{x}$ is an $\epsilon$-approximate solution of SDP, then there exists $\overline{Z} \in S$ such that $(\overline{x}, \overline{Z})$ is a $2\epsilon$-approximate solution of SDD.*

*Proof.* Let $\overline{x} \in A$ be an $\epsilon$-approximate solution of SDP. Then $f(x) \geqq f(\overline{x}) - \epsilon$, for any $x \in A$. By Lemma 3.1, there exists $\overline{Z} \in S$ such that

$$
f(x) - \operatorname{Tr}\left[\overline{Z}F(x)\right] \geqq f(\overline{x}) - \epsilon,
\tag{3.14}
$$

for any $x \in C$. Letting $x = \overline{x}$ in (3.14), $\operatorname{Tr}[\overline{Z}F(\overline{x})] \leqq \epsilon$. Since $F(\overline{x}) \in S$ and $\overline{Z} \in S$, $\operatorname{Tr}[\overline{Z}F(\overline{x})] \geqq 0$.

Thus from (3.14),

$$
f(x) - \operatorname{Tr}\left[\overline{Z}F(x)\right] + \epsilon \geqq f(\overline{x}) - \operatorname{Tr}\left[\overline{Z}F(\overline{x})\right]
\tag{3.15}
$$

for any $x \in C$. Hence $\overline{x}$ is an $\epsilon$-approximate solution of the following problem:

$$\begin{aligned} &\text{maximize} \quad f(x) - \operatorname{Tr}\left[\overline{Z}F(x)\right], \\ &\text{subject to} \quad x \in C, \end{aligned} \tag{3.16}$$

and so, $0 \in \partial_\epsilon(f - \widehat{F}^*(\overline{Z}) + \delta_C)\ (\overline{x})$, and hence, by Proposition 2.6, there exist $\epsilon_0$, $\epsilon_1 \in [0, \epsilon]$ such that $\epsilon_0 + \epsilon_1 = \epsilon$ and

$$0 \in \partial_{\epsilon_0} f(\overline{x}) - \widehat{F}^*\left(\overline{Z}\right) + N_C^{\epsilon_1}(\overline{x}). \tag{3.17}$$

So, $(\overline{x}, \overline{Z})$ is a feasible solution of SDD. For any feasible solution $(y, Z)$ of SDD,

$$\begin{aligned} f(\overline{x}) - \operatorname{Tr}\left[\overline{Z}F(\overline{x})\right] - \{f(y) - \operatorname{Tr}[ZF(y)]\} &= f(\overline{x}) - \{f(y) - \operatorname{Tr}[ZF(y)]\} \\ &\quad - \operatorname{Tr}\left[\overline{Z}F(\overline{x})\right] \\ &\geqq -\epsilon - \operatorname{Tr}\left[\overline{Z}F(\overline{x})\right] \\ &\quad (\text{by } \epsilon\text{-weak duality}) \\ &\geqq -\epsilon - \epsilon \\ &= -2\epsilon. \end{aligned} \tag{3.18}$$

Thus $(\overline{x}, \overline{Z})$ is a $2\epsilon$-approximate solution to SDD. □

Now we characterize the $\epsilon$-normal set to $\mathbb{R}^n_+$.

**Proposition 3.4.** *Let* $(x_1, \ldots, x_n) \in \mathbb{R}^n_+$ *and* $\epsilon \geqq 0$. *Then*

$$N^{\epsilon}_{\mathbb{R}^n_+}(x_1, \ldots, x_n) = \bigcup_{\substack{\epsilon_i \geqq 0 \\ \sum_{i=1}^n \epsilon_i = \epsilon}} \prod_{i=1}^{n} A(\epsilon_i), \tag{3.19}$$

*where*

$$A(\epsilon_i) = \begin{cases} -\mathbb{R}_+ & \text{if } x_i = 0, \\ \left[-\dfrac{\epsilon_i}{x_i}, 0\right] & \text{if } x_i > 0. \end{cases} \tag{3.20}$$

