

Research Article

Nonlinear Boundary Value Problem of First-Order Impulsive Functional Differential Equations

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This paper investigates the nonlinear boundary value problem for a class of first-order impulsive functional differential equations. By establishing a comparison result and utilizing the method of upper and lower solutions, some criteria on the existence of extremal solutions as well as the unique solution are obtained. Examples are discussed to illustrate the validity of the obtained results.

1. Introduction

It is now realized that the theory of impulsive differential equations provides a general framework for mathematical modelling of many real world phenomena. In particular, it serves as an adequate mathematical tool for studying evolution processes that are subjected to abrupt changes in their states. Some typical physical systems that exhibit impulsive behaviour include the action of a pendulum clock, mechanical systems subject to impacts, the maintenance of a species through periodic stocking or harvesting, the thrust impulse maneuver of a spacecraft, and the function of the heart. For an introduction to the theory of impulsive differential equations, refer to [1].

It is also known that the method of upper and lower solutions coupled with the monotone iterative technique is a powerful tool for obtaining existence results of nonlinear differential equations [2]. There are numerous papers devoted to the applications of this method to nonlinear differential equations in the literature, see [3–9] and references therein. The existence of extremal solutions of impulsive differential equations is considered in papers [3–11]. However, only a few papers have implemented the technique in nonlinear boundary value problem of impulsive differential equations [5, 12]. In this paper, we will investigate

nonlinear boundary value problem of a class of first-order impulsive functional differential equations. Such equations include the retarded impulsive differential equations as special cases [5, 12–14].

The rest of this paper is organized as follows. In Section 2, we establish a new comparison principle and discuss the existence and uniqueness of the solution for first order impulsive functional differential equations with linear boundary condition. We then obtain existence results for extremal solutions and unique solution in Section 3 by using the method of upper and lower solutions coupled with monotone iterative technique. To illustrate the obtained results, two examples are discussed in Section 4.

2. Preliminaries

Let $J = [0, T]$, $T > 0$, $J' = J - \{t_1, t_2, \dots, t_p\}$ with $0 < t_1 < t_2 < \dots < t_p < T$. We define that $PC(J) = \{x : J \rightarrow \mathbb{R} : x \text{ is continuous for any } t \in J'; x(t_k^+) \text{ and } x(t_k^-) \text{ exist and } x(t_k^-) = x(t_k)\}$, $PC^1(J) = \{x : J \rightarrow \mathbb{R} : x \text{ is continuously differentiable for any } t \in J'; x(t_k^+), x(t_k^-) \text{ exist and } x'(t_k^-) = x'(t_k)\}$. It is clear that $PC(J)$ and $PC^1(J)$ are Banach spaces with respective norms

$$\|x\|_{PC} = \sup_{t \in J} |x(t)|, \quad \|x\|_{PC^1} = \|x\|_{PC} + \|x'\|_{PC}. \quad (2.1)$$

Let us consider the following nonlinear boundary value problem (NBVP):

$$\begin{aligned} x'(t) &= f(t, x(t), [\varphi x](t)), \quad t \in J' = J - \{t_1, t_2, \dots, t_p\}, \\ \Delta x(t) &= I_k(t, x(t), [\varphi x](t)), \quad t = t_k, \quad k = 1, 2, \dots, p, \\ g(x(0), x(T)) &= 0, \end{aligned} \quad (2.2)$$

where $f : J \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous in the second and the third variables, and for fixed $x, y \in \mathbb{R}$, $f(\cdot, x, y) \in PC(J)$, $g \in C(\mathbb{R}^2, \mathbb{R})$, $I_k \in C(\mathbb{R}^3, \mathbb{R})$, $k = 1, 2, \dots, p$ and $\varphi : PC(J) \rightarrow PC(J)$ is continuous.

A function $x \in PC^1(J)$ is called a solutions of NBVP (2.2) if it satisfies (2.2).

Remark 2.1. (i) If $[\varphi x](t) = x(t)$ and the impulses I_k depend only on $x(t_k)$, the equation of NBVP (2.2) reduces to the simpler case of impulsive differential equations:

$$\begin{aligned} x'(t) &= f(t, x(t)), \quad t \in J', \\ \Delta x(t_k) &= I_k(x(t_k)), \quad k = 1, 2, \dots, p \end{aligned} \quad (2.3)$$

which have been studied in many papers. In some situation, the impulse I_k depends also on some other parameters (e.g., the control of the amount of drug ingested by a patient at certain moments in the model for drug distribution [1, 3]).

(ii) If $[\varphi x](t) = x(\theta(t))$, where $\theta \in C(J, J)$, the equation of NBVP (2.2) can be regarded as retarded differential equation which has been considered in [5, 12–14].

We will need the following lemma.

Lemma 2.2 (see [1]). *Assume that*

(B₀) *the sequence $\{t_k\}$ satisfies $0 \leq t_0 < t_1 < t_2 < \dots < t_k < \dots$ with $\lim_{k \rightarrow \infty} t_k = +\infty$,*

(B₁) *$m \in PC^1(\mathbb{R}^+)$ is left continuous at t_k for $k = 1, 2, \dots$,*

(B₂) *for $k = 1, 2, \dots$, $t \geq t_0$,*

$$\begin{aligned} m'(t) &\leq p(t)m(t) + q(t), \quad t \neq t_k \\ m(t_k^+) &\leq d_k m(t_k) + b_k, \end{aligned} \quad (2.4)$$

where $p, q \in C(\mathbb{R}^+, \mathbb{R})$, $d_k \geq 0$ and b_k are real constants.

