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Research Article

A Class of Logarithmically Completely Monotonic Functions Associated with a Gamma Function

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We show that the function $G_{\alpha,\beta}(x) = e^x\Gamma(x+a)/(x+\beta)^{x+\beta}$ is strictly logarithmically completely monotonic on $(0,\infty)$ if and only if $(\alpha,\beta) \in \{(\alpha,\beta) : 0 < \alpha \le \beta\}$ and $[G_{\alpha,\beta}(x)]^{-1}$ is strictly logarithmically completely monotonic on $(0,\infty)$ if and only if $(\alpha,\beta) \in \{(\alpha,\beta) : 0 < \beta \le \alpha - 1/2\}$.

1. Introduction

For real and positive values of x the Euler gamma function Γ and its logarithmic derivative ψ, the so-called digamma function, are defined as

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{1.1}$$

For extension of these functions to complex variables and for basic properties, see [1]. These functions play central roles in the theory of special functions and have lots of extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences. Over the past half century monotonicity properties of these functions have attracted the attention of many authors (see [2–22]).

Recall that a real-valued function $f : I \to \mathbb{R}$ is said to be completely monotonic on I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \ge 0 \tag{1.2}$$

for all $x \in I$ and $n \geq 0$. Moreover, f is said to be strictly completely monotonic if inequality (1.2) is strict.

Recall also that a positive real-valued function $f : I \to (0, \infty)$ is said to be logarithmically completely monotonic on I if f has derivatives of all orders on I and its logarithm $\log f$ satisfies

$$(-1)^k [\log f(x)]^{(k)} \geq 0 \tag{1.3}$$

for all $x \in I$ and $k \in \mathbb{N}$. Moreover, f is said to be strictly logarithmically completely monotonic if inequality (1.3) is strict.

Recently, the completely monotonic or logarithmically completely monotonic functions have been the subject of intensive research. In particular, many remarkable results for the complete monotonicity or logarithmically complete monotonicity involving the gamma, psi and polygamma functions can be found in the literature [18, 19, 23–42].

The Kershaw's inequality in [21] states that the double inequality

$$\left(x + \frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(x - \frac{1}{2} + \sqrt{s + \frac{1}{4}}\right)^{1-s} \tag{1.4}$$

holds for $0 < s < 1$ and $x \geq 1$. In [43], Laforgia extends the both sides of inequality in (1.4) as follows:

$$\frac{\Gamma(x+1)}{\Gamma(x+\lambda)} > \left(x + \frac{\lambda}{2}\right)^{1-\lambda} \tag{1.5}$$

for $0 < \lambda < 1$ or $\lambda > 2$ and $x \geq 0$, and inequality (1.5) is reversed for $1 < \lambda < 2$ and $x \geq 0$.

Let us define

$$z_{s,t}(x) = \begin{cases} \left[\dfrac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} - x, & s \neq t, \\ e^{\psi(x+s)} - x, & s = t \end{cases} \tag{1.6}$$

for $x \in (-\alpha, \infty)$ with $\alpha = \min\{s, t\}$ and $s, t \in \mathbb{R}$. In order to establish the best bounds in Kershaw's inequality (1.4), the following monotonicity and convexity properties of $z_{s,t}(x)$ are established in [13, 44, 45]: the function $z_{s,t}(x)$ is either convex and decreasing for $|t-s| < 1$ or concave and increasing for $|t-s| > 1$.

This work is motivated by an paper of Guo [46], who proved that the function

$$g(x) = \frac{e^x \Gamma(x+1)}{(x+1/2)^{x+1/2}} \tag{1.7}$$

is strictly logarithmically concave and strictly increasing from $(-1/2, \infty)$ onto $(\sqrt{\pi/e}, \sqrt{2\pi/e})$. It is natural to ask for an extension of this result: is f^{-1} logarithmically complete monotonic? We will give the positive answer. Actually, we investigate a more general problem. The goal of this article is to discuss the logarithmically complete monotonicity properties of the functions

$$G_{\alpha,\beta}(x) = \frac{e^x\Gamma(x+\alpha)}{(x+\beta)^{x+\beta}} \tag{1.8}$$

on $(0,\infty)$ and $[G_{\alpha,\beta}(x)]^{-1}$ for fixed $\alpha,\beta > 0$.