*Proof.* Let $(x_1, \ldots, x_n) \in \mathbb{R}_+^n$ and $\epsilon \geqq 0$. Then

$$\begin{aligned}
N_{\mathbb{R}_+^n}^{\epsilon}(x_1, \ldots, x_n) &= \partial_{\epsilon}\delta_{\mathbb{R}_+^n}(x_1, \ldots, x_n) \\
&= \partial_{\epsilon}\left(\sum_{i=1}^{n}\delta_{\mathbb{R}\times\cdots\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times\cdots\times\mathbb{R}}\right)(x_1, \ldots, x_n) \\
&= \bigcup_{\substack{\epsilon_i \geqq 0 \\ \sum_{i=1}^{n}\epsilon_i = \epsilon}}\sum_{i=1}^{n}\partial_{\epsilon_i}\delta_{\mathbb{R}\times\cdots\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times\cdots\times\mathbb{R}}(x_1, \ldots, x_n).
\end{aligned} \tag{3.21}$$

Let $(v_1, \ldots, v_n) \in \partial_{\epsilon_i}\delta_{\mathbb{R}\times\cdots\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times\cdots\times\mathbb{R}}(x_1, \ldots, x_n)$ (where $\mathbb{R}_+$ is at the $i$th position in $\mathbb{R}\times\cdots\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times\cdots\times\mathbb{R}$)

$$\begin{aligned}
\Longleftrightarrow \quad & \text{for any } (y_1, \ldots, y_n) \in \mathbb{R}\times\cdots\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times\cdots\times\mathbb{R}, \\
& \epsilon_i \geqq v_1(y_1 - x_1) + \cdots + v_i(y_i - x_i) + \cdots + v_n(y_n - x_n), \\
\Longleftrightarrow \quad & \text{for any } y_i \in \mathbb{R}_+, \quad \epsilon_i \geqq v_i(y_i - x_i), \quad v_j = 0, \\
& \text{for } j \in \{1, \ldots, n\} \setminus \{i\}, \\
\Longleftrightarrow \quad & v_i \in \begin{cases} -\mathbb{R}_+, & \text{if } x_i = 0, \\ \left[-\dfrac{\epsilon_i}{x_i}, 0\right], & \text{if } x_i > 0, \end{cases} \\
& v_j = 0 \text{ for } j \in \{1, \ldots, n\} \setminus \{i\}.
\end{aligned} \tag{3.22}$$

Thus, we have

$$\begin{aligned}
N_{\mathbb{R}_+^n}^{\epsilon_1}(x_1, \ldots, x_n) &= \bigcup_{\substack{\epsilon_i \geqq 0 \\ \sum_{i=1}^{n}\epsilon_i = \epsilon}}\sum_{i=1}^{n}\{0\}\times\cdots\times\{0\}\times A(\epsilon_i)\times\{0\}\times\cdots\times\{0\} \\
&= \bigcup_{\substack{\epsilon_i \geqq 0 \\ \sum_{i=1}^{n}\epsilon_i = \epsilon}}\prod_{i=1}^{n}A(\epsilon_i).
\end{aligned} \tag{3.23}$$

□

From Proposition 3.4, we can calculate $N_{\mathbb{R}_+^2}^{\epsilon}$.

**Corollary 3.5.** *Let* $(x_1, x_2) \in \mathbb{R}^2_+$ *and* $\epsilon \geqq 0$. *Then following hold.*

(i) *If* $(x_1, x_2) = (0,0)$, *then* $N^{\epsilon}_{\mathbb{R}^2_+}(0,0) = -\mathbb{R}^2_+$.

(ii) *If* $(x_1, x_2) = (x_1, 0)$ *and* $x_1 > 0$, *then* $N^{\epsilon}_{\mathbb{R}^2_+}(x_1, 0) = [-\epsilon/x_1, 0] \times (-\infty, 0]$.

(iii) *If* $(x_1, x_2) = (0, x_2)$ *and* $x_2 > 0$, *then* $N^{\epsilon}_{\mathbb{R}^2_+}(0, x_2) = (-\infty, 0] \times [-\epsilon/x_2, 0]$.

