

Research Article

Convergence Theorems Concerning Hybrid Methods for Strict Pseudocontractions and Systems of Equilibrium Problems

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Let $\{S_i\}_{i=1}^N$ be N strict pseudo-contractions defined on a closed and convex subset C of a real Hilbert space H . We consider the problem of finding a common element of fixed point set of these mappings and the solution set of a system of equilibrium problems by parallel and cyclic algorithms. In this paper, new iterative schemes are proposed for solving this problem. Furthermore, we prove that these schemes converge strongly by hybrid methods. The results presented in this paper improve and extend some well-known results in the literature.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty, closed, and convex subset of H .

Let $\{F_k\}$ be a countable family of bifunctions from $C \times C$ to \mathbb{R} , where \mathbb{R} is the set of real numbers. Combettes and Hirstoaga [1] considered the following system of equilibrium problems:

$$\text{finding } x \in C \text{ such that } F_k(x, y) \geq 0, \quad \forall k \in \Gamma, \forall y \in C, \quad (1.1)$$

where Γ is an arbitrary index set. If Γ is a singleton, then problem (1.1) becomes the following equilibrium problem:

$$\text{finding } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \quad (1.2)$$

The solution set of (1.2) is denoted by $EP(F)$.

A mapping S of C is said to be a κ -strict pseudocontraction if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2 \quad (1.3)$$

for all $x, y \in C$; see [2]. We denote the fixed point set of S by $F(S)$, that is, $F(S) = \{x \in C : Sx = x\}$.

Note that the class of strict pseudocontractions properly includes the class of nonexpansive mappings which are mapping S on C such that

$$\|Sx - Sy\| \leq \|x - y\| \quad (1.4)$$

for all $x, y \in C$. That is, S is nonexpansive if and only if S is a 0-strict pseudocontraction.

The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games, and others; see, for instance, [1, 3, 4] and the references therein. Some methods have been proposed to solve the equilibrium problem (1.1); related work can also be found in [5–8].

Recently, Acedo and Xu [9] considered the problem of finding a common fixed point of a finite family of strict pseudo-contractive mappings by the parallel and cyclic algorithms. Very recently, Duan and Zhao [10] considered new hybrid methods for equilibrium problems and strict pseudocontractions. In this paper, motivated by [5, 8–12], applying parallel and cyclic algorithms, we obtain strong convergence theorems for finding a common element of the fixed point set of a finite family of strict pseudocontractions and the solution set of the system of equilibrium problems (1.1) by the hybrid methods.

We will use the following notations:

- (1) \rightharpoonup for the weak convergence and \rightarrow for the strong convergence,
- (2) $\omega_\omega(x_n) = \{x : \exists x_{n_j} \rightarrow x\}$ denotes the weak ω -limit set of $\{x_n\}$.

2. Preliminaries

We will use the facts and tools in a real Hilbert space H which are listed below.

Lemma 2.1. *Let H be a real Hilbert space. Then the following identities hold:*

- (i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle$, for all $x, y \in H$,
- (ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$, for all $t \in [0, 1]$, for all $x, y \in H$.

Lemma 2.2 (see [6]). *Let H be a real Hilbert space. Given a nonempty, closed, and convex subset $C \subset H$, points $x, y, z \in H$, and a real number $a \in \mathbb{R}$, then the set*

$$\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v \rangle + a\} \quad (2.1)$$

is convex (and closed).

Recall that given a nonempty, closed, and convex subset C of a real Hilbert space H , for any $x \in H$, there exists the unique nearest point in C , denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad (2.2)$$

for all $y \in C$. Such a P_C is called the metric (or the nearest point) projection of H onto C . As we all know $y = P_C x$ if and only if there holds the relation

$$\langle x - y, y - z \rangle \geq 0 \quad \forall z \in C. \quad (2.3)$$

Lemma 2.3 (see [13]). *Let C be a nonempty, closed, and convex subset of H . Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. Suppose that $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the following condition:*

$$\|x_n - u\| \leq \|u - q\| \quad \forall n. \quad (2.4)$$

Then $x_n \rightarrow q$.