Recently Chen et al. [38, Theorem 1] proved the following result: let $a \in \mathbb{R}$ and $b \geq 0$ be real numbers, define for $x > -b$,

$$f_{a,b}(x) = \frac{e^x\Gamma(x+b)}{(x+b)^{x+a}}. \tag{1.9}$$

Then, the function $x \mapsto f_{a,b}(x)$ is strictly logarithmically completely monotonic on $(-b,\infty)$ if and only if $b - a \leq 1/2$. So is the function $x \mapsto [f_{a,b}(x)]^{-1}$ if and only if $b - a \geq 1$.

Our main results are summarized as follows.

Theorem 1.1. *Let $\alpha > 0$, $\beta > 0$, and $G_{\alpha,\beta}(x)$ is defined as* (1.8)*, then*

(1) *$G_{\alpha,\beta}(x)$ is strictly logarithmically completely monotonic on $(0,\infty)$ if and only if $(\alpha,\beta) \in \{(\alpha,\beta) : 0 < \alpha \leq \beta\}$;*

(2) *$[G_{\alpha,\beta}(x)]^{-1}$ is strictly logarithmically completely monotonic on $(0,\infty)$ if and only if $(\alpha,\beta) \in \{(\alpha,\beta) : 0 < \beta \leq \alpha - 1/2\}$.*

As applications of Theorem 1.1, one has the following corollaries.

Corollary 1.2. *For $\alpha > 0$ and $0 < x < y$, one has the double inequalities for the ratio of the gamma functions*

$$\frac{e^{y-x}(x+\alpha+1/2)^{x+\alpha+1/2}}{(y+\alpha+1/2)^{y+\alpha+1/2}} < \frac{\Gamma(x+\alpha+1/2)}{\Gamma(y+\alpha+1/2)} < \frac{e^{y-x}(x+\alpha)^{x+\alpha}}{(y+\alpha)^{y+\alpha}}. \tag{1.10}$$

In particular, one has

$$\frac{e^{s-1}(x+1/2)^{x+1/2}}{(x+s-1/2)^{x+s-1/2}} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \frac{e^{s-1}(x+1)^{x+1}}{(x+s)^{x+s}} \tag{1.11}$$

for $x \geq 1$ and $0 < s < 1$, and

$$\frac{e^{s-1}(x+1)^{x+1}}{(x+s)^{x+s}} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \frac{e^{s-1}(x+1/2)^{x+1/2}}{(x+s-1/2)^{x+s-1/2}} \tag{1.12}$$

for $x \geq 0$ and $s > 1$.

Corollary 1.3. *For $\alpha > 0$ and $(x, y) \in (0, \infty) \times (0, \infty)$, one has the following double inequality*

$$M\left(x+\alpha+\frac{1}{2}, y+\alpha+\frac{1}{2}\right) < \frac{\Gamma(x+\alpha+1/2)\Gamma(y+\alpha+1/2)}{\Gamma^2((x+y+1)/2+\alpha)} < M(x+\alpha, y+\alpha), \tag{1.13}$$

where $M(u, v) = 2^{u+v}u^u v^v/(u+v)^{u+v}$.

2. Lemmas

In order to prove our Theorem 1.1, we need serval lemmas which we collect in this section. In our second lemma we present the area of (α, β) to determine positive (or negative) for a function, which plays a crucial role in the proof of our result Theorem 1.1 given in Section 3.

Let $\mu(x, y)$ be a function defined on $(0, \infty) \times (0, \infty)$ as

$$\mu(x, y) = -3y^2 + (4x-3)y - (x-1)^2. \tag{2.1}$$

We will discuss the properties for this function and refer to view Figure 1 more clearly.

The function $\mu(x, y)$ can be interpreted as a quadric equation with respect to y, that is

$$\mu(x, y) = a(x)y^2 + b(x)y + c(x), \tag{2.2}$$

where $a(x) = -3$, $b(x) = 4x-3$, $c(x) = -(x-1)^2$ and its discriminant function

$$\Delta(x) = \sqrt{b^2(x) - 4a(x)c(x)} = 4x^2 - 3. \tag{2.3}$$

Obviously, if $0 < x < \sqrt{3}/2$, then $\Delta(x) < 0$. It follows from $a(x) = -3$ that $\mu(x) < 0$.

