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Research Article

On Some Integral Inequalities on Time Scales and Their Applications

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The purpose of this paper is to investigate some new dynamic inequalities on time scales. We establish some new dynamic inequalities; the results unify and extend some continuous inequalities and their corresponding discrete analogues. The inequalities given here can be used as tools in the qualitative theory of certain dynamic equations. Some examples are given in the end of this paper.

1. Introduction

The theory of time scales was introduced by Hilger [1] in 1988 in order to contain both difference and differential calculus in a consistent way. Recently, many authors have extended some fundamental integral inequalities used in the theory of differential and integral equations on time scales. For example, we refer the reader to the papers [2–12] and the references cited there in.

In this paper, we investigate some nonlinear integral inequalities on time scales, which extend some inequalities established by Li and Sheng [8] and Li [9]. The obtained inequalities can be used as important tools in the study of dynamic equations on time scales.

Throughout this paper, let us assume that we have already acquired the knowledge of time scales and time scales notation; for an excellent introduction to the calculus on time scales, we refer the reader to Bohner and Peterson [4] for general overview.

2. Some Preliminaries on Time Scales

In what follows, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{C}$ denotes the set of complex numbers, and $C(M, S)$ denotes the class of all continuous functions defined on set M with range in the set S. $\mathbb{T}$ is an arbitrary time scale. If $\mathbb{T}$ has a right-scattered maximum m, then the set $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$. C_{rd} denotes the set of rd-continuous functions; $\mathcal{R}$ denotes the set of all regressive and rd-continuous functions. We define the set of all positively regressive functions by $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, t \in \mathbb{T}\}$. Obviously, if $p \in C_{\mathrm{rd}}$ and $p(t) \geq 0$ for $t \in \mathbb{T}$, then $p \in \mathcal{R}^+$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define $f^{\Delta}(t)$ as follows (provided it exists):

$$f^{\Delta}(t) := \lim_{s \to t} \frac{f^{\sigma}(t) - f(s)}{\sigma(t) - s}; \tag{2.1}$$

we call $f^{\Delta}(t)$ the *delta derivative* of f at t.

The following lemmas are very useful in our main results.

Lemma 2.1 (see [4]). *If $p \in \mathcal{R}$ and fix $t_0 \in \mathbb{T}$, then the exponential function $e_p(\cdot, t_0)$ is for the unique solution of the initial value problem*

$$x^{\Delta} = p(t)x, \quad x(t_0) = 1 \quad on\ \mathbb{T}. \tag{2.2}$$

Lemma 2.2 (see [4]). *Let $t_0 \in \mathbb{T}^k$ and $w : \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$ be continuous at (t, t), where $t \geq t_0$. Assume that $w^{\Delta}(t, \cdot)$ is rd-continuous on $[t_0, \sigma(t)]$. If for any $\varepsilon > 0$, there exists a neighborhood U of t, independent of $\tau \in [t_0, \sigma(t)]$, such that*

$$\left| w(\sigma(t), \tau) - w(s, \tau) - w^{\Delta}(t, \tau)(\sigma(t) - s) \right| \leq \varepsilon |\sigma(t) - s|, \quad s \in U, \tag{2.3}$$

where w^{Δ} denotes the derivative of w with respect to the first variable, then

$$g(t) := \int_{t_0}^{t} w(t, \tau) \Delta\tau \tag{2.4}$$

implies

$$g^{\Delta}(t) := \int_{t_0}^{t} w^{\Delta}(t, \tau) \Delta\tau + w(\sigma(t), t). \tag{2.5}$$

The following theorem is a foundational result in dynamic inequalities.

Lemma 2.3 (Comparison Theorem [4]). *Suppose $u, b \in C_{\mathrm{rd}}$, $a \in \mathcal{R}^+$; then*

$$u^{\Delta}(t) \leq a(t)u(t) + b(t), \quad t \geq t_0,\ t \in \mathbb{T}^k, \tag{2.6}$$

implies

$$u(t) \leq u(t_0)e_a(t, t_0) + \int_{t_0}^{t} b(\tau)e_a(t, \sigma(\tau)) \Delta\tau, \quad t \geq t_0,\ t \in \mathbb{T}^k. \tag{2.7}$$

The following lemma is useful in our main results.