Proposition 2.4 (see [9]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H .*

(i) *If $T : C \rightarrow C$ is a κ -strict pseudocontraction, then T satisfies the Lipschitz condition*

$$\|Tx - Ty\| \leq \frac{1 + \kappa}{1 - \kappa} \|x - y\|, \quad \forall x, y \in C. \quad (2.5)$$

- (ii) *If $T : C \rightarrow C$ is a κ -strict pseudocontraction, then the mapping $I - T$ is demiclosed (at 0). That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightarrow x$ and $(I - T)x_n \rightarrow 0$, then $(I - T)x = 0$.*
- (iii) *If $T : C \rightarrow C$ is a κ -strict pseudocontraction, then the fixed point set $F(T)$ of T is closed and convex. Therefore the projection $P_{F(T)}$ is well defined.*
- (iv) *Given an integer $N \geq 1$, assume that, for each $1 \leq i \leq N$, $T_i : C \rightarrow C$ is a κ_i -strict pseudocontraction for some $0 \leq \kappa_i < 1$. Assume that $\{\lambda_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \lambda_i = 1$. Then $\sum_{i=1}^N \lambda_i T_i$ is a κ -strict pseudocontraction, with $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$.*
- (v) *Let $\{T_i\}_{i=1}^N$ and $\{\lambda_i\}_{i=1}^N$ be given as in item (iv). Suppose that $\{T_i\}_{i=1}^N$ has a common fixed point. Then*

$$F\left(\sum_{i=1}^N \lambda_i T_i\right) = \bigcap_{i=1}^N F(T_i). \quad (2.6)$$

Lemma 2.5 (see [2]). *Let $S : C \rightarrow H$ be a κ -strict pseudocontraction. Define $T : C \rightarrow H$ by $Tx = \lambda x + (1 - \lambda)Sx$ for any $x \in C$. Then, for any $\lambda \in [\kappa, 1)$, T is a nonexpansive mapping with $F(T) = F(S)$.*

For solving the equilibrium problem, let one assume that the bifunction F satisfies the following conditions:

- (A1) $F(x, x) = 0$ for all $x \in C$;
- (A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;
- (A3) for each $x, y, z \in C$, $\limsup_{t \rightarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;
- (A4) $F(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

Lemma 2.6 (see [3]). *Let C be a nonempty, closed, and convex subset of H , let F be bifunction from $C \times C$ to \mathbb{R} which satisfies conditions (A1)–(A4), and let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C. \quad (2.7)$$

Lemma 2.7 (see [1]). *For $r > 0$, $x \in H$, define the mapping $T_r : H \rightarrow C$ as follows:*

$$T_r(x) = \{z \in C \mid F(z, y) + (1/r) \langle y - z, z - x \rangle \geq 0, \forall y \in C\} \quad (2.8)$$

for all $x \in H$. Then, the following statements hold:

- (i) T_r is single valued;
- (ii) T_r is firmly nonexpansive, that is, for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle; \quad (2.9)$$

- (iii) $F(T_r) = \text{EP}(F)$;
- (iv) $\text{EP}(F)$ is closed and convex.

3. Parallel Algorithm

In this section, we apply the hybrid methods to the parallel algorithm for finding a common element of the fixed point set of strict pseudocontractions and the solution set of the problem (1.1) in Hilbert spaces.

Theorem 3.1. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H , and let $F_k, k \in \{1, 2, \dots, M\}$, be bifunctions from $C \times C$ to \mathbb{R} which satisfies conditions (A1)–(A4). Let, for each $1 \leq i \leq N$, $S_i : C \rightarrow C$ be a κ_i -strict pseudocontraction for some $0 \leq \kappa_i < 1$. Let $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$. Assume that $\Omega = \bigcap_{i=1}^N F(S_i) \cap (\bigcap_{k=1}^M \text{EP}(F_k)) \neq \emptyset$. Assume also that $\{\eta_i^{(n)}\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \eta_i^{(n)} = 1$ for all $n \in \mathbb{N}$ and $\inf_{n \geq 1} \eta_i^{(n)} > 0$ for all $1 \leq i \leq N$. Let the mapping A_n be defined by*

$$A_n = \sum_{i=1}^N \eta_i^{(n)} S_i. \quad (3.1)$$

Given $x_1 \in C$, let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$, be sequences which are generated by the following algorithm:

$$\begin{aligned}
 u_n &= T_{r_{M,n}}^{F_M} T_{r_{M-1,n}}^{F_{M-1}} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} x_n, \\
 A_n^{\lambda_n} &= \lambda_n I + (1 - \lambda_n) A_n, \\
 y_n &= \alpha_n x_n + (1 - \alpha_n) A_n^{\lambda_n} u_n, \\
 C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\
 Q_n &= \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\
 x_{n+1} &= P_{C_n \cap Q_n} x_1,
 \end{aligned}
 \tag{3.2}$$

where $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$, $\{\lambda_n\} \subset [\kappa, b]$ for some $b \in [\kappa, 1)$, and $\{r_{k,n}\} \subset (0, \infty)$ satisfies $\liminf_{n \rightarrow \infty} r_{k,n} > 0$ for all $k \in \{1, 2, \dots, M\}$. Then, $\{x_n\}$ converge strongly to $P_\Omega x_1$.