If $x \geq \sqrt{3}/2$, then $\Delta(x) \geq 0$. We can solve two roots of the equation $\mu(x, y) = 0$, which are

$$y_1(x) = \frac{4x-3-\sqrt{4x^2-3}}{6}, \qquad y_2(x) = \frac{4x-3+\sqrt{4x^2-3}}{6}. \tag{2.4}$$

It follows from the properties of the quadratic equation that $\mu(x, y) > 0$ for $y_1(x) < y < y_2(x)$ and $\mu(x, y) < 0$ for $0 < y < y_1(x)$ or $y > y_2(x)$.

Differentiating $y_1(x)$ with respect to x, one has

$$\begin{aligned}\frac{dy_1(x)}{dx} &= \frac{2}{3}\left(1 - \frac{x}{\sqrt{4x^2-3}}\right),\\ \frac{d^2y_1(x)}{dx^2} &= \frac{3}{(4x^2-3)^{3/2}} > 0.\end{aligned} \tag{2.5}$$

By (2.5) we know that the minimal value of $y_1(x)$ can be attained at $x = 1$, that is $y_1(x) \geq y_1(1) = 0$. Moreover, $y_1(x)$ is strictly decreasing on $(\sqrt{3}/2, 1)$ and strictly increasing on $(1, \infty)$.

Obviously, $y_2(x)$ is strictly increasing on $(\sqrt{3}/2, \infty)$. Note that

$$y_2(x) - \left(x - \frac{1}{2}\right) = -\frac{1}{2\left(\sqrt{4x^2 - 3} + 2x\right)} \longrightarrow 0 \tag{2.6}$$

as $x \longrightarrow +\infty$. In other words, $y_2(x) < x - 1/2$ and $y_2(x)$ has the asymptotic line $y = x - 1/2$.

Lemma 2.1. *The psi or digamma function, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed as*

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt, \tag{2.7}$$

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-xt} dt \tag{2.8}$$

for $x > 0$ and $n \in \mathbb{N} := \{1, 2, \ldots\}$, where $\gamma = 0.5772\cdots$ is Euler's constant.

Lemma 2.2. *Let $\alpha, \beta \in (0, \infty)$ and $g(t) = te^{-\alpha t} - (1 - e^{-t})e^{-\beta t}$. Then the following statements are true:*

(1) *if $0 < \alpha \le \beta$, then $g(t) > 0$ for $t \in (0, \infty)$;*

(2) *if $0 < \beta \le \alpha - 1/2$, then $g(t) < 0$ for $t \in (0, \infty)$;*

(3) *if $\beta > 0$ and $\alpha - 1/2 < \beta < \alpha$, then there exist $\delta_2 > \delta_1 > 0$ such that $g(t) > 0$ for $t \in (0, \delta_1)$ and $g(t) < 0$ for $t \in (\delta_2, \infty)$.*

Proof. Let $g_1(t) = e^{\alpha t} g'(t)$ and $g_2(t) = e^{(\beta - \alpha + 1)t} g_1''(t)$. Then simple computations lead to

$$g(0) = 0, \quad \lim_{t \to \infty} g(t) = 0, \tag{2.9}$$

$$g'(t) = (1 - \alpha t)e^{-\alpha t} + \beta e^{-\beta t} - (\beta + 1)e^{-(\beta+1)t}, \quad g_1(0) = g'(0) = 0, \tag{2.10}$$

$$g_1(t) = \beta e^{(\alpha - \beta)t} - (\beta + 1)e^{(\alpha - \beta - 1)t} + 1 - \alpha t, \tag{2.11}$$

$$g_1'(t) = \beta(\alpha - \beta)e^{(\alpha - \beta)t} + (\beta + 1)(\beta - \alpha + 1)e^{(\alpha - \beta - 1)t} - \alpha, \quad g_1'(0) = 2(\beta - \alpha) + 1, \tag{2.12}$$

$$g_1''(t) = \beta(\alpha - \beta)^2 e^{(\alpha - \beta)t} - (\beta + 1)(\beta - \alpha + 1)^2 e^{(\alpha - \beta - 1)t}, \quad g_2(0) = \mu(\alpha, \beta), \tag{2.13}$$

$$g_2(t) = \beta(\alpha - \beta)^2 e^t - (\beta + 1)(\beta - \alpha + 1)^2 \tag{2.14}$$

$$g_2'(t) = \beta(\alpha - \beta)^2 e^t. \tag{2.15}$$

(1) If $0 < \alpha \leq \beta$, then we divide the proof into two cases.