Lemma 2.4 (see [7]). *Let $a \geq 0$, $p \geq q > 0$, then*

$$a^{q/p} \leq \frac{q}{p} K^{(q-p)/p} a + \frac{p-q}{p} K^{q/p}, \quad K > 0. \tag{2.8}$$

3. Main Results

In this section, we study some integral inequalities on time scales. We always assume that p, q, r, m are constants, $p \geq q > 0$, $p \geq m > 0$, $p \geq r > 0$, and $t \geq t_0, t \in \mathbb{T}^k$.

Theorem 3.1. *Assume that $u, a, b, f, g, h \in C_{\mathrm{rd}}$; $u(t), a(t), b(t), f(t), g(t)$, and $h(t)$ are nonnegative; then*

$$u^p(t) \leq a(t) + b(t) \int_{t_0}^{t} \left[f(s) u^q(s) + g(s) u^r(s) + \int_{t_0}^{s} h(\tau) u^m(\tau) \Delta\tau \right] \Delta s, \quad t \in \mathbb{T}^k, \tag{3.1}$$

implies

$$u(t) \leq \left\{ a(t) + b(t) \int_{t_0}^{t} B(\tau) e_{A(\tau)}(t, \sigma(\tau)) \Delta\tau \right\}^{1/p}, \quad K > 0, \ t \in \mathbb{T}^k, \tag{3.2}$$

where

$$\begin{aligned} A(t) &= \left[\frac{q}{p} K^{(q-p)/p} f(t) + \frac{r}{p} K^{(r-p)/p} g(t) \right] b(t) + \frac{m}{p} K^{(m-p)/p} \int_{t_0}^{t} b(\tau) h(\tau) \Delta\tau, \\ B(t) &= f(t) \left[\frac{q}{p} K^{(q-p)/p} a(t) + \frac{p-q}{p} K^{q/p} \right] + g(t) \left[\frac{r}{p} K^{(r-p)/p} a(t) + \frac{p-r}{p} K^{p/r} \right] \\ &\quad + \int_{t_0}^{t} \left[\frac{m}{p} K^{(m-p)/p} a(\tau) + \frac{p-m}{p} K^{m/p} \right] h(\tau) \Delta\tau, \quad t \in \mathbb{T}^k. \end{aligned} \tag{3.3}$$

Proof. Define $z(t)$ by

$$z(t) = \int_{t_0}^{t} \left[f(s) u^q(s) + g(s) u^r(s) + \int_{t_0}^{s} h(\tau) u^m(\tau) \Delta\tau \right] \Delta s, \tag{3.4}$$

then $z(t_0) = 0$, and (3.1) can be restated as

$$u^p(t) \leq a(t) + b(t) z(t). \tag{3.5}$$

Using Lemma 2.1, for any $k > 0$, we obtain

$$\begin{aligned} u^q(t) &\le [a(t)+b(t)z(t)]^{q/p} \le \frac{q}{p}K^{(q-p)/p}[a(t)+b(t)z(t)] + \frac{p-q}{p}K^{q/p}, \\ u^r(t) &\le [a(t)+b(t)z(t)]^{r/p} \le \frac{r}{p}K^{(r-p)/p}[a(t)+b(t)z(t)] + \frac{p-r}{p}K^{r/p}, \\ u^m(t) &\le [a(t)+b(t)z(t)]^{m/p} \le \frac{m}{p}K^{(m-p)/p}[a(t)+b(t)z(t)] + \frac{p-m}{p}K^{m/p}. \end{aligned} \tag{3.6}$$

It follows from (3.4) and (3.6) that

$$\begin{aligned} z^{\Delta}(t) &= f(t)u^q(t) + g(t)u^r(t) + \int_{t_0}^{t} h(\tau)u^m(\tau)\Delta\tau \\ &\le f(t)\left[\frac{q}{p}K^{(q-p)/p}(a(t)+b(t)z(t)) + \frac{p-q}{p}K^{q/p}\right] \\ &\quad + g(t)\left[\frac{r}{p}K^{(r-p)/p}(a(t)+b(t)z(t)) + \frac{p-r}{p}K^{r/p}\right] \\ &\quad + \int_{t_0}^{t} h(\tau)\left[\frac{m}{p}K^{(m-p)/p}(a(\tau)+b(\tau)z(\tau)) + \frac{p-m}{p}K^{m/p}\right]\Delta\tau \\ &\le B(t) + A(t)z(t), \quad t \in \mathbb{T}^k, \end{aligned} \tag{3.7}$$

where $A(t),$ and $B(t)$ are defined as in (3.3) and $A(t)$ is regressive obviously.