Proof. Denote $\Theta_n^k = T_{r_{k,n}}^{F_k} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1}$ for every $k \in \{1, 2, \dots, M\}$ and $\Theta_n^0 = I$ for all $n \in \mathbb{N}$. Therefore $u_n = \Theta_n^M x_n$. The proof is divided into six steps.

Step 1. The sequence $\{x_n\}$ is well defined.

It is obvious that C_n is closed and Q_n is closed and convex for every $n \in \mathbb{N}$. From Lemma 2.2, we also get that C_n is convex.

Take $p \in \Omega$, since for each $k \in \{1, 2, \dots, M\}$, $T_{r_{k,n}}^{F_k}$ is nonexpansive, $p = T_{r_{k,n}}^{F_k} p$, and $u_n = \Theta_n^M x_n$, we have

$$\|u_n - p\| = \|\Theta_n^M x_n - \Theta_n^M p\| \leq \|x_n - p\|
 \tag{3.3}$$

for all $n \in \mathbb{N}$. From Proposition 2.4, Lemma 2.5, and (3.3), we get

$$\|y_n - p\| = \|\alpha_n x_n + (1 - \alpha_n) A_n^{\lambda_n} u_n - p\| \leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|A_n^{\lambda_n} u_n - p\| \leq \|x_n - p\|.
 \tag{3.4}$$

So $p \in C_n$ for all $n \in \mathbb{N}$. Thus $\Omega \subset C_n$. Next we will show by induction that $\Omega \subset Q_n$ for all $n \in \mathbb{N}$. For $n = 1$, we have $\Omega \subset C = Q_1$. Assume that $\Omega \subset Q_n$ for some $n \geq 1$. Since $x_{n+1} = P_{C_n \cap Q_n} x_1$, we obtain

$$\langle x_{n+1} - z, x_1 - x_{n+1} \rangle \geq 0, \quad \forall z \in C_n \cap Q_n.
 \tag{3.5}$$

As $\Omega \subset C_n \cap Q_n$ by induction assumption, the inequality holds, in particular, for all $z \in \Omega$. This together with the definition of Q_{n+1} implies that $\Omega \subset Q_{n+1}$. Hence $\Omega \subset Q_n$ holds for all $n \geq 1$. Thus $\Omega \subset C_n \cap Q_n$, and therefore the sequence $\{x_n\}$ is well defined.

Step 2.

$$\text{If } q = P_\Omega x_1, \quad \text{then } \|x_n - x_1\| \leq \|x_1 - q\|.
 \tag{3.6}$$

From the definition of Q_n we imply that $x_n = P_{Q_n}x_1$. This together with the fact that $\Omega \subset Q_n$ further implies that

$$\|x_n - x_1\| \leq \|x_1 - p\| \quad \forall p \in \Omega. \quad (3.7)$$

Then $\{x_n\}$ is bounded and (3.6) holds. From (3.3), (3.4), and Proposition 2.4 (i), we also obtain that $\{u_n\}$, $\{y_n\}$, and $\{S_i x_n\}$ are bounded.

Step 3. The following limit holds:

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (3.8)$$

From $x_n = P_{Q_n}x_1$ and $x_{n+1} \in Q_n$, we get $\langle x_{n+1} - x_n, x_n - x_1 \rangle \geq 0$. This together with Lemma 2.1 (i) implies that

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_1 - (x_n - x_1)\|^2 \\ &= \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - 2\langle x_{n+1} - x_n, x_n - x_1 \rangle \\ &\leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2. \end{aligned} \quad (3.9)$$

Then $\|x_n - x_1\| \leq \|x_{n+1} - x_1\|$, that is, the sequence $\{\|x_n - x_1\|\}$ is nondecreasing. Since $\{\|x_n - x_1\|\}$ is bounded, $\lim_{n \rightarrow \infty} \|x_n - x_1\|$ exists. Then (3.8) holds.

