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Research Article

Generalized Vector-Valued Sequence Spaces Defined by Modulus Functions

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We introduce the vector-valued sequence spaces $w_\infty(\Delta^m, F, Q, p, u)$, $w_1(\Delta^m, F, Q, p, u)$, and $w_0(\Delta^m, F, Q, p, u)$, S_u^q and S_{0u}^q, using a sequence of modulus functions and the multiplier sequence $u = (u_k)$ of nonzero complex numbers. We give some relations related to these sequence spaces. It is also shown that if a sequence is strongly $\Delta^m u_q$-Cesàro summable with respect to the modulus function f then it is $\Delta^m u_q$-statistically convergent.

1. Introduction

Let w be the set of all sequences real or complex numbers and ℓ_∞, c, and c_0 be, respectively, the Banach spaces of *bounded*, *convergent*, and *null sequences* $x = (x_k)$ with the usual norm $\|x\| = \sup |x_k|$, where $k \in \mathbb{N} = \{1, 2, \ldots\}$, the set of positive integers.

The studies on vector-valued sequence spaces are done by Das and Choudhury [1], Et [2], Et et al. [3], Leonard [4], Rath and Srivastava [5], J. K. Srivastava and B. K. Srivastava [6], Tripathy et al. [7, 8], and many others.

Let (E_k, q_k) be a sequence of seminormed spaces such that $E_{k+1} \subset E_k$ for each $k \in \mathbb{N}$. We define

$$w(E) = \{x = (x_k) : x_k \in E_k \text{ for each } k \in \mathbb{N}\}. \tag{1.1}$$

It is easy to verify that $w(E)$ is a linear space under usual coordinatewise operations defined by $x + y = (x_k + y_k)$ and $\alpha x = (\alpha x_k)$, where $\alpha \in \mathbb{C}$.

Let $u = (u_k)$ be a sequences of nonzero scalar. Then for a sequence space E, the multiplier sequence space $E(u)$, associated with the multiplier sequence u, is defined as $E(u) = \{(x_k) \in w : (u_k x_k) \in E\}$.

The notion of a modulus was introduced by Nakano [9]. We recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that

(i) $f(x)=0$ if and only if $x=0$,

(ii) $f(x+y)\le f(x)+f(y)$ for $x,y\ge 0$,

(iii) f is increasing,

(iv) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0,\infty)$. Maddox [10] and Ruckle [11] used a modulus function to construct some sequence spaces.

After then some sequence spaces defined by a modulus function were introduced and studied by Bilgin [12], Pehlivan and Fisher [13], Waszak [14], Bhardwaj [15], Altın [16], and many others.

The notion of difference sequence spaces was introduced by Kızmaz [17] and it was generalized by Et and Çolak [18]. Let m be a fixed positive integer. Then we write

$$X(\Delta^m)=\{x=(x_k):(\Delta^m x_k)\in X\} \tag{1.2}$$

for $X=\ell_\infty$, c and c_0, where $m\in\mathbb{N}$, $\Delta^m x=(\Delta^{m-1}x_k-\Delta^{m-1}x_{k+1})$, $\Delta^0 x=(x_k)$ and so we have

$$\Delta^m x_k=\sum_{\nu=0}^{m}(-1)^\nu\binom{m}{\nu}x_{k+\nu}. \tag{1.3}$$

2. Main Results

In this section, we prove some results involving the sequence spaces $w_0(\Delta^m,F,Q,p,u)$, $w_1(\Delta^m,F,Q,p,u)$, and $w_\infty(\Delta^m,F,Q,p,u)$.

Definition 2.1. Let (E_k,q_k) be a sequence of seminormed spaces such that $E_{k+1}\subset E_k$ for each $k\in\mathbb{N}$, $p=(p_k)$ a sequence of strictly positive real numbers, $Q=(q_k)$ a sequence of seminorms, $F=(f_k)$ a sequence of modulus functions, and $u=(u_k)$ any fixed sequence of nonzero complex numbers u_k. We define the following sequence spaces:

$$\begin{aligned}
w_0(\Delta^m,F,Q,p,u)&=\left\{x=(x_k):x_k\in E_k:\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q_k\left(u_k\Delta^m x_k\right)\right)\right]^{p_k}\longrightarrow 0,\ \text{as } n\longrightarrow\infty\right\},\\
w_1(\Delta^m,F,Q,p,u)&=\left\{x=(x_k):x_k\in E_k:\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q_k\left(u_k\Delta^m x_k-\ell\right)\right)\right]^{p_k}\longrightarrow 0,\right.\\
&\qquad\left.\text{as } n\longrightarrow\infty,\ \ell\in E_k\right\},\\
w_\infty(\Delta^m,F,Q,p,u)&=\left\{x=(x_k):x_k\in E_k:\sup_n\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q_k\left(u_k\Delta^m x_k\right)\right)\right]^{p_k}<\infty\right\}.
\end{aligned} \tag{2.1}$$

Throughout the paper Z will denote any one of the notation 0,1 or ∞.

If $f_k = f$ and $q_k = q$ for all $k \in \mathbb{N}$, we will write $w_Z(\Delta^m, f, q, p, u)$ instead of $w_Z(\Delta^m, F, Q, p, u)$.

If $f(x) = x$ and $p_k = 1$ for all $k \in \mathbb{N}$, we will write $w_Z(\Delta^m, q, u)$ instead of $w_Z(\Delta^m, f, q, p, u)$.

If $x \in w_1(\Delta^m, f, q, p, u)$, we say that x is strongly $\Delta^m u_q$-Cesàro summable with respect to the modulus function f and we will write $x_k \to \ell(w_1(\Delta^m, f, q, p, u))$ and ℓ will be called $\Delta^m u_q$-limit of x with respect to the modulus f.

The proofs of the following theorems are obtained by using the known standard techniques; therefore, we give them without proofs.

Theorem 2.2. *Let the sequence (p_k) be bounded. Then the spaces $w_Z(\Delta^m, F, Q, p, u)$ are linear spaces.*

Theorem 2.3. *Let f be a modulus function and the sequence (p_k) be bounded; then*

$$w_0(\Delta^m, f, q, p, u) \subset w_1(\Delta^m, f, q, p, u) \subset w_\infty(\Delta^m, f, q, p, u) \tag{2.2}$$

and the inclusions are strict.

Theorem 2.4. *$w_0(\Delta^m, F, Q, p, u)$ is a paranormed (need not total paranorm) space with*

$$g_\Delta(x) = \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} \left[f_k\left(q_k\left(u_k \Delta^m x_k\right)\right)\right]^{p_k} \right)^{1/M}, \tag{2.3}$$

where $M = \max(1, \sup p_k)$.

Theorem 2.5. *Let $F = (f_k)$ and $G = (g_k)$ be any two sequences of modulus functions. For any bounded sequences $p = (p_k)$ and $t = (t_k)$ of strictly positive real numbers and for any two sequences of seminorms $q = (q_k)$ and $r = (r_k)$, we have*

(i) *$w_Z(\Delta^m, f, Q, u) \subset w_Z(\Delta^m, f \circ g, Q, u)$;*

(ii) *$w_Z(\Delta^m, F, Q, p, u) \cap w_Z(\Delta^m, F, R, p, u) \subset w_Z(\Delta^m, F, Q + R, p, u)$;*

(iii) *$w_Z(\Delta^m, F, Q, p, u) \cap w_Z(\Delta^m, G, Q, p, u) \subset w_Z(\Delta^m, F + G, Q, p, u)$;*

(iv) *If q_k is stronger than r_k for each $k \in \mathbb{N}$, then $w_Z(\Delta^m, F, Q, p, u) \subset w_Z(\Delta^m, F, R, p, u)$;*

(v) *If q_k equivalent to r_k for each $k \in \mathbb{N}$, then $w_Z(\Delta^m, F, Q, p, u) = w_Z(\Delta^m, F, R, p, u)$;*

(vi) *$w_Z(\Delta^m, F, Q, p, u) \cap w_Z(\Delta^m, F, R, p, u) \neq \emptyset$.*