From Lemma 2.3 and (3.7), noting $z(t_0) = 0$, we obtain

$$z(t) \le \int_{t_0}^{t} B(\tau)e_{A(\tau)}(t,\sigma(\tau))\Delta\tau. \tag{3.8}$$

Therefore, the desired inequality (3.2) follows from (3.5) and (3.8). □

Remark 3.2. Theorem 3.1 extends some known inequalities on time scales. If $q = 1, r = 0, h(t) = 0$, then Theorem 3.1 reduces to [7, Theorem 3.1]. If $q = p, h(t) = 0$, then Theorem 3.1 reduces to [8, Theorem 3.2].

Remark 3.3. The result of Theorem 3.1 holds for an arbitrary time scale. If $\mathbb{T} = \mathbb{R}$, then Theorem 3.1 becomes the Theorem 1 established by Yuan et al. [13]. If $\mathbb{T} = \mathbb{Z}$, we can have the following Corollary.

Corollary 3.4. *Let $\mathbb{T} = \mathbb{Z}$ and assume that $u(t), a(t), b(t), f(t), g(t)$, and $h(t)$ are nonnegative functions defined for $t \in \mathbb{N}_0$. Then the inequality*

$$u^p(t) \le a(t) + b(t)\sum_{s=0}^{t-1}\left[f(s)u^q(s) + g(s)u^r(s) + \sum_{\tau=0}^{s-1}h(\tau)u^m(\tau)\right], \quad t \in \mathbb{N}_0, \tag{3.9}$$

implies

$$u(t) \le \left\{ a(t) + b(t) \sum_{s=0}^{t-1} \overline{B}(s) \prod_{\tau=s+1}^{t-1} \left(1 + \overline{A}(\tau)\right) \right\}^{1/p}, \quad K > 0,\ t \in \mathbb{N}_0, \tag{3.10}$$

where

$$\begin{aligned}
\overline{A}(t) &= \left[\frac{q}{p} K^{(q-p)/p} f(t) + \frac{r}{p} K^{(r-p)/p} g(t) \right] b(t) + \frac{m}{p} K^{(m-p)/p} \sum_{\tau=0}^{t-1} b(\tau) h(\tau), \\
\overline{B}(t) &= f(t) \left[\frac{q}{p} K^{(q-p)/p} a(t) + \frac{p-q}{p} K^{q/p} \right] + g(t) \left[\frac{r}{p} K^{(r-p)/p} a(t) + \frac{p-r}{p} K^{p/r} \right] \\
&\quad + \sum_{\tau=0}^{t-1} \left[\frac{m}{p} K^{(m-p)/p} a(\tau) + \frac{p-m}{p} K^{m/p} \right] h(\tau), \quad t \in \mathbb{N}_0.
\end{aligned} \tag{3.11}$$

Corollary 3.5. *Let* $\mathbb{T} = l\mathbb{Z} \cap [0, \infty)$*, where* $l\mathbb{Z} = \{lk : k \in \mathbb{Z}, l > 0\}$*. We assume that* $u(t), a(t), b(t), f(t), g(t)$*,and* $h(t)$ *are nonnegative functions defined for* $t \in \mathbb{T}$*. Then the inequality*

$$u^p(t) \le a(t) + b(t) \sum_{s=0}^{t/l-1} \left[f(ls) u^q(ls) + g(ls) u^r(ls) + \sum_{\tau=0}^{s/l-1} h(l\tau) u^m(l\tau) \right], \quad t \in \mathbb{T} \tag{3.12}$$

implies

$$u(t) \le \left\{ a(t) + b(t) \sum_{s=0}^{t/l-1} \overline{\overline{B}}(ls) \prod_{\tau=s/l+1}^{t/l-1} \left(1 + \overline{\overline{A}}(\tau)\right) \right\}^{1/p}, \quad K > 0,\ t \in \mathbb{T}, \tag{3.13}$$

