Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 498631, 10 pages
doi:10.1155/2010/498631

Research Article

A Note on Algorithms for Determining the Copositivity of a Given Symmetric Matrix

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Received 5 October 2009; Accepted 10 November 2009

Academic Editor: Shusen Ding

In the previous paper by the first and the third authors, we present six algorithms for determining whether a given symmetric matrix is strictly copositive, copositive (but not strictly), or not copositive. The algorithms for matrices of order $n \geq 8$ are not guaranteed to produce an answer. It also shows that for 1000 symmetric random matrices of order 8, 9, and 10 with unit diagonal and with positive entries all being less than or equal to 1 and negative entries all being greater than or equal to -1, there are 8, 6, and 2 matrices remaing undetermined, respectively. In this paper we give two more algorithms for $n = 8, 9$ and our experiment shows that no such matrix of order 8 or 9 remains undetermined; and almost always no such matrix of order 10 remains undetermined. We also do some discussion based on our experimental results.

1. Introduction

Reference [1] gives six algorithms for determining whether a given symmetric matrix is strictly copositive, copositive (but not strictly), or not copositive. The algorithms for matrices of order 3, 4, 5, 6 or 7 are efficient. But for matrices of order $n \geq 8$, it cannot guarantee to produce an answer. Table 1 of [1] shows that for 1000 symmetric random matrices of order n with unit diagonal and with positive entries all being less than or equal to 1 and negative entries all being greater than or equal to -1, there are 8, 6, and 2 matrices remaining undetermined when $n = 8, 9,$ and 10, respectively. In this paper we continue our study as in [1] and give two algorithms for $n = 8, 9$, and our experiment shows that no such matrix of order 8 or 9 remains undetermined; and almost always no such matrix of order 10 remains undetermined. We also do some discussion based on our experimental results.

In this paper we use all the concepts and notations of [1, 2] without explanation. Our main theorems will give the necessary and sufficient conditions for symmetric matrices of order 8 or 9 to be (strictly) copositive.

Let $A \in R^{n\times n}$ be symmetric and be partitioned into

$$A = \begin{pmatrix} a_{11} & \alpha^T \\ \alpha & A_2 \end{pmatrix}, \tag{1.1}$$

with $a_{11} \geq 0$, $B = a_{11}A_2 - \alpha\alpha^T$. As in [2], let

$$U = \left\{ u \in R^n : u = (u_1, u_2, \dots, u_n)^T \geq 0,\ \sum_1^n u_i = 1 \right\} \tag{1.2}$$

be the simplex of order $n-1$; and let

$$T = \left\{ u \in R^{n-1} : u = (u_2, u_3, \dots, u_n)^T \geq 0,\ \sum_2^n u_i = 1 \right\} \tag{1.3}$$

be the standard simplex of order $n-1$ whose vertices are all vertices of U. It is proved in [2] that an $n \times n$ symmetric matrix A is copositive if and only if $u^T A u \geq 0$ for all $u \in U$. Consider the polyhedron in $R^{n-1} : T^- = \{u \in T : \alpha^T u \leq 0\}$ (α is the given vector of dimension $n-1$ in (1.1)) which has some vertices being vertices of T, and all the other vertices being in the hyperplane $\Pi = \{u \in R^{n-1} : \alpha^T u = 0\}$. It is known (see [2, Section 2 and Lemma 3.1]) that the polyhedron T^- can be subdivided into l simplices S^i in R^{n-1} such that $T^- = \bigcup_{i=1,\dots,l} S^i$, $S^i \bigcap S^j \neq \emptyset$ is a subsimplex of S^i and S^j if $i \neq j$, and the vertices of S^i are all vertices of T^-. We mention this fact since that T^- is subdivided into simplices $S^1, \dots, S^l$.