(iv) *If* $(x_1, x_2) = (x_1, x_2)$, *and* $x_1 > 0$ *and* $x_2 > 0$, *then*

$$N^{\epsilon}_{\mathbb{R}^2_+}(x_1, x_2) = \bigcup_{\substack{\epsilon_1 \geqq 0, \epsilon_2 \geqq 0,\\ \epsilon_1 + \epsilon_2 = \epsilon}} \left[-\frac{\epsilon_1}{x_1}, 0\right] \times \left[-\frac{\epsilon_2}{x_2}, 0\right]. \tag{3.24}$$

Now we give an example illustrating our $\epsilon$-duality theorems.

*Example 3.6.* Consider the following convex semidefinite program.

$$\begin{aligned}
\text{(SDP)} \qquad & \text{Minimize} \quad x_1 + x_2^2, \\
& \text{subject to} \quad \begin{pmatrix} 0 & x_1 \\ x_1 & 0 \end{pmatrix} \succeq 0, \\
& \qquad\qquad (x_1, x_2) \in \mathbb{R}^2{}_+.
\end{aligned} \tag{3.25}$$

Let $f(x_1, x_2) = x_1 + x_2^2$,

$$F_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad F_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad F_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.26}$$

and $\epsilon \geqq 0$. Let $f(x_1, x_2) = x_1 + x_2^2$ and

$$F(x_1, x_2) = \begin{pmatrix} 0 & x_1 \\ x_1 & 0 \end{pmatrix}. \tag{3.27}$$

Then $A := \{(0, x_2) \in \mathbb{R}^2 \mid x_2 \geqq 0\}$ is the set of all feasible solutions of SDP and the set of all $\epsilon$-approximate solutions of SDP is $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0, 0 \leqq x_2 \leqq \sqrt{\epsilon}\}$. Let $F = \{((x_1, x_2), Z) \mid 0 \in \partial_{\epsilon_0} f(x_1, x_2) - \widehat{F}^*(Z) + N^{\epsilon_1}_{\mathbb{R}^2_+}(x_1, x_2), Z \succeq 0, \epsilon_0 + \epsilon_1 \in [0, \epsilon]\}$. Then $F$ is the set of all feasible solution of SDD. Now we calculate the set $F$.