where

$$\begin{aligned}
\overline{\overline{A}}(t) &= \left[\frac{q}{p} K^{(q-p)/p} f(t) + \frac{r}{p} K^{(r-p)/p} g(t) \right] b(t) + \frac{m}{p} K^{(m-p)/p} \sum_{\tau=0}^{t/l-1} b(l\tau) h(l\tau), \\
\overline{\overline{B}}(t) &= f(t) \left[\frac{q}{p} K^{(q-p)/p} a(t) + \frac{p-q}{p} K^{q/p} \right] + g(t) \left[\frac{r}{p} K^{(r-p)/p} a(t) + \frac{p-r}{p} K^{p/r} \right] \\
&\quad + \sum_{\tau=0}^{t/l-1} \left[\frac{m}{p} K^{(m-p)/p} a(l\tau) + \frac{p-m}{p} K^{m/p} \right] h(l\tau), \quad t \in \mathbb{T}.
\end{aligned} \tag{3.14}$$

Theorem 3.6. *Assume that u, a, b, f, h are defined as in Theorem 3.1, $L(t,y), M(t,y) : \mathbb{T}^k \times \mathbb{R} \to \mathbb{R}_+$ are continuous functions, and $L(t,y)$ is nondecreasing about the second variable and satisfies*

$$0 \le L(t,x) - L(t,y) \le M(t,y)(x-y) \tag{3.15}$$

for $t \in \mathbb{T}^k$ and $x \ge y \ge 0$; then

$$u^p(t) \le a(t) + b(t)\int_{t_0}^{t}\left[f(s)u^q(s) + L(s,u^r(s)) + \int_{t_0}^{s} h(\tau)u^m(\tau)\Delta\tau\right]\Delta s, \quad t \in \mathbb{T}^k \tag{3.16}$$

implies

$$u(t) \le \left\{a(t) + b(t)\int_{t_0}^{t} B_1(\tau)e_{A_1}(t,\sigma(\tau))\Delta\tau\right\}^{1/p}, \quad K > 0,\ t \in \mathbb{T}^k, \tag{3.17}$$

where

$$\begin{aligned}
A_1(t) &= \frac{q}{p}K^{(q-p)/p}f(t)b(t) + \frac{m}{p}K^{(m-p)/p}\int_{t_0}^{t} h(\tau)b(\tau)\Delta\tau \\
&\quad + \frac{r}{p}K^{(r-p)/p}M\left(t, \frac{r}{p}K^{(r-p)/p}a(t) + \frac{p-r}{p}K^{r/p}\right)b(t), \\
B_1(t) &= f(t)\left(\frac{q}{p}K^{(q-p)/p}a(t) + \frac{p-q}{p}K^{q/p}\right) \\
&\quad + \int_{t_0}^{t} h(\tau)\left(\frac{m}{p}K^{(m-p)/p}a(\tau) + \frac{p-m}{p}K^{m/p}\right)\Delta\tau \\
&\quad + L\left(t, \frac{r}{p}K^{(r-p)/p}a(t) + \frac{p-r}{p}K^{r/p}\right), \quad t \in \mathbb{T}^k.
\end{aligned} \tag{3.18}$$

Proof. Define $z(t)$ by

$$z(t) = \int_{t_0}^{t}\left[f(s)u^q(s) + L(s,u^r(s)) + \int_{t_0}^{s} h(\tau)u^m(\tau)\Delta\tau\right]\Delta s, \tag{3.19}$$

then $z(t_0) = 0$, and (3.16) can be written as (3.5).