Denote the vertices of S^i by $V_1^i, \dots, V_{n-1}^i$, then V_j^i is a vertex of T, or a common point of the line connecting two vertices of T and the hyperplane Π and should be presented in the barycenter coordinates of T. If V_j^i is the kth vertex of T, then it is represented by the coordinate vector $e_k \in R^{n-1}$ with a 1 in the kth position and all 0's elsewhere; otherwise write $V_j^i = V^{km}$ to denote that it is the common point of line $e_k - e_m$ and the hyperplane Π. Each S^i determines a matrix $W^i \in R^{(n-1)\times(n-1)}$ (see [2, Lemma 3.1]), to simplify the notation we still write $W^i = (V_1^i, \dots, V_{n-1}^i)$ with $V_j^i = e_k$ or V^{km}. For example, if S^i share only one vertex $V_1^i = e_k$ with T and the other vertices are $\{V_1^i, \dots, V_1^i\} = \{V^{k,u_1}, \dots, V^{k,u_{n-2}}\}$, then

$$\begin{gathered} W = \left(e_k, V^{k,u_1}, \dots, V^{k,u_{n-2}}\right), \quad \{u_1, \dots, u_{n-2}\} = \{1, 2, \dots, n-1\} \setminus \{k\}, \\ \left(V^{k,u}\right)_m = \begin{cases} a_{1,u+1}, & m = k, \\ a_{1,k+1}, & m = u, \quad \text{for any } u \in \{1, 2, \dots, n-1\} \setminus \{k\}, \\ 0, & \text{else.} \end{cases} \end{gathered} \tag{1.4}$$

Lemma 1.1 (see [2]). *Let $A \in R^{n\times n}$ be symmetric and partitioned as in* (1.1) *with $a_{11} \geq 0$, $B = a_{11}A_2 - \alpha\alpha^T$ being copositive and T^- is subdivided into simplices $S^1, \dots, S^l$ which determine matrices $W^1, W^2, \dots, W^l$. Then A is copositive if and only if $(W^i)^T B W^i$, $i = 1, \dots, l$ are all copositive (see [2, Lemma 3.1]); A is strictly copositive if and only if $(W^i)^T B W^i$, $i = 1, \dots, l$ are all strictly copositive and $a_{11} > 0$ and A_2 is strictly copositive (see [1]).*

It is noticed from [2] that if the polyhedron $T^- \subseteq R^{n-1}$ contains $f(\leq n-1)$ vertices (coordinate vectors of the standard simplex T) not in the hyperplane $\Pi = \{u \in R^{n-1} : \alpha^T u = 0\}$, then T^- contains exact $g = f(n-1-f)$ vertices in the hyperplane Π, and that T^- can be subdivided into $l = f+g-n+2$ simplices $\{S^1, S^2, \dots, S^l\}$ of dimension $n-2$ such that $S^i \bigcap S^{i+1}$ is a simplex of dimension $n-3$ for $i = 1, 2, \dots, l-1$, and $S^i \bigcap S^j$ is a simplex of dimension $< n-3$ when $j \notin \{i-1, i, i+1\}$.

Lemma 1.2 (see [1]). *Let $n \geq 3$. If there are $l = f(n-f) - n + 2(n-1)$-triples of pairwise different vertices of T^- : $\{S_1, S_2, \dots, S_l\}$ satisfying the following two conditions:*

(i) *each S_i contains at least one coordinate vector vertex;*

(ii) *$S_i \bigcap S_{i+1}$ has exactly $n-2$ vertices for $i = 1, \dots, l-1$, and $S_i \bigcap S_j$ has less than $n-2$ vertices when $j \notin \{i-1, i, i+1\}$,*

then T^- can be subdivided into l simplices $\{S^1, S^2, \dots, S^l\}$, where S^i is the simplex whose vertices are the elements of S_i.

These two lemmas are basic for proving Theorems 2.5, 2.6, 2.7, and 2.8 in [1]; they are also basic for proving Theorems 2.1 and 2.2 of this paper.

2. Main Theorems and Algorithms

The following two theorems give two algorithms for determining the copositivity of a given symmetric matrix of order 8 or 9. These two theorems can be proved by Lemma 1.1 and Lemma 1.2 following the same pattern as in [1].