Case 1. If $\alpha = \beta$, then $g_1'(t) = (\alpha+1)e^{-t} - \alpha$ implies that $g_1'(t) > 0$ for $t \in (0, \log(1+1/\alpha))$ and $g_1'(t) < 0$ for $t \in (\log(1+1/\alpha), \infty)$. Thus $g_1(t)$ is strictly increasing on $(0, \log(1+1/\alpha))$ and strictly decreasing on $(\log(1+1/\alpha), \infty)$. From (2.10) and $\lim_{t\to+\infty} g_1(t) = -\infty$ we clearly see that there exists $\zeta_1 > \log(1+1/\alpha) > 0$ such that $g_1(t) > 0$ for $t \in (0, \zeta_1)$ and $g_1(t) < 0$ for $t \in (\zeta_1, \infty)$, which implies that $g(t)$ is strictly increasing on $(0, \zeta_1)$ and strictly decreasing on (ζ_1, ∞). It follows from (2.9) that

$$g(t) > \min\left\{g(0), \lim_{t\to+\infty} g(t)\right\} = 0 \tag{2.16}$$

for $t \in (0, \infty)$.

Case 2. If $0 < \alpha < \beta$, then we know $\mu(\alpha, \beta) < 0$ since $\beta > \alpha - 1/2 > y_2(\alpha)$. It follows from (2.13) and (2.15) that

$$\begin{gathered} g_2(0) < 0, \\ g_2'(t) > 0. \end{gathered} \tag{2.17}$$

Therefore, there exists $\zeta_2 > 0$ such that $g_2(t) < 0$ for $t \in (0, \zeta_2)$ and $g_2(t) > 0$ for $t \in (\zeta_2, \infty)$ follows from (2.17), which implies that $g_1'(t)$ is strictly decreasing on $(0, \zeta_2)$ and strictly increasing on (ζ_2, ∞). It follows from (2.12) and $\lim_{t\to+\infty} g_1'(t) = -\alpha < 0$ that there exists $\zeta_3 > \zeta_2 > 0$ such that $g_1'(t) > 0$ for $t \in (0, \zeta_3)$ and $g_1'(t) < 0$ for $t \in (\zeta_3, \infty)$. By the same argument, it follows from (2.10) and $\lim_{t\to+\infty} g_1(t) = -\infty$ that there exists $\zeta_4 > \zeta_3$ such that $g_1(t) > 0$ for $t \in (0, \zeta_4)$ and $g_1(t) < 0$ for $t \in (\zeta_4, \infty)$.

Therefore, $g(t) > 0$ for $t \in (0, \infty)$ follows from (2.9).

(2) If $0 < \beta \leq \alpha - 1/2$, then from Figure 1 we know that $\mu(x, y)$ could be positive or negative. We divide the proof into two cases.

Case 1. If $\mu(\alpha, \beta) \geq 0$, then from (2.13) and (2.15) we clearly know that $g_2(t) > 0$ for $t \in (0, \infty)$, which implies that ${g'}_1(t)$ is strictly increasing on $(0, \infty)$. Then the properties of $\mu(x, y)$ and $\mu(\alpha, \beta) \geq 0$ lead to

$$\beta < y_2(\alpha) < \alpha - \frac{1}{2}. \tag{2.18}$$

It follows from (2.12) and (2.18) that there exists $\zeta_5 > 0$ such that $g_1'(t) < 0$ for $t \in (0, \zeta_5)$ and $g_1'(t) > 0$ for $t \in (\zeta_5, \infty)$ since $g_1'(t) \to +\infty$ as $t \to +\infty$. Hence $g_1(t)$ is strictly decreasing on $(0, \zeta_5)$ and strictly increasing on (ζ_5, ∞). From (2.10) and $\lim_{t\to+\infty} g_1(t) = +\infty$ we know that there exists $\zeta_6 > \zeta_5$ such that $g_1(t) < 0$ for $t \in (0, \zeta_6)$ and $g_1(t) > 0$ for $t \in (\zeta_6, \infty)$.Therefore, it