Then

$$\begin{aligned} m(t) &\leq m(t_0) \prod_{t_0 < t_k < t} d_k \exp\left(\int_{t_0}^t p(s) ds\right) \\ &\quad + \int_{t_0}^t \prod_{t_0 < t_k < t} d_k \exp\left(\int_s^t p(\sigma) d\sigma\right) q(s) ds \\ &\quad + \sum_{t_0 < t_k < t} \prod_{t_k < t_j < t} d_j \exp\left(\int_{t_k}^t p(s) ds\right) b_k. \end{aligned} \quad (2.5)$$

In order to establish a comparison result and some lemmas, we will make the following assumptions on the function φ .

(H1) There exists a constant $R > 0$ such that

$$[\varphi x](t) \geq R \inf_{t \in J} x(t), \quad \text{for any } x \in PC(J), \quad \forall t \in J. \quad (2.6)$$

(H2) The function φ satisfies Lipschitz condition, that is, there exists a $L > 0$ such that

$$\|\varphi x - \varphi y\|_{PC} \leq L \|x - y\|_{PC}, \quad \forall x, y \in PC(J). \quad (2.7)$$

Inspired by the ideas in [5, 6], we shall establish the following comparison result.

Theorem 2.3. *Let $m \in PC^1(J)$ such that*

$$\begin{aligned} m'(t) &\leq -Mm(t) - N[\varphi m](t), \quad t \in J', \\ \Delta m(t_k) &\leq -L_k m(t_k), \quad k = 1, 2, \dots, p, \\ m(0) &\leq \mu m(T), \end{aligned} \quad (2.8)$$

where $M > 0$, $N \geq 0$, $0 \leq L_k < 1$, $k = 1, 2, \dots, p$, and $0 < \mu e^{-MT} \leq 1$.

Suppose in addition that condition (H1) holds and

$$\frac{NR(e^{MT} + \mu)}{\mu} \int_0^T \prod_{0 < t_k < T} (1 - L_k) e^{Ms} ds \leq \sum_{k=1}^p (1 - L_k)^2 \quad (2.9)$$

then $m(t) \leq 0$, for $t \in J$.

Proof. For simplicity, we let $c_k = 1 - L_k$, $k = 1, 2, \dots, p$. Set $v(t) = m(t)e^{Mt}$, then we have

$$\begin{aligned} v'(t) &\leq -Ne^{Mt}[\varphi m](t), \quad t \in J', \\ v(t_k^+) &\leq c_k v(t_k), \quad k = 1, 2, \dots, p, \\ v(0) &\leq \mu e^{-MT} v(T). \end{aligned} \quad (2.10)$$

Obviously, $v(t) \leq 0$ implies $m(t) \leq 0$.

To show $v(t) \leq 0$, we suppose, on the contrary, that $v(t) > 0$ for some $t \in J$. It is enough to consider the following cases.

- (i) there exists a $\bar{t} \in J$, such that $v(\bar{t}) > 0$, and $v(t) \geq 0$ for all $t \in J$;
- (ii) there exist $t_*, t^* \in J$, such that $v(t_*) < 0$, $v(t^*) > 0$.

Case (i). By (2.10), we have $v'(t) \leq 0$ for $t \neq t_k$ and $\Delta v(t_k) \leq 0$, $k = 1, 2, \dots, m$, hence $v(t)$ is nonincreasing in J , that is, $v(T) \leq v(0)$. If $\mu < e^{MT}$, then $v(0) < v(T)$, which is a contradiction. If $\mu = e^{MT}$, then $v(0) \leq v(T)$ which implies $v(t) \equiv C > 0$. But from (2.10), we get $v'(t) < 0$ for $t \in J'$. Hence, $v(T) < v(0)$. It is again a contradiction.

Case (ii). Let $\inf_{t \in J} v(t) = -\lambda$, then $\lambda > 0$. For some $i \in \{1, 2, \dots, p\}$, there exists $t_* \in (t_i, t_{i+1}]$ such that $v(t_*) = -\lambda$ or $v(t_*^+) = -\lambda$. We only consider $v(t_*) = -\lambda$, as for the case $v(t_*^+) = -\lambda$, the proof is similar.

From (2.10) and condition (H1), we get

$$\begin{aligned} v'(t) &\leq -Ne^{Mt}[\varphi m](t) = -Ne^{Mt} \left[\varphi(v(t)e^{-Mt}) \right](t) \\ &\leq -NRe^{Mt} \inf_{t \in J} \{v(t)e^{-Mt}\} \leq -NRe^{Mt} \inf_{t \in J} \{v(t)\} \\ &\leq \lambda NRe^{Mt}, \quad t \in J'. \end{aligned} \quad (2.11)$$

Consider the inequalities

$$\begin{aligned} v'(t) &\leq \lambda NRe^{Mt}, \quad t \in J', \\ v(t_k^+) &\leq c_k v(t_k), \quad k = 1, 2, \dots, p. \end{aligned} \quad (2.12)$$

By Lemma 2.2, we have

$$v(t) \leq v(t_*) \left(\prod_{t_* < t_k < t} c_k \right) + \int_{t_*}^t \left(\prod_{s < t_k < t} c_k \right) \lambda N R e^{Ms} ds, \quad (2.13)$$

that is

$$v(t) \leq -\lambda \left(\prod_{t_* < t_k < t} c_k \right) + \lambda N R \int_{t_*}^t \left(\prod_{s < t_k < t} c_k \right) e^{Ms} ds. \quad (2.14)$$