Theorem 2.1. *Let $A \in R^{8\times 8}$ be symmetric and be partitioned as in* (1.1) *and $B = a_{11}A_2 - \alpha\alpha^T$, then at least one of the following cases must happen:*

(a) *If one 7×7 principal submatrix of A is not copositive, then A is not copositive. Otherwise it holds that $a_{11} \geq 0$ and A_2 is copositive.*

(b) *If $\alpha \geq 0$ then A is copositive; if $\alpha \geq 0$ with $a_{11} > 0$ and A_2 is strictly copositive, then A is strictly copositive.*

(c) *If $\alpha \leq 0$, then A is copositive if and only if B is copositive; A is strictly copositive if and only if B is strictly copositive and $a_{11} > 0$ and A_2 is strictly copositive.*

(d) *If α has exactly one negative entry: $a_{1,k+1}$, then A is copositive if and only if $W^T BW$ is copositive; A is strictly copositive if and only if $W^T BW$ is strictly copositive, and $a_{11} > 0$ and A_2 is strictly copositive, where*

$$W = \left(e_k, V^{k,u_1}, \dots, V^{k,u_6}\right), \quad \{u_1, \dots, u_6\} = \{1, 2, \dots, 7\} \setminus \{k\},$$
$$\left(V^{k,u}\right)_m = \begin{cases} a_{1,u+1}, & m = k, \\ a_{1,k+1}, & m = u, \quad \text{for any } u \in \{1, 2, \dots, 7\} \setminus \{k\}, \\ 0 & \text{else}. \end{cases} \tag{2.1}$$

(e) *If α has exactly two negative entries:$a_{1,i+1}, a_{1,j+1}$, and $\{r, s, t, u, v\} = \{1, 2, 3, 4, 5, 6, 7\} \setminus \{i, j\}$, then A is copositive if and only if $W_1^T BW_1, W_2^T BW_2, W_3^T BW_3, W_4^T BW_4, W_5^T BW_5$ and*

$W_6^T BW_6$ are all copositive; A is strictly copositive if and only if $W_1^T BW_1, \ldots, W_6^T BW_6$ are all strictly copositive and $a_{11} > 0$ and A_2 is strictly copositive, where

$$\begin{aligned}
&W_1 = \left(e_i, e_j, V^{i,r}, V^{i,s}, V^{i,t}, V^{i,u}, V^{i,v}\right), \qquad W_2 = \left(e_j, V^{i,r}, V^{i,s}, V^{i,t}, V^{i,u}, V^{i,v}, V^{j,r}\right),\\
&W_3 = \left(e_j, V^{i,s}, V^{i,t}, V^{i,u}, V^{i,v}, V^{j,r}, V^{j,s}\right), \qquad W_4 = \left(e_j, V^{i,t}, V^{i,u}, V^{i,v}, V^{j,r}, V^{j,s}, V^{j,t}\right),\\
&W_5 = \left(e_j, V^{i,u}, V^{i,v}, V^{j,r}, V^{j,s}, V^{j,t}, V^{j,u}\right), \qquad W_6 = \left(e_j, V^{i,v}, V^{j,r}, V^{j,s}, V^{j,t}, V^{j,u}, V^{j,v}\right).
\end{aligned} \tag{2.2}$$

(f) *If α has exactly three negative entries: $a_{1,i+1}, a_{1,j+1}, a_{1,k+1}$ and $\{r,s,t,u\} = \{1,2,3,4,5,6,7\} \setminus \{i,j,k\}$, then A is copositive if and only if $W_1^T BW_1, \ldots, W_9^T BW_9$ are all copositive; A is strictly copositive if and only if $W_1^T BW_1, \ldots, W_9^T BW_9$ are all strictly copositive and $a_{11} > 0$ and A_2 is strictly copositive, where*