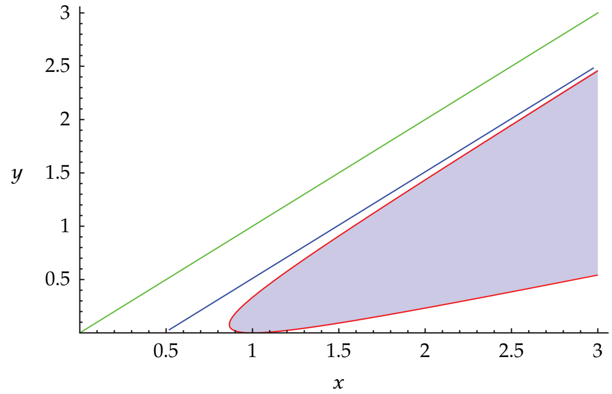

Figure 1: The shading area is denoted by $\mu(x,y) > 0$. Otherwise, $\mu(x,y) < 0$. The red curve is the graph of $\mu(x,y) = 0$ with an asymptotic line $y = x - 1/2$.

follows from (2.9) that

$$g(t) < \max\left\{g(0), \lim_{t\to+\infty} g(t)\right\} = 0 \tag{2.19}$$

for $t \in (0,\infty)$.

Case 2. If $\mu(\alpha,\beta) < 0$, then from (2.13) and (2.15) we know that there exists $\zeta_7 > 0$ such that $g_2(t) < 0$ for $t \in (0,\zeta_7)$ and $g_2(t) > 0$ for $t \in (\zeta_7,\infty)$. Thus $g_1'(t)$ is strictly decreasing on $(0,\zeta_7)$ and strictly increasing on (ζ_7,∞). It follows from (2.12) and $\lim_{t\to+\infty} g_1'(t) = +\infty$ that there exists $\zeta_8 > 0$ such that $g_1'(t) < 0$ for $t \in (0,\zeta_8)$ and $g_1'(t) > 0$ for $t \in (\zeta_8,\infty)$. By the same argument as Case 1, $g(t) < 0$ for $t \in (0,\infty)$ follows from (2.9) and (2.10).

(3) If $\beta > 0$ and $\alpha - 1/2 < \beta < \alpha$, then from (2.12) we clearly know that $g_1'(0) > 0$. Thus there exists $\delta_1 > 0$ such that $g_1'(t) > 0$ for $t \in (0,\delta_1)$. It follows from (2.10) that $g_1(t) > 0$, $t \in (0,\delta_1)$. Since $g_1(t) \to +\infty$ as $t \to +\infty$, we know that there exists $\delta_2 > \delta_1$ such that $g_1(t) > 0$ for $t \in (\delta_2,\infty)$, which implies that $g(t)$ is strictly increasing on $(0,\delta_1)$ and (δ_2,∞). Therefore, $g(t) > g(0) = 0$ for $t \in (0,\delta_1)$ and $g(t) < \lim_{t\to+\infty} g(t) = 0$ for $t \in (\delta_2,\infty)$. □

We state a simple lemma as the results of [12, 47].

Lemma 2.3. *Inequality*

$$\log x - \frac{1}{2x} - \frac{1}{12x^2} < \psi(x) < \log x - \frac{1}{2x} \tag{2.20}$$

holds for $x > 0$.

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Taking the logarithm of (1.8) and differentiating, then we have

$$-\left[\log G_{\alpha,\beta}(x)\right]' = \log(x+\beta) - \psi(x+\alpha). \tag{3.1}$$

For $n \geq 1$, it follows from (2.8) that

$$\begin{aligned}(-1)^{n+1}\left[\log G_{\alpha,\beta}(x)\right]^{(n+1)} &= (-1)^{n+1}\left[\psi^{(n)}(x+\alpha) - (-1)^{n-1}\frac{(n-1)!}{(x+\beta)^n}\right] \\ &= \int_0^\infty \frac{t^n}{1-e^{-t}}e^{-(x+\alpha)t}dt - \int_0^\infty t^{n-1}e^{-(x+\beta)t}dt \\ &= \int_0^\infty \frac{t^{n-1}e^{-xt}}{1-e^{-t}}g(t)dt,\end{aligned} \tag{3.2}$$

where

$$g(t) = te^{-\alpha t} - \left(1-e^{-t}\right)e^{-\beta t}. \tag{3.3}$$

(1) If $0 < \alpha \leq \beta$, then it follows from (3.1) and (2.20) that

$$-\left[\log G_{\alpha,\beta}(x)\right]' > \log(x+\beta) - \log(x+\alpha) + \frac{1}{2(x+\alpha)} > 0. \tag{3.4}$$

From (3.2) and (3.3) together with Lemma 2.2(1) we clearly see that

$$(-1)^{n+1}\left[\log G_{\alpha,\beta}(x)\right]^{(n+1)} > 0 \tag{3.5}$$

holds for $n \geq 1$. Therefore, $G_{\alpha,\beta}(x)$ is strictly logarithmically completely monotonic on $(0,\infty)$ that follows from (3.4) and (3.5).