Therefore, from (3.6) and (3.19), we have

$$\begin{aligned}
z^{\Delta}(t) &= f(t)u^{q}(t) + L(t,u^{r}(t)) + \int_{t_0}^{t} h(\tau)u^{m}(\tau)\Delta\tau \\
&\leq f(t)\left[\frac{q}{p}K^{(q-p)/p}(a(t)+b(t)z(t)) + \frac{p-q}{p}K^{q/p}\right] \\
&\quad + L\left(t,\frac{r}{p}K^{(r-p)/p}(a(t)+b(t)z(t)) + \frac{p-r}{p}K^{r/p}\right) \\
&\quad - L\left(t,\frac{r}{p}K^{(r-p)/p}a(t) + \frac{p-r}{p}K^{r/p}\right) \\
&\quad + \int_{t_0}^{t} h(\tau)\left[\frac{m}{p}K^{(m-p)/p}(a(\tau)+b(\tau)z(\tau)) + \frac{p-m}{p}K^{m/p}\right]\Delta\tau \\
&\quad + L\left(t,\frac{r}{p}K^{(r-p)/p}a(t) + \frac{p-r}{p}K^{r/p}\right) \\
&\leq f(t)\left[\frac{q}{p}K^{(q-p)/p}a(t) + \frac{p-q}{p}K^{q/p}\right] + \int_{t_0}^{t} h(\tau)\left(\frac{m}{p}K^{(m-p)/p}a(\tau) + \frac{p-m}{p}K^{m/p}\right)\Delta\tau \\
&\quad + \left[\frac{q}{p}K^{(q-p)/p}f(t)b(t) + \frac{m}{p}K^{(m-p)/p}\int_{t_0}^{t} h(\tau)b(\tau)\Delta\tau\right]z(t) \\
&\quad + M\left(t,\frac{r}{p}K^{(r-p)/p}a(t) + \frac{p-r}{p}K^{r/p}\right)\frac{r}{p}K^{(r-p)/p}b(t)z(t) \\
&\quad + L\left(t,\frac{r}{p}K^{(r-p)/p}a(t) + \frac{p-r}{p}K^{r/p}\right) = A_1(t)z(t) + B_1(t), \quad t \in \mathbb{T}^{k},
\end{aligned} \tag{3.20}$$

where $A_1(t)$, and $B_1(t)$ are defined as in (3.18) and $A_1(t)$ is regressive obviously.

From Lemma 2.3 and (3.20), noting $z(t_0) = 0$, we obtain

$$z(t) \leq \int_{t_0}^{t} B_1(\tau)e_{A_1(\tau)}(t,\sigma(\tau))\Delta\tau. \tag{3.21}$$

Therefore, the desired inequality (3.17) follows from (3.5) and (3.21). □

Remark 3.7. If $\mathbb{T} = \mathbb{R}$, then Theorem 3.6 becomes [13, Theorem 3]. If $\mathbb{T} = \mathbb{Z}$, we can have the following Corollary.

Corollary 3.8. *Let $\mathbb{T} = \mathbb{Z}$ and assume that $u(t), a(t), b(t), f(t), g(t)$, and $h(t)$ are nonnegative functions defined for $t \in \mathbb{N}_0$. $L, M \in C(\mathbb{R}_+^2, \mathbb{R}_+)$ satisfy*

$$0 \le L(t,x) - L(t,y) \le M(t,y)(x-y) \tag{3.22}$$

for $x \ge y \ge 0$ and $L(t,y)$ is nondecreasing about the second variable. Then the inequality

$$u^p(t) \le a(t) + b(t)\sum_{s=0}^{t-1}\left[f(s)u^q(s) + L(s, u^r(s)) + \sum_{\tau=0}^{s-1} h(\tau)u^m(\tau)\right], \quad t \in \mathbb{N}_0 \tag{3.23}$$

implies

$$u(t) \le \left\{a(t) + b(t)\sum_{s=0}^{t-1}\overline{B}_1(\tau)\prod_{\tau=s+1}^{t-1}\left(1+\overline{A}_1(\tau)\right)\right\}^{1/p}, \quad K > 0,\ t \in \mathbb{N}_0, \tag{3.24}$$

where

$$\begin{aligned}
\overline{A}_1(t) &= \frac{q}{p}K^{(q-p)/p}f(t)b(t) + \frac{m}{p}K^{(m-p)/p}\sum_{\tau=0}^{t-1}h(\tau)b(\tau) \\
&\quad + \frac{r}{p}K^{(r-p)/p}M\left(t, \frac{r}{p}K^{(r-p)/p}a(t) + \frac{p-r}{p}K^{r/p}\right)b(t), \\
\overline{B}_1(t) &= f(t)\left(\frac{q}{p}K^{(q-p)/p}a(t) + \frac{q-p}{p}K^{q/p}\right) + \sum_{\tau=0}^{t-1}h(\tau)\left(\frac{m}{p}K^{(m-p)/p}a(\tau) + \frac{p-m}{p}K^{m/p}\right) \\
&\quad + L\left(t, \frac{r}{p}K^{(r-p)/p}a(t) + \frac{p-r}{p}K^{r/p}\right), \quad t \in \mathbb{N}_0.
\end{aligned} \tag{3.25}$$