Proof. (i) We will only prove (i) for $Z = 0$ and the other cases can be proved by using similar arguments. Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \leq t \leq \delta$ and for all $k \in \mathbb{N}$. Write $y_k = g(q_k(u_k \Delta^m x_k))$ and consider

$$\sum_{k=1}^{n} [f(y_k)] = \sum_{1} [f(y_k)] + \sum_{2} [f(y_k)], \tag{2.4}$$

where the first summation is over $y_k \leq \delta$ and second summation is over $y_k > \delta$. Since f is continuous, we have

$$\sum_1 [f(y_k)] < n\varepsilon. \tag{2.5}$$

By the definition of f, we have for $y_k > \delta$,

$$f(y_k) < 2f(1)\frac{y_k}{\delta}. \tag{2.6}$$

Hence

$$\frac{1}{n}\sum_2 [f(y_k)] \leq 2\delta^{-1} f(1)\frac{1}{n}\sum_{k=1}^{n} y_k. \tag{2.7}$$

From (2.5) and (2.7), we obtain $w_0(\Delta^m, f, Q, u) \subset w_0(\Delta^m, f \circ g, Q, u)$. □

The following result is a consequence of Theorem 2.5(i).

Corollary 2.6. *Let f be a modulus function. Then $w_Z(\Delta^m, Q, u) \subset w_Z(\Delta^m, f, Q, u)$.*

Theorem 2.7. *Let $0 < p_k \leq t_k$ and (t_k/p_k) be bounded; then $w_Z(\Delta^m, F, Q, t, u) \subset w_Z(\Delta^m, F, Q, p, u)$.*

Proof. If we take $w_k = [f_k(q_k(u_k\Delta^m x_k))]^{t_k}$ for all k and using the same technique of Theorem 5 of Maddox [19], it is easy to prove the theorem. □

Theorem 2.8. *Let f be a modulus function; if $\lim_{t\to\infty}(f(t)/t) = \beta > 0$, then $w_1(\Delta^m, Q, p, u) = w_1(\Delta^m, f, Q, p, u)$.*

Proof. Omitted. □

3. $\Delta^m u_q$-Statistical Convergence

The notion of statistical convergence were introduced by Fast [20] and Schoenberg [21], independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, and number theory. Later on it was further investigated from the sequence space point of view and linked with summability theory by Šalát [22], Fridy [23], Connor [24], Mursaleen [25], Işik [26], Malkowsky and Savas [27], and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Čech compactification of the natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability. The notion depends on the density of subsets of the set $\mathbb{N}$ of natural numbers.

A subset E of $\mathbb{N}$ is said to have density positive integers which is defined by $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_E(k) \text{ exists}, \tag{3.1}$$

where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ have zero natural density and $\delta(E^c) = 1 - \delta(E)$.

In this section, we introduce $\Delta^m u_q$-statistically convergent sequences and give some inclusion relations between $\Delta^m u_q$-statistically convergent sequences and $w_1(f, q, p, u)$-summable sequences.

Definition 3.1. A sequence $x = (x_k)$ is said to be $\Delta^m u_q$-statistically convergent to ℓ if for every $\varepsilon > 0$,

$$\delta\left(\left\{k \in \mathbb{N} : q(u_k\Delta^m x_k - \ell) \geq \varepsilon\right\}\right) = 0. \tag{3.2}$$

In this case, we write $x_k \to \ell(S_u^q(\Delta^m))$. The set of all $\Delta^m u_q$-statistically convergent sequences is denoted by $S_u^q(\Delta^m)$. In the case $\ell = 0$, we will write $S_{0u}^q(\Delta^m)$ instead of $S_u^q(\Delta^m)$.

Theorem 3.2. *Let f be a modulus function; then*

(i) *If $x_k \to \ell(w_1(\Delta^m, q, u))$, then $x_k \to \ell(S_u^q(\Delta^m))$;*

(ii) *If $x \in \ell_\infty(\Delta^m u_q)$ and $x_k \to \ell(S_u^q(\Delta^m))$, then $x_k \to \ell(w_1(\Delta^m, q, u))$;*

(iii) *$S_u^q(\Delta^m) \cap \ell_\infty(\Delta^m u_q) = w_1(\Delta^m, q, u) \cap \ell_\infty(\Delta^m u_q)$,*

where $\ell_\infty(\Delta^m u_q) = \{x \in w(X) : \sup_k q(u_k\Delta^m x_k) < \infty\}$.