$$\begin{aligned}
&W_1 = \left(e_i, e_j, e_k, V^{i,r}, V^{i,s}, V^{i,t}, V^{i,u}\right), \qquad W_2 = \left(e_j, e_k, V^{i,r}, V^{i,s}, V^{i,t}, V^{i,u}, V^{j,r}\right),\\
&W_3 = \left(e_k, V^{i,r}, V^{i,s}, V^{i,t}, V^{i,u}, V^{j,r}, V^{j,s}\right), \qquad W_4 = \left(e_k, V^{i,s}, V^{i,t}, V^{i,u}, V^{j,r}, V^{j,s}, V^{j,t}\right),\\
&W_5 = \left(e_k, V^{i,t}, V^{i,u}, V^{j,r}, V^{j,s}, V^{j,t}, V^{j,u}\right), \qquad W_6 = \left(e_k, V^{i,u}, V^{j,r}, V^{j,s}, V^{j,t}, V^{j,u}, V^{k,r}\right),\\
&W_7 = \left(e_k, V^{j,r}, V^{j,s}, V^{j,t}, V^{j,u}, V^{k,r}, V^{k,s}\right), \qquad W_8 = \left(e_k, V^{j,s}, V^{j,t}, V^{j,u}, V^{k,r}, V^{k,s}, V^{k,t}\right),\\
&W_9 = \left(e_k, V^{j,t}, V^{j,u}, V^{k,r}, V^{k,s}, V^{k,t}, V^{k,u}\right).
\end{aligned} \tag{2.3}$$

(g) *If α has exactly four negative entries: $a_{1,i+1}, a_{1,j+1}, a_{1,k+1}, a_{1,h+1}$ and $\{r,s,t\} = \{1,2,3,4,5,6,7\} \setminus \{i,j,k,h\}$, then A is copositive if and only if $W_1^T BW_1, \ldots, W_{10}^T BW_{10}$ are all copositive; A is strictly copositive if and only if $W_1^T BW_1, \ldots, W_{10}^T BW_{10}$ are all strictly copositive and $a_{11} > 0$ and A_2 is strictly copositive, where*

$$\begin{aligned}
&W_1 = \left(e_i, e_j, e_k, e_h, V^{i,r}, V^{i,s}, V^{i,t}\right), \qquad W_2 = \left(e_j, e_k, e_h, V^{i,r}, V^{i,s}, V^{i,t}, V^{j,r}\right),\\
&W_3 = \left(e_k, e_h, V^{i,r}, V^{i,s}, V^{i,t}, V^{j,r}, V^{j,s}\right), \qquad W_4 = \left(e_h, V^{i,r}, V^{i,s}, V^{i,t}, V^{j,r}, V^{j,s}, V^{j,t}\right),\\
&W_5 = \left(e_h, V^{i,s}, V^{i,t}, V^{j,r}, V^{j,s}, V^{j,t}, V^{k,r}\right), \qquad W_6 = \left(e_h, V^{i,t}, V^{j,r}, V^{j,s}, V^{j,t}, V^{k,r}, V^{k,s}\right),\\
&W_7 = \left(e_h, V^{j,r}, V^{j,s}, V^{j,t}, V^{k,r}, V^{k,s}, V^{k,t}\right), \qquad W_8 = \left(e_h, V^{j,s}, V^{j,t}, V^{k,r}, V^{k,s}, V^{k,t}, V^{h,r}\right),\\
&W_9 = \left(e_h, V^{j,t}, V^{k,r}, V^{k,s}, V^{k,t}, V^{h,r}, V^{h,s}\right), \qquad W_{10} = \left(e_h, V^{k,r}, V^{k,s}, V^{k,t}, V^{h,r}, V^{h,s}, V^{h,t}\right).
\end{aligned} \tag{2.4}$$