Theorem 3.9. *Assume that $u(t), a(t), b(t), f(t), g(t)$, and $h(t)$ are defined as in Theorem 3.1, $w(t,s)$ is defined as in Lemma 2.2 such that $w(\sigma(t),t) \ge 0$, $w^{\Delta}(t,s) \ge 0$ for $t,s \in \mathbb{T}$ with $s \le t$; then*

$$u^p(t) \le a(t) + b(t)\int_{t_0}^{t} w(t,s)\left[f(s)u^q(s) + g(s)u^r(s) + \int_{t_0}^{s} h(\tau)u^m(\tau)\Delta\tau\right]\Delta s, \quad t \in \mathbb{T}^k, \tag{3.26}$$

implies

$$u(t) \leq \left\{ a(t) + b(t) \int_{t_0}^{t} B_2(\tau) e_{A_2}(t, \sigma(\tau)) \Delta\tau \right\}^{1/p}, \quad K > 0,\ t \in \mathbb{T}^k, \tag{3.27}$$

where

$$\begin{aligned}
A_2(t) = {}& w(\sigma(t), t) \left[\left(\frac{q}{p} K^{(q-p)/p} f(t) + \frac{r}{p} K^{(r-p)/p} g(t) \right) b(t) + \frac{m}{p} K^{(m-p)/p} \int_{t_0}^{t} b(\tau) h(\tau) \Delta\tau \right] \\
& + \int_{t_0}^{t} w^{\Delta}(t, s) \left[\left(\frac{q}{p} K^{(q-p)/p} f(s) + \frac{r}{p} K^{(r-p)/p} g(s) \right) b(s) \right. \\
& \left. + \frac{m}{p} K^{(m-p)/p} \int_{t_0}^{s} b(\tau) h(\tau) \Delta\tau \right] \Delta s, \\
B_2(t) = {}& w(\sigma(t), t) \left[f(t) \left(\frac{q}{p} K^{(q-p)/p} a(t) + \frac{p-q}{p} K^{q/p} \right) + g(t) \left(\frac{r}{p} K^{(r-p)/p} a(t) + \frac{p-r}{p} K^{p/r} \right) \right. \\
& \left. + \int_{t_0}^{t} h(\tau) \left(\frac{m}{p} K^{(m-p)/p} a(\tau) + \frac{p-m}{p} K^{m/p} \right) \Delta\tau \right] \\
& + \int_{t_0}^{t} w^{\Delta}(t, s) \left[f(s) \left(\frac{q}{p} K^{(q-p)/p} a(s) + \frac{p-q}{p} K^{q/p} \right) \right. \\
& + g(s) \left(\frac{r}{p} K^{(r-p)/p} a(s) + \frac{p-r}{p} K^{p/r} \right) \\
& \left. + \int_{t_0}^{s} h(\tau) \left(\frac{m}{p} K^{(m-p)/p} a(\tau) + \frac{p-m}{p} K^{m/p} \right) \Delta\tau \right] \Delta s, \quad t \in \mathbb{T}^k.
\end{aligned} \tag{3.28}$$

Proof. Define $z(t)$ by

$$z(t) = \int_{t_0}^{t} w(t, s) \left[f(s) u^q(s) + g(s) u^r(s) + \int_{t_0}^{s} h(\tau) u^m(\tau) \Delta\tau \right] \Delta s, \tag{3.29}$$

then $z(t_0) = 0$, and (3.26) can be written as (3.5).