First, we assume that $t^* > t_*$. Let $t = t^*$ in (2.14), then

$$v(t^*) \leq -\lambda \left(\prod_{t_* < t_k < t^*} c_k \right) + \lambda N R \int_{t_*}^{t^*} \left(\prod_{s < t_k < t^*} c_k \right) e^{Ms} ds. \quad (2.15)$$

Noting that $v(t^*) > 0$, we have

$$\prod_{t_* < t_k < t^*} c_k < N R \int_{t_*}^{t^*} \left(\prod_{s < t_k < t^*} c_k \right) e^{Ms} ds. \quad (2.16)$$

Hence

$$\left(\prod_{k=1}^p c_k \right)^2 \leq \prod_{k=1}^p c_k < N R \int_0^T \left(\prod_{s < t_k < T} c_k \right) e^{Ms} ds \quad (2.17)$$

which is a contradiction.

Next, we assume that $t^* < t_*$. By Lemma 2.2 and (2.10), we have

$$\begin{aligned} 0 < v(t^*) &\leq v(0) \left(\prod_{0 < t_k < t^*} c_k \right) + \int_0^{t^*} \left(\prod_{s < t_k < t^*} c_k \right) \lambda N R e^{Ms} ds \\ &\leq \mu e^{-MT} v(T) \left(\prod_{0 < t_k < t^*} c_k \right) + \lambda N R \int_0^{t^*} \left(\prod_{s < t_k < t^*} c_k \right) e^{Ms} ds, \end{aligned} \quad (2.18)$$

then

$$0 < \mu e^{-MT} v(T) \left(\prod_{k=1}^p c_k \right) + \lambda N R \int_0^T \left(\prod_{s < t_k < T} c_k \right) e^{Ms} ds. \quad (2.19)$$

Setting $t = T$ in (2.14), we have

$$\begin{aligned}
 v(T) &\leq v(t_*) \left(\prod_{t_* < t_k < T} c_k \right) + \int_{t_*}^T \left(\prod_{s < t_k < T} c_k \right) \lambda N R e^{Ms} ds \\
 &= -\lambda \left(\prod_{t_* < t_k < T} c_k \right) + \lambda N R \int_{t_*}^T \left(\prod_{s < t_k < T} c_k \right) e^{Ms} ds \\
 &\leq -\lambda \prod_{k=1}^p c_k + \lambda N R \int_0^T \left(\prod_{s < t_k < T} c_k \right) e^{Ms} ds
 \end{aligned} \tag{2.20}$$

with (2.19), we obtain that

$$\begin{aligned}
 0 &< \mu e^{-MT} v(T) \left(\prod_{k=1}^p c_k \right) + \lambda N R \int_0^T \left(\prod_{s < t_k < T} c_k \right) e^{Ms} ds \\
 &\leq \left[-\lambda \left(\prod_{k=1}^p c_k \right) + \lambda N R \int_0^T \left(\prod_{s < t_k < T} c_k \right) e^{Ms} ds \right] \mu e^{-MT} \left(\prod_{k=1}^p c_k \right) + \lambda N R \int_0^T \left(\prod_{s < t_k < T} c_k \right) e^{Ms} ds \\
 &= -\mu \lambda e^{-MT} \left(\prod_{k=1}^p c_k \right)^2 + \mu \lambda N R e^{-MT} \left(\prod_{k=1}^p c_k \right) \int_0^T \left(\prod_{s < t_k < T} c_k \right) e^{Ms} ds \\
 &\quad + \lambda N R \int_0^T \left(\prod_{s < t_k < T} c_k \right) e^{Ms} ds,
 \end{aligned} \tag{2.21}$$

that is,

$$\begin{aligned}
 \mu e^{-MT} \left(\prod_{k=1}^p c_k \right)^2 &\leq \left[\mu N R e^{-MT} \left(\prod_{k=1}^p c_k \right) + N R \right] \int_0^T \left(\prod_{s < t_k < T} c_k \right) e^{Ms} ds \\
 &< N R (\mu e^{-MT} + 1) \int_0^T \left(\prod_{s < t_k < T} c_k \right) e^{Ms} ds.
 \end{aligned} \tag{2.22}$$

Therefore,

$$\sum_{k=1}^p (1 - L_k)^2 < \frac{N R (e^{MT} + \mu)}{\mu} \int_0^T \prod_{s < t_k < T} (1 - L_k) e^{Ms} ds \tag{2.23}$$

which is a contradiction. The proof of Theorem 2.3 is complete. \square

The following corollary is an easy consequence of Theorem 2.3.

Corollary 2.4. Assume that there exist $M > 0$, $N \geq 0$, $0 \leq L_k < 1$, for $k = 1, 2, \dots, p$ such that $m \in PC^1(J)$ satisfies (2.8) with $0 < \mu e^{-MT} \leq 1$ and

$$\frac{RN(e^{MT} + \mu)e^{MT}}{\mu} \leq \frac{\sum_{k=1}^p (1 - L_k)^2}{\int_0^T \sum_{s < t_k < T} (1 - L_k) ds} \tag{2.24}$$

then $m(t) \leq 0$, for $t \in J$.