Step 4. The following limit holds:

$$\lim_{n \rightarrow \infty} \|A_n x_n - x_n\| = 0. \quad (3.10)$$

From $x_{n+1} \in C_n$, we have

$$\|y_n - x_n\| \leq \|x_{n+1} - x_n\| + \|y_n - x_{n+1}\| \leq 2\|x_{n+1} - x_n\|. \quad (3.11)$$

By (3.6), we obtain

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \quad (3.12)$$

Next we will show that

$$\lim_{n \rightarrow \infty} \|\Theta_n^k x_n - \Theta_n^{k-1} x_n\| = 0, \quad k = 1, 2, \dots, M. \quad (3.13)$$

Indeed, for $p \in \Omega$, it follows from the firm nonexpansivity of $T_{r_{k,n}}^{F_k}$ that for each $k \in \{1, 2, \dots, M\}$, we have

$$\begin{aligned} \left\| \Theta_n^k x_n - p \right\|^2 &= \left\| T_{r_{k,n}}^{F_k} \Theta_n^{k-1} x_n - T_{r_{k,n}}^{F_k} p \right\|^2 \\ &\leq \left\langle \Theta_n^k x_n - p, \Theta_n^{k-1} x_n - p \right\rangle \\ &= \frac{1}{2} \left(\left\| \Theta_n^k x_n - p \right\|^2 + \left\| \Theta_n^{k-1} x_n - p \right\|^2 - \left\| \Theta_n^k x_n - \Theta_n^{k-1} x_n \right\|^2 \right). \end{aligned} \quad (3.14)$$

Thus we get

$$\left\| \Theta_n^k x_n - p \right\|^2 \leq \left\| \Theta_n^{k-1} x_n - p \right\|^2 - \left\| \Theta_n^k x_n - \Theta_n^{k-1} x_n \right\|^2, \quad k = 1, 2, \dots, M, \quad (3.15)$$

which implies that, for each $k \in \{1, 2, \dots, M\}$,

$$\begin{aligned} \left\| \Theta_n^k x_n - p \right\|^2 &\leq \left\| \Theta_n^0 x_n - p \right\|^2 - \left\| \Theta_n^k x_n - \Theta_n^{k-1} x_n \right\|^2 - \left\| \Theta_n^{k-1} x_n - \Theta_n^{k-2} x_n \right\|^2 \\ &\quad - \dots - \left\| \Theta_n^2 x_n - \Theta_n^1 x_n \right\|^2 - \left\| \Theta_n^1 x_n - \Theta_n^0 x_n \right\|^2 \\ &\leq \left\| x_n - p \right\|^2 - \left\| \Theta_n^k x_n - \Theta_n^{k-1} x_n \right\|^2. \end{aligned} \quad (3.16)$$

Therefore, by the convexity of $\|\cdot\|^2$ and Lemma 2.5, we get

$$\begin{aligned} \left\| y_n - p \right\|^2 &\leq \alpha_n \left\| x_n - p \right\|^2 + (1 - \alpha_n) \left\| A_n^{\lambda_n} u_n - p \right\|^2 \\ &\leq \alpha_n \left\| x_n - p \right\|^2 + (1 - \alpha_n) \left\| u_n - p \right\|^2 \\ &\leq \alpha_n \left\| x_n - p \right\|^2 + (1 - \alpha_n) \left\| \Theta_n^k x_n - p \right\|^2 \\ &\leq \alpha_n \left\| x_n - p \right\|^2 + (1 - \alpha_n) \left(\left\| x_n - p \right\|^2 - \left\| \Theta_n^k x_n - \Theta_n^{k-1} x_n \right\|^2 \right) \\ &= \left\| x_n - p \right\|^2 - (1 - \alpha_n) \left\| \Theta_n^k x_n - \Theta_n^{k-1} x_n \right\|^2. \end{aligned} \quad (3.17)$$

It follows that

$$(1 - \alpha_n) \left\| \Theta_n^k x_n - \Theta_n^{k-1} x_n \right\|^2 \leq \left\| x_n - p \right\|^2 - \left\| y_n - p \right\|^2 \leq \left\| x_n - y_n \right\| (\left\| x_n - p \right\| + \left\| y_n - p \right\|). \quad (3.18)$$