Therefore, from (3.6) and (3.29) we have

$$
\begin{aligned}
z^{\Delta}(t) &= w(\sigma(t),t)\left[f(t)u^{q}(t)+g(t)u^{r}(t)+\int_{t_0}^{t}h(\tau)u^{m}(\tau)\Delta\tau\right] \\
&\quad+\int_{t_0}^{t}w^{\Delta}(t,s)\left[f(s)u^{q}(s)+g(s)u^{r}(s)+\int_{t_0}^{s}h(\tau)u^{m}(\tau)\Delta\tau\right]\Delta s \\
&\leq w(\sigma(t),t)\Bigg[f(t)\left(\frac{q}{p}K^{(q-p)/p}(a(t)+b(t)z(t))+\frac{p-q}{p}K^{q/p}\right) \\
&\qquad+g(t)\left(\frac{r}{p}K^{(r-p)/p}(a(t)+b(t)z(t))+\frac{p-r}{p}K^{r/p}\right) \\
&\qquad+\int_{t_0}^{t}h(\tau)\left(\frac{m}{p}K^{(m-p)/p}(a(\tau)+b(\tau)z(\tau))+\frac{p-m}{p}K^{m/p}\right)\Bigg]\Delta\tau \\
&\quad+\int_{t_0}^{t}w^{\Delta}(t,s)\Bigg[f(s)\left(\frac{q}{p}K^{(q-p)/p}(a(s)+b(s)z(s))+\frac{p-q}{p}K^{q/p}\right) \\
&\qquad+g(s)\left(\frac{r}{p}K^{(r-p)/p}(a(s)+b(s)z(s))+\frac{p-r}{p}K^{r/p}\right) \\
&\qquad+\int_{t_0}^{s}h(\tau)\left(\frac{m}{p}K^{(m-p)/p}(a(\tau)+b(\tau)z(\tau))+\frac{p-m}{p}K^{m/p}\right)\Delta\tau\Bigg]\Delta s \\
&\leq B_2(t)+A_2(t)z(t), \qquad t\in\mathbb{T}^{k},
\end{aligned}
\tag{3.30}
$$

where $A_2(t)$, and $B_2(t)$ are defined by (3.28) and $A_2(t)$ is regressive obviously.

From Lemma 2.3 and (3.30), noting $z(t_0)=0$, we obtain

$$
z(t)\leq\int_{t_0}^{t}B_2(\tau)e_{A_2(\tau)}(t,\sigma(\tau))\Delta\tau. \tag{3.31}
$$

Therefore, the desired inequality (3.27) follows from (3.5) and (3.31). □

Remark 3.10. If $q=p, h(t)=0$, then Theorem 3.9 reduces to [8, Theorem 3.8].

Using our results, we can also obtain many dynamic inequalities for some peculiar time scales; here, we omit them.

4. Some Applications

In this section, we present some applications of Theorem 3.9 to investigate certain properties of solution $u(t)$ of the following dynamic equation:

$$
[u^{p}(t)]^{\Delta}=F\left(t,U(t,u(t)),\int_{t_0}^{t}H(s,u(s))\Delta s\right), \quad u^{p}(t_0)=C, \quad t\in\mathbb{T}^{k}, \tag{4.1}
$$

where C is a constant, $F : \mathbb{T}^k \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $U : \mathbb{T}^k \times \mathbb{R} \to \mathbb{R}$, $H : \mathbb{T}^k \times \mathbb{R} \to \mathbb{R}$ are also continuous functions.

Example 4.1. Assume that

$$\begin{aligned} |F(t,U,V)| &\leq |U| + |V|, \\ |U(t,u)| &\leq f(t)|u|^q + g(t)|u|^r, \\ |H(t,u)| &\leq h(t)|u|^m, \quad t \in \mathbb{T}^k, \end{aligned} \tag{4.2}$$

where $p, q, r,$ and m are constants, $p \geq q > 0,$ and $p \geq m > 0, p \geq r > 0.$ $f, g, h \in C_{\mathrm{rd}}, f(t), g(t)$ and $h(t)$ are nonnegative. Then every solution $u(t)$ of (4.1) satisfies

$$|u(t)| \leq \left\{ |C| + \int_{t_0}^{t} B(\tau) e_{A(\tau)}(t, \sigma(\tau)) \Delta\tau \right\}^{1/p}, \quad K > 0, \ t \in \mathbb{T}^k, \tag{4.3}$$

where A, B are defined as in (3.3) with $a(t) = |C|, b(t) = 1$.

Indeed, the solution $u(t)$ of (4.1) satisfies the following equivalent equation

$$u^p(t) = C + \int_{t_0}^{t} F\left(\tau, U(\tau, u(\tau)), \int_{t_0}^{\tau} H(s, u(s)) \Delta s \right) \Delta\tau, \quad t \in \mathbb{T}^k. \tag{4.4}$$

It follows from (4.2) and (4.4) that

$$\begin{aligned} |u^p(t)| &\leq |C| + \int_{t_0}^{t} \left| F\left(\tau, U(\tau, u(\tau)), \int_{t_0}^{\tau} H(s, u(s)) \Delta s \right) \right| \Delta\tau \\ &\leq |C| + \int_{t_0}^{t} \left[f(\tau)|u(\tau)|^q + g(\tau)|u(\tau)|^r + \int_{t_0}^{\tau} h(s)|u(s)|^m \Delta s \right] \Delta\tau. \end{aligned} \tag{4.5}$$

Using Theorem 3.1, the inequality (4.3) is obtained from (4.5).