Remark 2.5. Setting $\mu \equiv 1$, Corollary 2.4 reduces to the Theorem 2.3 of Li and Shen [6]. Therefore, Theorem 2.3 and Corollary 2.4 develops and generalizes the result in [6].

Remark 2.6. We show some examples of function φ satisfying (H1).

(i) $[\varphi x](t) = x(\theta(t))$, where $\theta \in C(J \times J)$, satisfies (H1) with $R = 1$,

$$[\varphi x](t) = x(\theta(t)) \geq \inf_{t \in J} x(t), \quad \text{for } t \in J. \tag{2.25}$$

(ii) $[\varphi x](t) = \int_0^{t+T} x(s) ds$, satisfies (H1) with $R = T$,

$$[\varphi x](t) = \int_0^{t+T} x(s) ds \geq (t + T) \inf_{t \in J} x(t) \geq T \inf_{t \in J} x(t), \quad \text{for } t \in J. \tag{2.26}$$

Consider the linear boundary value problem (LBVP)

$$\begin{aligned} y'(t) + My(t) + N[\varphi y](t) &= \sigma(t), \quad t \in J', \\ \Delta y(t_k) &= -L_k y(t_k) + I_k(t_k, \eta(t_k), [\varphi \eta](t_k)) + L_k \eta(t_k), \quad k = 1, 2, \dots, p, \\ g(\eta(0), \eta(T)) + M_1(y(0) - \eta(0)) - M_2(y(T) - \eta(T)) &= 0, \end{aligned} \tag{2.27}$$

where $M > 0$, $N \geq 0$, $0 \leq L_k < 1$, $k = 1, 2, \dots, p$, and $\eta, \sigma \in PC(J)$.

By direct computation, we have the following result.

Lemma 2.7. $y \in PC^1(J)$ is a solution of LBVP (2.27) if and only if y is a solution of the impulsive integral equation

$$\begin{aligned} y(t) &= Ce^{-Mt} B\eta + \int_0^T G(t, s) \{ \sigma(s) - N[\varphi y](s) \} ds \\ &+ \sum_{0 < t_k < T} G(t, t_k) \{ -L_k y(t_k) + I_k(t_k, \eta(t_k), [\varphi \eta](t_k)) + L_k \eta(t_k) \}, \quad t \in J, \end{aligned} \tag{2.28}$$

where $B\eta = -g(\eta(0), \eta(T)) + M_1\eta(0) - M_2\eta(T)$, $C = (M_1 - M_2e^{-MT})^{-1}$, $M_1 \neq M_2e^{-MT}$ and

$$G(t, s) = \begin{cases} CM_2e^{M(s-t-T)} + e^{M(s-t)}, & 0 \leq s < t \leq T \\ CM_2e^{M(s-t-T)}, & 0 \leq t \leq s \leq T \end{cases} \tag{2.29}$$

Lemma 2.8. *Let (H2) hold. Suppose further*

$$\left(NLT + \sum_{k=1}^p |L_k| \right) r < 1, \quad r = \max\{|CM_1|, |CM_2|\}, \quad C = (M_1 - M_2 e^{-MT})^{-1}, \quad (2.30)$$

where $M > 0$, $N \geq 0$, $M_1 \neq M_2 e^{-MT}$, then LBVP (2.27) has a unique solution.

By Lemma 2.7 and Banach fixed point theorem, the proof of Lemma 2.8 is apparent, so we omit the details.

3. Main Results

In this section, we use monotone iterative technique to obtain the existence results of extremal solutions and the unique solution of NBVP (2.2). We shall need the following definition.

Definition 3.1. A function $\alpha \in PC^1(J)$ is said to be a lower solution of NBVP (2.2) if it satisfies

$$\begin{aligned} \alpha'(t) &\leq f(t, \alpha(t), [\varphi\alpha](t)), \quad t \in J', \\ \Delta\alpha(t_k) &\leq I_k(t_k, \alpha(t_k), [\varphi\alpha](t_k)), \quad k = 1, 2, \dots, p, \\ g(\alpha(0), \alpha(T)) &\leq 0. \end{aligned} \quad (3.1)$$

Analogously, $\beta \in PC^1(J)$ is an upper solution of NBVP (2.2) if

$$\begin{aligned} \beta'(t) &\geq f(t, \beta(t), [\varphi\beta](t)), \quad t \in J', \\ \Delta\beta(t_k) &\geq I_k(t_k, \beta(t_k), [\varphi\beta](t_k)), \quad k = 1, 2, \dots, p, \\ g(\beta(0), \beta(T)) &\geq 0. \end{aligned} \quad (3.2)$$

For convenience, let us list the following conditions.

(H3) There exist constants $M > 0$, $N \geq 0$ such that

$$f(t, x, \varphi x) - f(t, \bar{x}, \varphi\bar{x}) \geq -M(x - \bar{x}) - N(\varphi x - \varphi\bar{x}) \quad (3.3)$$

wherever $\alpha_0(t) \leq \bar{x} \leq x \leq \beta_0(t)$.

(H4) There exist constants $0 \leq L_k < 1$ for $k = 1, 2, \dots, p$ such that

$$I_k(t_k, x, \varphi x) - I_k(t_k, \bar{x}, \varphi\bar{x}) \geq -L_k(x - \bar{x}), \quad k = 1, 2, \dots, p \quad (3.4)$$

wherever $\alpha_0(t_k) \leq \bar{x} \leq x \leq \beta_0(t_k)$.