Conversely, if $0 < \beta < \alpha$, then we can divide the set $\{(\alpha,\beta) : 0 < \beta < \alpha\}$ into two subsets: $\mathbf{\Omega_1} = \{(\alpha,\beta) : 0 < \beta \leq \alpha - 1/2\}$ and $\mathbf{\Omega_2} = \{(\alpha,\beta) : \beta > 0, \alpha - 1/2 < \beta < \alpha\}$. Therefore, it follows from Lemma 2.2(2) and (3) that $G_{\alpha,\beta}(x)$ is not strictly logarithmically completely monotonic on $(0,\infty)$ for $(\alpha,\beta) \in \mathbf{\Omega_1} \cup \mathbf{\Omega_2}$.

(2) If $0 < \beta \leq \alpha - 1/2$, then from (3.1) and (2.20) we get

$$\begin{aligned}-\left\{\log\left[G_{\alpha,\beta}(x)\right]^{-1}\right\}' &> \log\frac{x+\alpha}{x+\beta} - \frac{1}{2(x+\alpha)} - \frac{1}{12(x+\alpha)^2} \\ &\geq \log\frac{x+\alpha}{x+\alpha-1/2} - \frac{1}{2(x+\alpha)} - \frac{1}{12(x+\alpha)^2} \\ &\triangleq \eta(x) > 0,\end{aligned} \tag{3.6}$$

since

$$\lim_{x \to +\infty} \eta(x) = 0, \qquad \frac{d\eta(x)}{dx} = -\frac{x+\alpha+1}{6(x+\alpha)^3(2x+2\alpha-1)} < 0. \tag{3.7}$$

For $n \geq 1$, it follows from (3.2) that

$$(-1)^{n+1}\left\{\log\left[G_{\alpha,\beta}(x)\right]^{-1}\right\}^{(n+1)} = -\int_0^{\infty} \frac{t^{n-1}e^{-xt}}{1-e^{-t}} g(t)dt, \tag{3.8}$$

where $g(t)$ is defined as (3.3).

Therefore, $[G_{\alpha,\beta}(x)]^{-1}$ is strictly logarithmically completely monotonic on $(0,\infty)$ that follows from (3.6), (3.8), and Lemma 2.2(2).

Conversely, if $\beta > 0$ and $\beta > \alpha - 1/2$, then we can divide the set $\{(\alpha,\beta) : \beta > 0, \beta > \alpha - 1/2\}$ into two subsets: $\mathbf{\Omega}_1' = \{(\alpha,\beta) : 0 < \alpha \leq \beta\}$ and $\mathbf{\Omega}_2' = \{(\alpha,\beta) : \beta > 0, \alpha - 1/2 < \beta < \alpha\}$. Therefore, it follows from Lemma 2.2(1) and (3) that $[G_{\alpha,\beta}(x)]^{-1}$ is not strictly logarithmically completely monotonic on $(0,\infty)$ for $(\alpha,\beta) \in \mathbf{\Omega}_1' \cup \mathbf{\Omega}_2'$. □

Remark 1. Although the upper and lower bounds of Kershaw's inequalities given in (1.11) and (1.12) are not better than those of inequalities in (1.4) and (1.5), the difference between them is close to zero as x is large enough. For example,

$$\begin{aligned} &\log\left[\frac{e^{s-1}(x+1)^{x+1}}{(x+s)^{x+s}} \Big/ \left(x - \frac{1}{2} + \sqrt{s+\frac{1}{4}}\right)^{1-s}\right] \\ &= s - 1 + (1-s)\log\left(1 + \frac{1-s}{x+s}\right)^{(x+s)/(1-s)} + (1-s)\log\frac{x+1}{x-1/2+\sqrt{s+1/4}} \\ &\Longrightarrow s - 1 + 1 - s = 0 \quad (x \longrightarrow \infty). \end{aligned} \tag{3.9}$$

Furthermore, the advantage of our inequalities is to give the upper and lower bounds of Kershaw's inequality for $s > 1$ and $x \geq 0$ while Laforgia established only one side of Kershaw's inequality.

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