Example 4.2. Assume that

$$\begin{aligned} |F(t, U_1, V_1) - F(t, U_2, V_2)| &\leq |U_1 - U_2| + |V_1 - V_2|, \\ |U(t, u_1) - U(t, u_2)| &\leq f(t) \left| u_1^p - u_2^p \right|, \\ |H(t, u_1) - H(t, u_2)| &\leq h(t) \left| u_1^p - u_2^p \right|, \quad t \in \mathbb{T}^k, \end{aligned} \tag{4.6}$$

p, f, and h are defined as in Example 4.1. If $p = (m/n)$ $(m, n \in N)$ and m is odd, then (4.1) has at most one solution; otherwise, the two solutions $u_1(t)$, and $u_2(t)$ of (4.1) have the relation $u_1^p(t) = u_2^p(t)$.

Proof. Let $u_1(t),$ and $u_2(t)$ be two solutions of (4.1). Then we have

$$u_1^p(t) - u_2^p(t) = \int_{t_0}^{t} \left[F\left(\tau, U(\tau, u_1(\tau)), \int_{t_0}^{\tau} H(s, u_1(s))\Delta s\right) - F\left(\tau, U(\tau, u_2(\tau)), \int_{t_0}^{\tau} H(s, u_2(s))\Delta s\right)\right]\Delta\tau, \quad t \in \mathbb{T}^k. \tag{4.7}$$

It follows from (4.6) and (4.7) that

$$\left|u_1^p(t) - u_2^p(t)\right| \le \int_{t_0}^{t} \left[f(\tau)\left|u_1^p(\tau) - u_2^p(\tau)\right| + \int_{t_0}^{\tau} h(s)\left|u_1^p(s) - u_2^p(s)\right|\Delta s\right]\Delta\tau, \quad t \in \mathbb{T}^k. \tag{4.8}$$

By Theorem 3.1, we have $u_1^p(t) - u_2^p(t) \equiv 0$, $t \in \mathbb{T}^k$. The results are obtained. □

Example 4.3. Consider the equation

$$u^p = a(t) + b(t)\int_{t_0}^{t} F\left(t, s, U(s, u), \int_{t_0}^{s} H(\tau, u)\Delta\tau\right)\Delta s, \quad t \in \mathbb{T}^k. \tag{4.9}$$

If

$$\begin{aligned} |F(t, s, U, V)| &\le w(t, s)(|U| + |V|), \\ |U(t, u)| &\le f(t)|u|^q + g(t)|u|^r, \\ |H(t, u)| &\le h(t)|u|^m, \quad t \in \mathbb{T}^k, \end{aligned} \tag{4.10}$$

where p, q, r, m are constants, $p \ge q > 0$, $p \ge m > 0$, $p \ge r > 0$. $a, b, f, g, h \in C_{\mathrm{rd}}$, $a(t), b(t), f(t), g(t)$ and $h(t)$ are nonnegative, $w(t, s)$ is defined as in Lemma 2.2 such that $w(\sigma(t), t) \ge 0$, $w^{\Delta}(t, s) \ge 0$ for $t, s \in \mathbb{T}$ with $s \le t$.

Then we have the estimate of the solution $u(t)$ of (4.9) that

$$|u(t)| \le \left\{ a(t) + b(t)\int_{t_0}^{t} B_2(\tau)e_{A_2}(t, \sigma(\tau))\Delta\tau \right\}^{1/p}, \quad K > 0,\ t \in \mathbb{T}^k, \tag{4.11}$$

where A_2, B_2 are defined as in (3.28).

Proof. From (4.10) and (4.9), we have

$$|u(t)|^p \le a(t) + b(t)\int_{t_0}^{t} w(t, s)\left[f(s)|u(s)|^q + g(s)|u(s)|^r + \int_{t_0}^{s} h(\tau)|u(\tau)|^m\Delta\tau\right]\Delta s, \quad t \in \mathbb{T}^k. \tag{4.12}$$

By Theorem 3.9 and (4.12), we have that (4.11) holds. □

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