Since $\{\alpha_n\} \subset [0, a]$, we get from (3.12) that (3.13) holds; then we have

$$\left\| u_n - x_n \right\| \leq \left\| u_n - \Theta_n^{M-1} x_n \right\| + \left\| \Theta_n^{M-1} x_n - \Theta_n^{M-2} x_n \right\| + \dots + \left\| \Theta_n^1 x_n - x_n \right\| \rightarrow 0. \quad (3.19)$$

Observe that $\|y_n - u_n\| \leq \|y_n - x_n\| + \|x_n - u_n\|$; we also have $\|y_n - u_n\| \rightarrow 0$ as $n \rightarrow \infty$. On the other hand, from $y_n = \alpha_n x_n + (1 - \alpha_n) A_n^{\lambda_n} u_n$, we observe that

$$\begin{aligned} (1 - \alpha_n) \|A_n^{\lambda_n} u_n - u_n\| &= \|(1 - \alpha_n)(A_n^{\lambda_n} u_n - u_n)\| \\ &= \|y_n - u_n - \alpha_n(x_n - u_n)\| \\ &\leq \|y_n - u_n\| + \alpha_n \|x_n - u_n\|. \end{aligned} \quad (3.20)$$

From $\{\alpha_n\} \subset [0, a]$, (3.19), and $\|y_n - u_n\| \rightarrow 0$, we obtain $\|A_n^{\lambda_n} u_n - u_n\| \rightarrow 0$ as $n \rightarrow \infty$. It is easy to see that

$$\|A_n^{\lambda_n} x_n - x_n\| \leq \|A_n^{\lambda_n} x_n - A_n^{\lambda_n} u_n\| + \|A_n^{\lambda_n} u_n - u_n\| + \|u_n - x_n\| \leq 2\|u_n - x_n\| + \|A_n^{\lambda_n} u_n - u_n\|. \quad (3.21)$$

Combining the above arguments and (3.2), we have

$$\|A_n^{\lambda_n} x_n - x_n\| = \|\lambda_n x_n + (1 - \lambda_n) A_n x_n - x_n\| = (1 - \lambda_n) \|A_n x_n - x_n\|. \quad (3.22)$$

Now, it follows from $\{\lambda_n\} \subset [\kappa, b]$ that $\|A_n x_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Step 5. The following implication holds:

$$\omega_w(x_n) \subset \Omega. \quad (3.23)$$

We first show that $\omega_w(x_n) \subset \bigcap_{i=1}^N F(S_i)$. To this end, we take $\omega \in \omega_w(x_n)$ and assume that $x_{n_j} \rightarrow \omega$ as $j \rightarrow \infty$ for some subsequence $\{x_{n_j}\}$ of x_n . Without loss of generality, we may assume that

$$\eta_i^{(n_j)} \rightarrow \eta_i \quad (\text{as } j \rightarrow \infty), \quad \forall 1 \leq i \leq N. \quad (3.24)$$

It is easily seen that each $\eta_i > 0$ and $\sum_{i=1}^N \eta_i = 1$. We also have

$$A_{n_j} x \rightarrow Ax \quad (\text{as } j \rightarrow \infty) \quad \forall x \in C, \quad (3.25)$$

where $A = \sum_{i=1}^N \eta_i S_i$. Note that, by Proposition 2.4, A is a κ -strict pseudocontraction and $F(A) = \bigcap_{i=1}^N F(S_i)$. Since

$$\begin{aligned} \|Ax_{n_j} - x_{n_j}\| &\leq \|A_{n_j} x_{n_j} - Ax_{n_j}\| + \|A_{n_j} x_{n_j} - x_{n_j}\| \\ &\leq \sum_{i=1}^N \left| \eta_i^{(n_j)} - \eta_i \right| \|S_i x_{n_j}\| + \|A_{n_j} x_{n_j} - x_{n_j}\|, \end{aligned} \quad (3.26)$$

we obtain by virtue of (3.10) and (3.24) that

$$\lim_{n \rightarrow \infty} \|Ax_{n_j} - x_{n_j}\| = 0. \quad (3.27)$$

So by the demiclosedness principle (Proposition 2.4 (ii)), it follows that $\omega \in F(A) = \bigcap_{i=1}^N F(S_i)$, and hence $\omega_\omega(x_n) \subset \bigcap_{i=1}^N F(S_i)$.