(H5) The function φ satisfies

$$\varphi x - \varphi\bar{x} \geq \varphi(x - \bar{x}), \quad \text{for } x, \bar{x} \in PC(J), \quad x \geq \bar{x}. \quad (3.5)$$

(H6) There exist constants M_1, M_2 with $0 < M_2 e^{-MT} \leq M_1$ such that

$$g(x, y) - g(\bar{x}, \bar{y}) \leq M_1(x - \bar{x}) - M_2(y - \bar{y}) \quad (3.6)$$

wherever $\alpha_0(0) \leq \bar{x} \leq x \leq \beta_0(0)$, and $\alpha_0(T) \leq \bar{y} \leq y \leq \beta_0(T)$.

Let $[\alpha_0, \beta_0] = \{x \in PC^1(J) : \alpha_0(t) \leq x(t) \leq \beta_0(t), \text{ for all } t \in J\}$. Now we are in the position to establish the main results of this paper.

Theorem 3.2. *Let (H_1) – (H_6) and inequalities (2.9) and (2.30) hold. Assume further that there exist lower and upper solutions α_0 and β_0 of NBVP (2.2), respectively, such that $\alpha_0 \leq \beta_0$ on J . Then there exist monotone sequences $\{\alpha_n\}, \{\beta_n\} \subset PC^1(J)$ with $\alpha_0 \leq \dots \leq \alpha_n \leq \dots \leq \beta_n \leq \dots \leq \beta_0$, such that $\lim_{n \rightarrow \infty} \alpha_n = x_*(t)$, $\lim_{n \rightarrow \infty} \beta_n = x^*(t)$ uniformly on J . Moreover, $x_*(t)$, $x^*(t)$ are minimal and maximal solutions of NBVP (2.2) in $[\alpha_0, \beta_0]$, respectively.*

Proof. For any $\eta \in [\alpha_0, \beta_0]$, consider LVBP (2.27) with

$$\sigma(t) = f(t, \eta(t), [\varphi\eta](t)) + M\eta(t) + N[\varphi\eta](t). \quad (3.7)$$

By Lemma 2.8, we know that LBVP (2.27) has a unique solution $y \in PC^1(J)$. Define an operator $A : PC(J) \rightarrow PC(J)$ by $y = A\eta$, then the operator A has the following properties:

- (a) $\alpha_0 \leq A\alpha_0$, $A\beta_0 \leq \beta_0$,
- (b) $A\eta_1 \leq A\eta_2$, if $\alpha_0 \leq \eta_1 \leq \eta_2 \leq \beta_0$.

To prove (a), let $\alpha_1 = A\alpha_0$ and $m(t) = \alpha_0(t) - \alpha_1(t)$.

$$\begin{aligned} m'(t) &= \alpha_0'(t) - \alpha_1'(t) \\ &= f(t, \alpha_0(t), [\varphi\alpha_0](t)) \\ &\quad - \{-M\alpha_1(t) - N[\varphi\alpha_1](t) + f(t, \alpha_0(t), [\varphi\alpha_0](t)) + M\alpha_0(t) + N[\varphi\alpha_0](t)\} \\ &\leq -Mm(t) - N[\varphi m](t), \\ \Delta m(t_k) &= \Delta\alpha_0(t_k) - \Delta\alpha_1(t_k) \\ &\leq I_k(t_k, \alpha_0(t_k), [\varphi\alpha_0](t_k)) - \{-L_k\alpha_1(t_k) + I_k(t_k, \alpha_0(t_k), [\varphi\alpha_0](t_k)) + L_k\alpha_0(t_k)\} \\ &\leq -L_k m(t_k), \\ m(0) &= \alpha_0(0) - \alpha_1(0) \\ &= \alpha_0(0) - \left\{ -\frac{1}{M_1} g(\alpha_0(0), \alpha_0(T)) + \alpha_0(0) + \frac{M_2}{M_1} (\alpha_1(T) - \alpha_0(T)) \right\} \\ &\leq \frac{M_2}{M_1} m(T). \end{aligned} \quad (3.8)$$

By Theorem 2.3, we get $m(t) \leq 0$ for $t \in J$, that is, $\alpha_0 \leq A\alpha_0$. Similarly, we can show that $A\beta_0 \leq \beta_0$.

To prove (b), set $m(t) = x_1(t) - x_2(t)$, where $x_1 = A\eta_1$ and $x_2 = A\eta_2$. Using (H3), (H4) and (H6), we get

$$\begin{aligned}
 m'(t) &= x_1'(t) - x_2'(t) \\
 &= M(\eta_1(t) - x_1(t)) + N([\varphi\eta_1](t) - [\varphi x_1](t)) + f(t, \eta_1(t), [\varphi\eta_1](t)) \\
 &\quad - M(\eta_2(t) - x_2(t)) - N([\varphi\eta_2](t) - [\varphi x_2](t)) - f(t, \eta_2(t), [\varphi\eta_2](t)) \\
 &\leq -Mm(t) - N[\varphi m](t), \\
 \Delta m(t_k) &= \Delta x_1(t_k) - \Delta x_2(t_k) \\
 &\leq L_k(\eta_1(t_k) - x_1(t_k)) + I_k(t_k, \eta_1(t_k), [\varphi\eta_1](t_k)) \\
 &\quad - L_k(\eta_2(t_k) - x_2(t_k)) - I_k(t_k, \eta_2(t_k), [\varphi\eta_2](t_k)) \\
 &\leq -L_k m(t_k), \\
 m(0) &= x_1(0) - x_2(0) \\
 &= -\frac{1}{M_1}g(\eta_1(0), \eta_1(T)) + \eta_1(0) + \frac{M_2}{M_1}(x_1(T) - \eta_1(T)) \\
 &\quad + \frac{1}{M_1}g(\eta_2(0), \eta_2(T)) - \eta_2(0) - \frac{M_2}{M_1}(x_2(T) - \eta_2(T)) \\
 &\leq \frac{M_2}{M_1}m(T).
 \end{aligned} \tag{3.9}$$

By Theorem 2.3, we get $m(t) \leq 0$ for $t \in J$, that is, $A\eta_1 \leq A\eta_2$, then (b) is proved.