$$\begin{aligned}
\widetilde{A} :=& \left\{ \left( (0,0), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid 0 \in \partial_{\epsilon_0} f(0,0) - \widehat{F}^* \begin{pmatrix} a & b \\ b & c \end{pmatrix} + N^{\epsilon_1}_{\mathbb{R}^2_+}(0,0), \right. \\
&\qquad \left. a \geqq 0,\ c \geqq 0,\ b^2 \leqq ac,\ \epsilon_0 + \epsilon_1 \in [0,\epsilon] \right\} \\
=& \left\{ \left( (0,0), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid 0 \in \{1\} \times \left[-2\sqrt{\epsilon}_0, 2\sqrt{\epsilon}_0\right] - (2b,0) - \mathbb{R}^2_+, \right. \\
&\qquad \left. a \geqq 0,\ c \geqq 0,\ b^2 \leqq ac,\ \epsilon_0 + \epsilon_1 \in [0,\epsilon] \right\} \\
=& \left\{ \left( (0,0), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid (2b,0) \in (-\infty,1] \times \left(-\infty, 2\sqrt{\epsilon}_0\right], \right. \\
&\qquad \left. a \geqq 0,\ c \geqq 0,\ b^2 \leqq ac,\ \epsilon_0 + \epsilon_1 \in [0,\epsilon] \right\} \\
=& \left\{ \left( (0,0), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid a \geqq 0,\ c \geqq 0,\ b \leqq \frac{1}{2},\ b^2 \leqq ac \right\}, \\
\widetilde{B} :=& \left\{ \left( (0,x_2), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid x_2 > 0,\ 0 \in \partial_{\epsilon_0} f(0,x_2) - \widehat{F}^* \begin{pmatrix} a & b \\ b & c \end{pmatrix} + N^{\epsilon_1}_{\mathbb{R}^2_+}(0,x_2) \right. \\
&\qquad \left. a \geqq 0,\quad c \geqq 0,\ b^2 \leqq ac,\ \epsilon_0 + \epsilon_1 \in [0,\epsilon] \right\} \\
=& \left\{ \left( (0,x_2), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid x_2 > 0,\ 0 \in \{1\} \times \left[2x_2 - 2\sqrt{\epsilon}_0, 2x_2 + 2\sqrt{\epsilon}_0\right] - (2b,0) \right. \\
&\qquad \left. + (-\infty,0] \times \left[-\frac{\epsilon_1}{x_2}, 0\right],\quad a \geqq 0,\quad c \geqq 0,\ b^2 \leqq ac,\ \epsilon_0 + \epsilon_1 \in [0,\epsilon] \right\} \\
=& \left\{ \left( (0,x_2), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid x_2 > 0,\ (2b,0) \in (-\infty,1] \times \left[2x_2 - \frac{\epsilon_1}{x_2} - 2\sqrt{\epsilon}_0, 2x_2 + 2\sqrt{\epsilon}_0\right], \right. \\
&\qquad \left. a \geqq 0,\ c \geqq 0,\quad b^2 \leqq ac,\ \epsilon_0 + \epsilon_1 \in [0,\epsilon] \right\} \\
=& \left\{ \left( (0,x_2), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid 0 < x_2 \leqq \frac{\sqrt{\epsilon}_0 + \sqrt{\epsilon_0 + 2\epsilon_1}}{2},\ a \geqq 0,\ c \geqq 0,\ b \leqq \frac{1}{2},\ b^2 \leqq ac, \right. \\
&\qquad \left. \epsilon_0 + \epsilon_1 \in [0,\epsilon] \right\},
\end{aligned}$$

$$\widetilde{C} := \left\{ \left( (x_1, 0), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid x_1 > 0,\ 0 \in \partial_{\epsilon_0} f(x_1, 0) - \widehat{F}^* \begin{pmatrix} a & b \\ b & c \end{pmatrix} + N^{\epsilon_1}_{\mathbb{R}^2_+}(x_1, 0), \right.$$

$$\left. a \geqq 0,\ c \geqq 0,\ b^2 \leqq ac,\ \epsilon_0 + \epsilon_1 \in [0, \epsilon] \right\}$$

$$= \left\{ \left( (x_1, 0), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid x_1 > 0,\ 0 \in \{1\} \times \left[ -2\sqrt{\epsilon_0}, +2\sqrt{\epsilon_0} \right] - (2b, 0) \right.$$

$$\left. + \left[ -\frac{\epsilon_1}{x_1}, 0 \right] \times (-\infty, 0],\ a \geqq 0,\ c \geqq 0,\ b^2 \leqq ac,\ \epsilon_0 + \epsilon_1 \in [0, \epsilon] \right\}$$

$$= \left\{ \left( (x_1, 0), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid x_1 > 0,\ (2b, 0) \in \left[ 1 - \frac{\epsilon_1}{x_1}, 1 \right] \times \left( -\infty, 2\sqrt{\epsilon_0} \right], \right.$$

$$\left. a \geqq 0,\ c \geqq 0,\ b^2 \leqq ac,\ \epsilon_0 + \epsilon_1 \in [0, \epsilon] \right\}$$

$$= \left\{ \left( (x_1, 0), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid 0 < x_1,\ -\epsilon_1 \leqq -x_1 + 2bx_1,\ a \geqq 0,\ c \geqq 0,\ b \leqq \frac{1}{2},\ b^2 \leqq ac, \right.$$

$$\left. \epsilon_0 + \epsilon_1 \in [0, \epsilon] \right\},$$

$$\widetilde{D} := \left\{ \left( (x_1, x_2), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid x_1 > 0,\ x_2 > 0,\ 0 \in \partial_{\epsilon_0} f(x_1, x_2) - \widehat{F}^* \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right.$$