Next we will show that $\omega \in \bigcap_{k=1}^M \text{EP}(F_k)$. Indeed, by Lemma 2.6, we have that, for each $k = 1, 2, \dots, M$,

$$F_k(\Theta_n^k x_n, y) + \frac{1}{r_n} \langle y - \Theta_n^k x_n, \Theta_n^k x_n - \Theta_n^{k-1} x_n \rangle \geq 0, \quad \forall y \in C. \quad (3.28)$$

From (A2), we get

$$\frac{1}{r_n} \langle y - \Theta_n^k x_n, \Theta_n^k x_n - \Theta_n^{k-1} x_n \rangle \geq F_k(y, \Theta_n^k x_n), \quad \forall y \in C. \quad (3.29)$$

Hence,

$$\left\langle y - \Theta_{n_j}^k x_{n_j}, \frac{\Theta_{n_j}^k x_{n_j} - \Theta_{n_j}^{k-1} x_{n_j}}{r_{n_j}} \right\rangle \geq F_k(y, \Theta_{n_j}^k x_{n_j}), \quad \forall y \in C. \quad (3.30)$$

From (3.13), we obtain that $\Theta_{n_j}^k x_{n_j} \rightarrow \omega$ as $j \rightarrow \infty$ for each $k = 1, 2, \dots, M$ (especially, $u_{n_j} = \Theta_{n_j}^M x_{n_j}$). Together with (3.13) and (A4) we have, for each $k = 1, 2, \dots, M$, that

$$0 \geq F_k(y, \omega), \quad \forall y \in C. \quad (3.31)$$

For any $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)\omega$. Since $y \in C$ and $\omega \in C$, we obtain that $y_t \in C$, and hence $F_k(y_t, \omega) \leq 0$. So, we have

$$0 = F_k(y_t, y_t) \leq tF_k(y_t, y) + (1-t)F_k(y_t, \omega) \leq tF_k(y_t, y). \quad (3.32)$$

Dividing by t , we get, for each $k = 1, 2, \dots, M$, that

$$F_k(y_t, y) \geq 0, \quad \forall y \in C. \quad (3.33)$$

Letting $t \rightarrow 0$ and from (A3), we get

$$F_k(\omega, y) \geq 0 \quad (3.34)$$

for all $y \in C$ and $\omega \in \text{EP}(F_k)$ for each $k = 1, 2, \dots, M$, that is, $\omega \in \bigcap_{k=1}^M \text{EP}(F_k)$. Hence (3.23) holds.

Step 6. From (3.6) and Lemma 2.3, we conclude that $x_n \rightarrow q$, where $q = P_\Omega x_1$. \square

Remark 3.2. In 2007, Acedo and Xu studied the following CQ method [9]:

$$\begin{aligned} x_0 &\in C \text{ chosen arbitrarily,} \\ y_n &= \alpha_n x_n + (1 - \alpha_n) A_n x_n, \\ C_n &= \left\{ z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1 - \alpha_n)(\alpha_n - \kappa) \|x_n - A_n x_n\|^2 \right\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0. \end{aligned} \quad (3.35)$$

In this paper, we first turn the strict pseudocontraction A_n into nonexpansive mapping $A_n^{\lambda_n}$ then replace C_n with a more simple form in the iterative algorithm.

Remark 3.3. If $F_k(x, y) = 0$, $N = 1$, and $\lambda_n = \delta$, we can obtain [14, Theorem 1].

Remark 3.4. If $M = 1$, $N = 1$, $\kappa = 0$, and $\lambda_n = 0$ and we use $1 - \alpha_n$ to replace α_n , we can get the result that has been studied by Tada and Takahashi in [8] for nonexpansive mappings. If $F_k(x, y) = 0$, $N = 1$, $\kappa = 0$, and $\lambda_n = 0$, we can get [7, Theorem 3.1].

Theorem 3.5. Let C be a nonempty, closed, and convex subset of a real Hilbert space H , and let $F_k, k \in \{1, 2, \dots\}$, be bifunctions from $C \times C$ to \mathbb{R} which satisfies conditions (A1)–(A4). Let, for each $1 \leq i \leq N$, $S_i : C \rightarrow C$ be a κ_i -strict pseudocontraction for some $0 \leq \kappa_i < 1$. Let $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$. Assume that $\Omega = \bigcap_{i=1}^N F(S_i) \cap (\bigcap_{k=1}^M EP(F_k)) \neq \emptyset$. Assume also that $\{\eta_i^{(n)}\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \eta_i^{(n)} = 1$ for all n and $\inf_{n \geq 1} \eta_i^{(n)} > 0$ for all $1 \leq i \leq N$. Let the mapping A_n be defined by