Proof. Omitted. □

In the following theorems, we will assume that the sequence $p = (p_k)$ is bounded and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$.

Theorem 3.3. *Let f be a modulus function. Then $w_1(\Delta^m, f, q, p, u) \subset S_u^q(\Delta^m)$.*

Proof. Let $x \in w_1(\Delta^m, f, q, p, u)$ and let $\varepsilon > 0$ be given. Let $\sum_1$ and $\sum_2$ denote the sums over $k \leq n$ with $q(u_k\Delta^m x_k - \ell) \geq \varepsilon$ and $q(u_k\Delta^m x_k - \ell) < \varepsilon$, respectively. Then

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n}\left[f\left(q(u_k\Delta^m x_k - \ell)\right)\right]^{p_k} &= \frac{1}{n}\sum_{1}\left[f\left(q(u_k\Delta^m x_k - \ell)\right)\right]^{p_k} \\ &\geq \frac{1}{n}\sum_{1}\left[f(\varepsilon)\right]^{p_k} \\ &\geq \frac{1}{n}\sum_{1}\min\left(\left[f(\varepsilon)\right]^{h}, \left[f(\varepsilon)\right]^{H}\right) \\ &\geq \frac{1}{n}\left|\left\{k \leq n : q(u_k\Delta^m x_k - \ell) \geq \varepsilon\right\}\right| \min\left(\left[f(\varepsilon)\right]^{h}, \left[f(\varepsilon)\right]^{H}\right). \end{aligned} \tag{3.3}$$

Hence $x \in S_u^q(\Delta^m)$. □

Theorem 3.4. *Let f be bounded; then $S_u^q(\Delta^m) \subset w_1(\Delta^m, f, q, p, u)$.*

Proof. Suppose that f is bounded. Let $\varepsilon > 0$ and $\sum_1$ and $\sum_2$ be denoted in previous theorem. Since f is bounded, there exists an integer K such that $f(x) < K$, for all $x \geq 0$. Then

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\left[f\left(q(u_k\Delta^m x_k - \ell)\right)\right]^{p_k} &\leq \frac{1}{n}\left(\sum_{1}\left[f\left(q(u_k\Delta^m x_k - \ell)\right)\right]^{p_k} + \sum_{2}\left[f\left(q(u_k\Delta^m x_k - \ell)\right)\right]^{p_k}\right) \\ &\leq \frac{1}{n}\sum_{1}\max\left(K^h, K^H\right) + \frac{1}{n}\sum_{2}\left[f(\varepsilon)\right]^{p_k} \\ &\leq \max\left(K^h, K^H\right)\frac{1}{n}\left|\left\{k \leq n : q(u_k\Delta^m x_k - \ell) \geq \varepsilon\right\}\right| \\ &\quad + \max\left(f(\varepsilon)^h, f(\varepsilon)^H\right).\end{aligned} \tag{3.4}$$

Hence $x \in w_1(\Delta^m, f, q, p, u)$. □

Theorem 3.5. *$S_u^q(\Delta^m) = w_1(\Delta^m, f, q, p, u)$ if and only if f is bounded.*

Proof. Let f be bounded. By Theorems 3.3 and 3.4, we have $S_u^q(\Delta^m) = w_1(\Delta^m, f, q, p, u)$.

Conversely suppose that f is unbounded. Then there exists a sequence (t_k) of positive numbers with $f(t_k) = k^2$, for $k = 1, 2, \ldots$. If we choose

$$u_i\Delta^m x_i = \begin{cases} t_k, & i = k^2,\ i = 1, 2, \ldots, \\ 0, & \text{otherwise}, \end{cases} \tag{3.5}$$

then we have

$$\frac{1}{n}\left|\left\{k \leq n : |u_k\Delta^m x_k| \geq \varepsilon\right\}\right| \leq \frac{\sqrt{n}}{n} \tag{3.6}$$

for all n and so $x \in S_u^q(\Delta^m)$, but $x \notin w_1(\Delta^m, f, q, p, u)$ for $X = \mathbb{C}$, $q(x) = |x|$ and $p_k = 1$ for all $k \in \mathbb{N}$. This contradicts to $S_u^q(\Delta^m) = w_1(\Delta^m, f, q, p, u)$. □

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