Let $\alpha_n = A\alpha_{n-1}$ and $\beta_n = A\beta_{n-1}$ for $n = 1, 2, 3, \dots$. By the properties (a) and (b), we have

$$\alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_n \leq \dots \leq \beta_n \leq \dots \leq \beta_1 \leq \beta_0. \tag{3.10}$$

By the definition of operator A , we have that $\{\alpha'_n\}$ and $\{\beta'_n\}$ are uniformly bounded in $[\alpha_0, \beta_0]$. Thus $\{\alpha_n\}$ and $\{\beta_n\}$ are uniformly bounded and equicontinuous in $[\alpha_0, \beta_0]$. By Arzela-Ascoli Theorem and (3.10), we know that there exist x_* , x^* in $[\alpha_0, \beta_0]$ such that

$$\lim_{n \rightarrow \infty} \alpha_n(t) = x_*(t), \quad \lim_{n \rightarrow \infty} \beta_n(t) = x^*(t) \quad \text{uniformly on } J \tag{3.11}$$

Moreover, $x_*(t)$, $x^*(t)$ are solutions of NBVP (2.2) in $[\alpha_0, \beta_0]$.

To prove that x_* , x^* are extremal solutions of NBVP (2.2), let $u(t) \in [\alpha_0, \beta_0]$ be any solution of NBVP (2.2), that is,

$$\begin{aligned}
 u'(t) &= f(t, u(t), [\varphi u](t)), \quad t \in J', \\
 \Delta u(t_k) &= I_k(t_k, u(t_k), [\varphi u](t_k)), \quad k = 1, 2, \dots, p, \\
 g(u(0), u(T)) &= 0.
 \end{aligned} \tag{3.12}$$

By Theorem 2.3 and Induction, we get $\alpha_n(t) \leq u(t) \leq \beta_n(t)$ with $t \in J$ and $n = 1, 2, 3, \dots$ which implies that $x_*(t) \leq u(t) \leq x^*(t)$, that is, x_* and x^* are minimal and maximal solution of NBVP (2.2) in $[\alpha_0, \beta_0]$, respectively. The proof is complete. \square

Theorem 3.3. *Let the assumptions of Theorem 3.2 hold and assume the following.*

(H7) *There exist constants $\widetilde{M} > 0$, $\widetilde{N} \geq 0$ such that*

$$f(t, x, \varphi x) - f(t, \bar{x}, \varphi \bar{x}) \leq -\widetilde{M}(x - \bar{x}) - \widetilde{N}(\varphi x - \varphi \bar{x}), \quad (3.13)$$

where $\alpha_0(t) \leq \bar{x} \leq x \leq \beta_0(t)$.

(H8) *There exist constants $0 \leq \widetilde{L}_k < 1$, $k = 1, 2, \dots, p$ such that*

$$I_k(t_k, x, \varphi x) - I_k(t_k, \bar{x}, \varphi \bar{x}) \leq -\widetilde{L}_k(x - \bar{x}), \quad k = 1, 2, \dots, p, \quad (3.14)$$

where $\alpha_0(t_k) \leq \bar{x} \leq x \leq \beta_0(t_k)$.

(H9) *There exist constants $\widetilde{M}_1, \widetilde{M}_2$ with $0 < \widetilde{M}_2 e^{-\widetilde{M}T} < \widetilde{M}_1$ such that*

$$g(x, y) - g(\bar{x}, \bar{y}) \geq \widetilde{M}_1(x - \bar{x}) - \widetilde{M}_2(y - \bar{y}) \quad (3.15)$$

whenever $\alpha_0(0) \leq \bar{x} \leq x \leq \beta_0(0)$, and $\alpha_0(T) \leq \bar{y} \leq y \leq \beta_0(T)$.

Then NBVP (2.2) has a unique solution in $[\alpha_0, \beta_0]$.

Proof. By Theorem 3.2, we know that there exist $x_*, x^* \in [\alpha_0, \beta_0]$, which are minimal and maximal solutions of NBVP (2.2) with $x_*(t) \leq x^*(t)$, $t \in J$.