(h) *If α has exactly five negative entries: $a_{1,i+1}, a_{1,j+1}, a_{1,k+1}, a_{1,h+1}, a_{1,f+1}$ and $\{r,s\} = \{1,2,3,4,5,6,7\} \setminus \{i,j,k,h,f\}$, then A is copositive if and only if $W_1^T BW_1, \ldots, W_9^T BW_9$ are all* *copositive; A is strictly copositive if and only if* $W_1^T BW_1, \ldots, W_9^T BW_9$ *are all strictly copositive and* $a_{11} > 0$ *and* A_2 *is strictly copositive, where*

$$\begin{aligned}
W_1 &= \left(e_i, e_j, e_k, e_h, e_f, V^{i,r}, V^{i,s}\right), \qquad W_2 = \left(e_j, e_k, e_h, e_f, V^{i,r}, V^{i,s}, V^{j,r}\right),\\
W_3 &= \left(e_k, e_h, e_f, V^{i,r}, V^{i,s}, V^{j,r}, V^{j,s}\right), \qquad W_4 = \left(e_k, e_f, V^{i,r}, V^{i,s}, V^{j,r}, V^{j,s}, V^{k,r}\right),\\
W_5 &= \left(e_f, V^{i,r}, V^{i,s}, V^{j,r}, V^{j,s}, V^{k,r}, V^{k,s}\right), \qquad W_6 = \left(e_f, V^{i,s}, V^{j,r}, V^{j,s}, V^{k,r}, V^{k,s}, V^{h,r}\right),\\
W_7 &= \left(e_f, V^{j,r}, V^{j,s}, V^{k,r}, V^{k,s}, V^{h,r}, V^{h,s}\right), \qquad W_8 = \left(e_f, V^{j,s}, V^{k,r}, V^{k,s}, V^{h,r}, V^{h,s}, V^{f,r}\right),\\
W_9 &= \left(e_f, V^{k,r}, V^{k,s}, V^{h,r}, V^{h,s}, V^{f,r}, V^{f,s}\right).
\end{aligned} \tag{2.5}$$

(i) *If* α *has exactly six negative entries:* $a_{1,i+1}, a_{1,j+1}, a_{1,k+1}, a_{1,h+1}, a_{1,f+1}, a_{1,g+1}$ *and* $\{r\} = \{1,2,3,4,5,6,7\} \setminus \{i,j,k,h,f,g\}$*, then A is copositive if and only if* $W_1^T BW_1, \ldots, W_6^T BW_6$ *are all copositive; A is strictly copositive if and only if* $W_1^T BW_1, \ldots, W_6^T BW_6$ *are all strictly copositive and* $a_{11} > 0$ *and* A_2 *is strictly copositive, where*

$$\begin{aligned}
W_1 &= \left(e_i, e_j, e_k, e_h, e_f, e_g, V^{i,r}\right), \qquad W_2 = \left(e_j, e_k, e_h, e_f, e_g, V^{i,r}, V^{j,r}\right),\\
W_3 &= \left(e_k, e_h, e_f, e_g, V^{i,r}, V^{j,r}, V^{k,r}\right), \qquad W_4 = \left(e_k, e_f, e_g, V^{i,r}, V^{j,r}, V^{k,r}, V^{h,r}\right),\\
W_5 &= \left(e_f, e_g, V^{i,r}, V^{j,r}, V^{k,r}, V^{h,r}, V^{f,r}\right), \qquad W_6 = \left(e_g, V^{i,r}, V^{j,r}, V^{k,r}, V^{h,r}, V^{f,r}, V^{g,r}\right).
\end{aligned} \tag{2.6}$$

It is clear (see [1, Remark 2.1]) that if n is odd, then a copositive matrix $A \in \mathbf{R}^{\mathbf{n}\times\mathbf{n}}$ must have a row with an even number of negative entries. In other words, if a symmetric matrix of odd order has row with an even number of negative entries, then some $(n-1)\times(n-1)$ principal submatrices of it are not copositive. This fact will be used in Theorem 2.2.