$$A_n = \sum_{i=1}^N \eta_i^{(n)} S_i. \quad (3.36)$$

Given $x_1 \in C = C_1$, let x_n, u_n , and y_n be sequences which are generated by the following algorithm:

$$\begin{aligned} u_n &= T_{r_{M,n}}^{F_M} T_{r_{M-1,n}}^{F_{M-1}} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} x_n, \\ A_n^{\lambda_n} &= \lambda_n I + (1 - \lambda_n) A_n, \\ y_n &= \alpha_n x_n + (1 - \alpha_n) A_n^{\lambda_n} u_n, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} &= P_{C_{n+1}} x_1, \end{aligned} \quad (3.37)$$

where $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$, $\{\lambda_n\} \subset [\kappa, b]$ for some $b \in [\kappa, 1)$, and, $\{r_{k,n}\} \subset (0, \infty)$ satisfies $\liminf_{n \rightarrow \infty} r_{k,n} > 0$ for all $k \in \{1, 2, \dots, M\}$. Then, $\{x_n\}$ converge strongly to $P_\Omega x_1$.

Proof. The proof of this theorem is similar to that of Theorem 3.1.

Step 1. The sequence $\{x_n\}$ is well defined. We will show by induction that C_n is closed and convex for all n . For $n = 1$, we have $C = C_1$ which is closed and convex. Assume that C_n for some $n \geq 1$ is closed and convex; from Lemma 2.2, we have that C_{n+1} is also closed and convex; The proof of $\Omega \subset C_n$ is similar to the one in Step 1 of Theorem 3.1.

Step 2. $\|x_n - x_1\| \leq \|q - x_1\|$ for all n , where $q = P_\Omega x_1$.

Step 3. $\|x_{n+1} - x_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Step 4. $\|A_n x_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Step 5. $\omega_w(x_n) \subset \Omega$.

Step 6. $x_n \rightarrow q$.

The proof of Step 2–Step 6 is similar to that of Theorem 3.1. \square

Remark 3.6. If $M = 1$, we can obtain the two corresponding theorems in [10].

4. Cyclic Algorithm

Let C be a closed, and convex subset of a Hilbert space H , and let $\{S_i\}_{i=0}^{N-1}$ be $N\kappa_i$ -strict pseudocontractions on C such that the common fixed point set

$$\bigcap_{i=0}^{N-1} F(S_i) \neq \emptyset. \quad (4.1)$$

Let $x_0 \in C$, and let $\{\alpha_n\}_{n=0}^\infty$ be a sequence in $(0, 1)$. The cyclic algorithm generates a sequence $\{x_n\}_{n=1}^\infty$ in the following way:

$$\begin{aligned} x_1 &= \alpha_0 x_0 + (1 - \alpha_0) S_0 x_0, \\ x_2 &= \alpha_1 x_1 + (1 - \alpha_1) S_1 x_1, \\ &\dots \\ x_N &= \alpha_{N-1} x_{N-1} + (1 - \alpha_{N-1}) S_{N-1} x_{N-1}, \\ x_{N+1} &= \alpha_N x_N + (1 - \alpha_N) S_0 x_N, \\ &\dots \end{aligned} \quad (4.2)$$

In general, x_{n+1} is defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S_{[n]} x_n, \quad (4.3)$$

where $S_{[n]} = S_i$, with $i = n \pmod{N}$, $0 \leq i \leq N - 1$.

Theorem 4.1. Let C be a nonempty, closed, and convex subset of a real Hilbert space H , and let $F_k, k \in \{1, 2, \dots, M\}$, be bifunctions from $C \times C$ to \mathbb{R} which satisfies conditions (A1)–(A4). Let, for each $0 \leq i \leq N - 1$, $S_i : C \rightarrow C$ be a κ_i -strict pseudocontraction for some $0 \leq \kappa_i < 1$. Let $\kappa = \max\{\kappa_i : 0 \leq i \leq N - 1\}$. Assume that $\Omega = \bigcap_{i=0}^{N-1} F(S_i) \cap (\bigcap_{k=1}^M \text{EP}(F_k)) \neq \emptyset$. Given $x_0 \in C$, let x_n, u_n , and y_n be sequences which are generated by the following algorithm:

$$\begin{aligned} u_n &= T_{r_{M,n}}^{F_M} T_{r_{M-1,n}}^{F_{M-1}} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} x_n, \\ S_{[n]}^{\lambda_n} &= \lambda_n I + (1 - \lambda_n) S_{[n]}, \\ y_n &= \alpha_n x_n + (1 - \alpha_n) S_{[n]}^{\lambda_n} u_n, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \end{aligned} \tag{4.4}$$

where $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$, $\{\lambda_n\} \subset [\kappa, b]$ for some $b \in [\kappa, 1)$, and $\{r_{k,n}\} \subset (0, \infty)$ satisfies $\liminf_{n \rightarrow \infty} r_{k,n} > 0$ for all $k \in \{1, 2, \dots, M\}$. Then, $\{x_n\}$ converge strongly to $P_{\Omega} x_0$.

Proof. The proof of this theorem is similar to that of Theorem 3.1. The main points are the following.

Step 1. The sequence $\{x_n\}$ is well defined.

Step 2. $\|x_n - x_0\| \leq \|q - x_0\|$ for all n , where $q = P_{\Omega} x_0$.

Step 3. $\|x_{n+1} - x_n\| \rightarrow 0$.

Step 4. $\|S_{[n]} x_n - x_n\| \rightarrow 0$. To prove the above steps, one simply replaces A_n with $S_{[n]}$ in the proof of Theorem 3.1.

Step 5. $\omega_w(x_n) \subset \Omega$. Indeed, let $\omega \in \omega_w(x_n)$ and $x_{n_m} \rightarrow \omega$ for some subsequence $\{x_{n_m}\}$ of $\{x_n\}$. We may assume that $l = n_m \pmod{N}$ for all m . Since, by $\|x_{n+1} - x_n\| \rightarrow 0$, we also have $x_{n_m+j} \rightarrow \omega$ for all $j \geq 0$, we deduce that

$$\|x_{n_m+j} - S_{[l+j]} x_{n_m+j}\| = \|x_{n_m+j} - S_{[n_m+j]} x_{n_m+j}\| \rightarrow 0. \tag{4.5}$$

Then the demiclosedness principle implies that $\omega \in F(S_{[l+j]})$ for all j . This ensures that $\omega \in \bigcap_{i=1}^N F(S_i)$. The Proof of $\omega \in \bigcap_{k=1}^M \text{EP}(F_k)$ is similar to that of Theorem 3.1.

Step 6. The sequence x_n converges strongly to q .

The strong convergence to q of $\{x_n\}$ is a consequence of Step 2, Step 5, and Lemma 2.3. □

Theorem 4.2. Let C be a nonempty, closed, and convex subset of a real Hilbert space H , and let $F_k, k \in \{1, 2, \dots, M\}$, be bifunctions from $C \times C$ to \mathbb{R} which satisfies conditions (A1)–(A4). Let, for each $0 \leq i \leq N - 1$, $S_i : C \rightarrow C$ be a κ_i -strict pseudocontraction for some $0 \leq \kappa_i < 1$. Let

$\kappa = \max\{\kappa_i : 0 \leq i \leq N-1\}$. Assume that $\Omega = \bigcap_{i=0}^{N-1} F(S_i) \cap (\bigcap_{k=1}^M \text{EP}(F_k)) \neq \emptyset$. Given $x_0 \in C = C_0$, let x_n , u_n , and y_n be sequences which are generated by the following algorithm:

$$\begin{aligned} u_n &= T_{r_{M,n}}^{F_M} T_{r_{M-1,n}}^{F_{M-1}} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} x_n, \\ S_{[n]}^{\lambda_n} &= \lambda_n I + (1 - \lambda_n) S_{[n]}, \\ y_n &= \alpha_n x_n + (1 - \alpha_n) S_{[n]}^{\lambda_n} u_n, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} &= P_{C_{n+1}} x_0, \end{aligned} \tag{4.6}$$

where $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$, $\{\lambda_n\} \subset [\kappa, b]$ for some $b \in [\kappa, 1)$, and $\{r_{k,n}\} \subset (0, \infty)$ satisfies $\liminf_{n \rightarrow \infty} r_{k,n} > 0$ for all $k \in \{1, 2, \dots\}$. Then, $\{x_n\}$ converge strongly to $P_\Omega x_0$.

Proof. The proof of this theorem is similar to that of Step 1 in Theorem 3.5 and Step 2–Step 6 in Theorem 4.1. \square

Remark 4.3. If $M = 1$, we can obtain the two corresponding theorems in [10].

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