Let $m(t) = x^*(t) - x_*(t)$. Using (H7), (H8), and (H9), we get

$$\begin{aligned} m'(t) &= (x^*(t))' - (x_*(t))' = f(t, x^*(t), [\varphi x^*](t)) - f(t, x_*(t), [\varphi x_*](t)) \\ &\leq -\widetilde{M}m(t) - \widetilde{N}[\varphi m](t), \\ \Delta m(t_k) &= \Delta x^*(t_k) - \Delta x_*(t_k) = I_k(t_k, x^*(t_k), [\varphi x^*](t_k)) - I_k(t_k, x_*(t_k), [\varphi x_*](t_k)) \\ &\leq \widetilde{L}_k m(t_k), \\ m(0) &= x^*(0) - x_*(0) \leq \frac{\widetilde{M}_2}{\widetilde{M}_1}(x^*(T) - x_*(T)) = \frac{\widetilde{M}_2}{\widetilde{M}_1}m(T). \end{aligned} \quad (3.16)$$

By Theorem 2.3, we have that $m(t) < 0$, $t \in J$, that is, $x^*(t) \leq x_*(t)$. Hence $x^*(t) = x_*(t)$, this completes the proof. \square

4. Examples

To illustrate our main results, we shall discuss in this section some examples.

Example 4.1. Consider the problem

$$\begin{aligned}
 x'(t) &= -\frac{1}{10}(-|\sin t| + x - 2)^5 - \frac{e^{-2\pi}}{3} \left(\int_t^{t+1} x(s) ds - \sin \frac{t}{4} \right)^2 + \frac{4}{3} e^{-2\pi}, \quad t \in [0, T], \quad t \neq t_k, \\
 \Delta x(t_k) &= -\frac{1}{2} e^{-\pi/2} (x(t_k) - 3) + \left(\int_{t_k}^{t_k+1} x(s) ds - \sin \frac{t_k}{4} \right)^{1/7} + \frac{4}{17} e^{\pi/2} \cos t_k, \quad k = 1, \\
 x(0) - \frac{1}{2} x(T) - \frac{1}{6\pi} \int_0^T x(s) ds &= 0,
 \end{aligned} \tag{4.1}$$

where $T = 2\pi$, $k = 1$, $t_1 = \pi$.

Let

$$\begin{aligned}
 [\varphi x](t) &= \int_t^{t+1} x(s) ds - \sin \frac{t}{4}, \\
 f(t, x, y) &= -\frac{1}{10}(-|\sin t| + x - 2)^5 - \frac{e^{-2\pi}}{3} y^2 + \frac{4}{3} e^{-2\pi}, \\
 I_1(t, x, y) &= -\frac{1}{2} e^{-\pi/2} (x - 3) + y^{1/7} + \frac{4}{17} e^{\pi/2} \cos t.
 \end{aligned} \tag{4.2}$$

Setting $\alpha_0(t) \equiv 2$ and $\beta_0(t) \equiv 3$, it is easy to verify that $\alpha_0(t)$ is a lower solution, and $\beta_0(t)$ is an upper solution with $\alpha_0(t) \leq \beta_0(t)$.

For $t \in J$, and $2 \leq \bar{x}(t) \leq x(t) \leq 3$, we have

$$\begin{aligned}
 \varphi x &\geq 1 \geq \frac{1}{3} \inf_{t \in J} x(t), \\
 \varphi x - \varphi \bar{x} &\geq \varphi(x - \bar{x}), \\
 \|\varphi x - \varphi \bar{x}\| &= \left\| \int_t^{t+1} [x(s) - \bar{x}(s)] ds \right\| \leq \|x - \bar{x}\|.
 \end{aligned} \tag{4.3}$$

Setting $M = 1/2$, $N = e^{-2\pi}/6$, $L = 1$, $R = 1/3$, $L_1 = (1/2)e^{-\pi/2}$ and $M_1 = 1$, $M_2 = 1/2$, then conditions (H1)–(H6) are all satisfied:

$$\int_0^{2\pi} \prod_{s < t_k < 2\pi} (1 - L_k) e^{Ms} ds = \int_0^{\pi} (1 - L_1) e^{Ms} ds + \int_{\pi}^{2\pi} e^{Ms} ds \approx 43.4893,$$

$$\prod_{k=1}^p (1 - L_k)^2 = (1 - L_1)^2 = \left(1 - \frac{1}{2}e^{-\pi/2}\right)^2 \approx 0.8029,$$

$$\frac{NR(e^{MT} + (M_2/M_1))}{M_2/M_1} = \frac{2e^{\pi} + 1}{18e^{2\pi}} \approx 0.0049, \tag{4.4}$$

$$\frac{NR(e^{MT} + (M_2/M_1))}{M_2/M_1} \int_0^{2\pi} \prod_{s < t_k < 2\pi} (1 - L_k) e^{Ms} ds \approx 0.2131 < 0.8029,$$

$$\left(NLT + \sum_{k=1}^p |L_k|\right) r = (NLT + L_1)CM_1 \approx 0.1082 < 1,$$

then inequalities (2.9) and (2.30) are satisfied. By Theorem 3.2, problem (4.1) has extremal solutions $x_*, x^* \in [\alpha_0, \beta_0]$.

Example 4.2. Consider the problem

$$x'(t) = -\frac{1}{2}x(t) - \frac{e^{-3\pi}}{e^{\pi} + 1} (e^{2x} - 1) + \frac{3}{2}, \quad t \in [0, T], \quad t \neq t_k,$$

$$\Delta x(t_k) = -2e^{-\pi} (x(t_k) - 2) + (e^{2x(t_k)} - 1)^{1/50} + \cos t_k, \quad k = 1, \tag{4.5}$$

$$x(0) = \frac{1}{2}x(T),$$

where $T = 2\pi$, $k = 1$, $t_1 = \pi$.