$$\left. + N^{\epsilon_1}_{\mathbb{R}^2_+}(x_1, x_2),\ a \geqq 0,\ c \geqq 0,\ b^2 \leqq ac,\ \epsilon_0 + \epsilon_1 \in [0, \epsilon] \right\}$$

$$= \left\{ \left( (x_1, x_2), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid x_1 > 0, x_2 > 0, \right.$$

$$0 \in \{1\} \times \left[ 2x_2 - 2\sqrt{\epsilon_0}, 2x_2 + 2\sqrt{\epsilon_0} \right] - (2b, 0) + \left[ -\frac{\epsilon_1^1}{x_1}, 0 \right] \times \left[ -\frac{\epsilon_1^2}{x_2}, 0 \right],$$

$$\left. a \geqq 0,\ c \geqq 0,\ b^2 \leqq ac,\ \epsilon_1^1 + \epsilon_1^2 = \epsilon_1,\ \epsilon_0 + \epsilon_1 \in [0,\ \epsilon] \right\}$$

$$= \left\{ \left( (x_1, x_2), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid x_1 > 0,\ x_2 > 0, \right.$$

$$(2b, 0) \in \left[ 1 - \frac{\epsilon_1^1}{x_1}, 1 \right] \times \left[ 2x_2 - \frac{\epsilon_1^2}{x_2} - 2\sqrt{\epsilon_0}, 2x_2 + 2\sqrt{\epsilon_0} \right],$$

$$\left. a \geqq 0,\ c \geqq 0,\ b^2 \leqq ac,\ \epsilon_1^1 + \epsilon_1^2 = \epsilon_1,\ \epsilon_0 + \epsilon_1 \in [0, \epsilon] \right\}$$

$$= \Bigg\{ \left( (x_1, x_2), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid 0 < x_1,\ -\epsilon_1^1 \leqq -x_1 + 2bx_1,$$
$$0 < x_2 \leqq \frac{\sqrt{\epsilon_0} + \sqrt{\epsilon_0 + 2\epsilon_1^2}}{2},\ a \geqq 0,\ c \geqq 0,\ b \leqq \frac{1}{2},\ b^2 \leqq ac,\ \epsilon_1^1 + \epsilon_1^2 = \epsilon_1,$$
$$\epsilon_0 + \epsilon_1 \in [0, \epsilon] \Bigg\}. \tag{3.28}$$

Thus $F = \widetilde{A} \cup \widetilde{B} \cup \widetilde{C} \cup \widetilde{D}$. We can check that for any $(x_1, x_2) \in A$ and any $((y_1, y_2), \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)) \in F$,

$$f(x_1, x_2) \geqq f(y_1, y_2) - \operatorname{Tr}\left( \begin{pmatrix} a & b \\ b & c \end{pmatrix} F(y_1, y_2) \right) - \epsilon, \tag{3.29}$$

that is, $\epsilon$-weak duality holds.

Let $(\overline{x}_1, \overline{x}_2) \in A$ be an $\epsilon$-approximate solution of SDP. Then $\overline{x}_1 = 0$ and $0 \leqq \overline{x}_2 \leqq \sqrt{\epsilon}$. So, we can easily check that $((\overline{x}_1, \overline{x}_2), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) \in F$.

Since $\operatorname{Tr}(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) F(\overline{x}_1, \overline{x}_2))) = 0$, from (3.29),

$$f(\overline{x}_1, \overline{x}_2) \geqq f(y_1, y_2) - \operatorname{Tr}\left( \begin{pmatrix} a & b \\ b & c \end{pmatrix} F(y_1, y_2) \right) - \epsilon, \tag{3.30}$$

for any $((y_1, y_2), \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)) \in F$. So $((\overline{x}_1, \overline{x}_2), \left(\begin{smallmatrix} \overline{a} & \overline{b} \\ \overline{b} & \overline{c} \end{smallmatrix}\right))$ is an $\epsilon$-approximate solution of SDD. Hence $\epsilon$-strong duality holds.

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