Let

$$\varphi x = e^{2x} - 1,$$

$$f(t, x, y) = -\frac{1}{10}(-|\sin t| + x - 2)^5 - \frac{e^{-3\pi}}{16(e^{\pi} + 1)} y^2 + \frac{(e^4 - 1)^2}{16e^{3\pi}(e^{\pi} + 1)}, \tag{4.6}$$

$$I_1(t, x, y) = -2e^{-\pi} (x - 2) + y^{1/50} + \cos t.$$

Setting $\alpha_0(t) \equiv 2$ and $\beta_0(t) \equiv 3$, then $\alpha_0(t)$ is a lower solution, and $\beta_0(t)$ is an upper solution with $\alpha_0(t) \leq \beta_0(t)$.

For $t \in J$, and $2 \leq \bar{x}(t) \leq x(t) \leq 3$, we have $\varphi x = e^{2x} - 1 \geq x$, $\varphi x - \varphi \bar{x} \geq \varphi(x - \bar{x})$, and $|\varphi x - \varphi \bar{x}| = |e^{2x} - e^{2\bar{x}}| = |e^x + e^{\bar{x}}| \cdot |e^x - e^{\bar{x}}| \leq 2e^6 |e^{x-\bar{x}} - 1| \leq 14e^6 |x - \bar{x}|$. Setting $M = 1/2$,

$N = e^{-3\pi}/(e^\pi + 1)$, $L = 14e^6$, $R = 1$, $L_1 = 2e^{-\pi}$ and $M_1 = 1$, $M_2 = 1/2$, then conditions (H1)–(H6) are all satisfied:

$$\begin{aligned} \frac{\prod_{k=1}^p (1 - L_k)^2}{\int_0^{2\pi} \prod_{s < t_k < 2\pi} (1 - L_k) ds} &= \frac{(e^\pi - 2)^2}{2\pi e^\pi (e^\pi - 1)} \approx 0.1388, \\ \frac{NR(e^{MT} + (M_2/M_1))e^{MT}}{M_2/M_1} &= \frac{2e^\pi + 1}{e^{2\pi}(e^\pi + 1)} \approx 0.0037 < 0.1388, \\ \left(NLT + \sum_{k=1}^p |L_k| \right) r &= (NLT + L_1)CM_1 \approx 0.2095 < 1, \end{aligned} \quad (4.7)$$

then inequalities (2.24) and (2.30) are satisfied. By Corollary 2.4 and Theorem 3.2, problem (4.5) has extremal solutions x_* , $x^* \in [\alpha_0, \beta_0]$.

Moreover, let $\widetilde{M} = 1/2$, $\widetilde{N} = e^{-3\pi}/(e^\pi + 1)$, $\widetilde{L}_1 = 2e^{-\pi}$ and $\widetilde{M}_1 = 1$, $\widetilde{M}_2 = 1/2$. It is easy to see that conditions (H7)–(H9) are satisfied. By Corollary 2.4 and Theorem 3.3, problem (4.5) has a unique solution in $[\alpha_0, \beta_0]$.

References

- [1] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.
- [2] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5 of *Notes and Reports in Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1988.
- [3] D. Franco, E. Liz, J. J. Nieto, and Y. V. Rogovchenko, "A contribution to the study of functional differential equations with impulses," *Mathematische Nachrichten*, vol. 218, pp. 49–60, 2000.
- [4] J. J. Nieto and R. Rodríguez-López, "Boundary value problems for a class of impulsive functional equations," *Computers & Mathematics with Applications*, vol. 55, no. 12, pp. 2715–2731, 2008.
- [5] L. Chen and J. Sun, "Nonlinear boundary value problem of first order impulsive functional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 2, pp. 726–741, 2006.
- [6] L. Li and J. Shen, "Periodic boundary value problems for functional differential equations with impulses," *Mathematica Scientia*, vol. 25A, pp. 237–244, 2005.
- [7] X. Liu and D. Guo, "Periodic boundary value problems for a class of second-order impulsive integro-differential equations in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 216, no. 1, pp. 284–302, 1997.
- [8] D. Guo and X. Liu, "Periodic boundary value problems for impulsive integro-differential equations in Banach spaces," *Nonlinear World*, vol. 3, no. 3, pp. 427–441, 1996.
- [9] X. Liu and D. Guo, "Initial value problems for first order impulsive integro-differential equations in Banach spaces," *Communications on Applied Nonlinear Analysis*, vol. 2, no. 1, pp. 65–83, 1995.
- [10] D. Guo and X. Liu, "First order impulsive integro-differential equations on unbounded domain in a Banach space," *Dynamics of Continuous, Discrete and Impulsive Systems*, vol. 2, no. 3, pp. 381–394, 1996.
- [11] D. Franco and J. J. Nieto, "First-order impulsive ordinary differential equations with anti-periodic and nonlinear boundary conditions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 42, no. 2, pp. 163–173, 2000.
- [12] X. Yang and J. Shen, "Nonlinear boundary value problems for first order impulsive functional differential equations," *Applied Mathematics and Computation*, vol. 189, no. 2, pp. 1943–1952, 2007.
- [13] W. Ding, J. Mi, and M. Han, "Periodic boundary value problems for the first order impulsive functional differential equations," *Applied Mathematics and Computation*, vol. 165, no. 2, pp. 433–446, 2005.
- [14] Z. Luo and Z. Jing, "Periodic boundary value problem for first-order impulsive functional differential equations," *Computers & Mathematics with Applications*, vol. 55, no. 9, pp. 2094–2107, 2008.