Theorem 2.2. *If* $A \in R^{9\times 9}$ *is symmetric, then at least one of the following cases must happen:*

(a) *If one* 7×7 *principal submatrix of A is not copositive, then A is not copositive.*

Otherwise (A must have a row with an even number of negative entries and $a_{11} \geq 0$, A_2 *is copositive) find a row of A which has exactly* m ($m \in \{0,2,4,6,8\}$) *negative entries. If the* i*th row does, then interchange the* i*th row and column with the first row and column, and partition A into* (1.1) *as in Theorem 2.1.*

(b) *If* $m = 0$*, then* $\alpha \geq 0$ *and A is copositive; if* $m = 0$ *with* $a_{11} > 0$ *and* A_2 *is strictly copositive, then A is strictly copositive.*

(c) *If* $m = 8$*, then* $\alpha \leq 0$*, then A is copositive if and only if B is copositive; A is strictly copositive if and only if B is strictly copositive and* $a_{11} > 0$ *and* A_2 *is strictly copositive.*

(d) *If* $m = 2$*, then* α *has exactly two negative entries:* $a_{1,i+1}, a_{1,j+1}$*, and* $\{r,s,t,u,v,w\} = \{1,2,\ldots,8\} \setminus \{i,j\}$*, then A is copositive if and only if* $W_1^T BW_1, \ldots, W_7^T BW_7$ *are all copositive; A is* *strictly copositive if and only if* $W_1^T BW_1, \dots, W_7^T BW_7$ *are all strictly copositive and* $a_{11} > 0$ *and* A_2 *is strictly copositive, where*

$$W_1 = \left(e_i, e_j, V^{i,r}, V^{i,s}, V^{i,t}, V^{i,u}, V^{i,v}, V^{i,w}\right), \qquad W_2 = \left(e_j, V^{i,r}, V^{i,s}, V^{i,t}, V^{i,u}, V^{i,v}, V^{i,w}, V^{j,r}\right),$$
$$W_3 = \left(e_j, V^{i,s}, V^{i,t}, V^{i,u}, V^{i,v}, V^{i,w}, V^{j,r}, V^{j,s}\right), \qquad W_4 = \left(e_j, V^{i,t}, V^{i,u}, V^{i,v}, V^{i,w}, V^{j,r}, V^{j,s}, V^{j,t}\right),$$
$$W_5 = \left(e_j, V^{i,u}, V^{i,v}, V^{i,w}, V^{j,r}, V^{j,s}, V^{j,t}, V^{j,u}\right), \qquad W_6 = \left(e_j, V^{i,v}, V^{i,w}, V^{j,r}, V^{j,s}, V^{j,t}, V^{j,u}, V^{j,v}\right),$$
$$W_7 = \left(e_j, V^{i,w}, V^{j,r}, V^{j,s}, V^{j,t}, V^{j,u}, V^{j,v}, V^{j,w}\right). \tag{2.7}$$

(e) *If* $m = 4$, *then* α *has exactly four negative entries:* $a_{1,i+1}, a_{1,j+1}, a_{1,k+1}, a_{1,h+1}$ *and* $\{r, s, t, u\} = \{1, 2, \dots, 8\} \setminus \{i, j, k, h\}$, *then* A *is copositive if and only if* $W_1^T BW_1, \dots, W_{13}^T BW_{13}$ *are all copositive;* A *is strictly copositive if and only if* $W_1^T BW_1, \dots, W_{13}^T BW_{13}$ *are all strictly copositive and* $a_{11} > 0$ *and* A_2 *is strictly copositive, where*

$$\begin{aligned}
W_1 &= \left(e_i, e_j, e_k, e_h, V^{i,r}, V^{i,s}, V^{i,t}, V^{i,u}\right),\\
W_2 &= \left(e_j, e_k, e_h, V^{i,r}, V^{i,s}, V^{i,t}, V^{i,u}, V^{j,r}\right),\\
W_3 &= \left(e_k, e_h, V^{i,r}, V^{i,s}, V^{i,t}, V^{i,u}, V^{j,r}, V^{j,s}\right),\\
W_4 &= \left(e_h, V^{i,r}, V^{i,s}, V^{i,t}, V^{i,u}, V^{j,r}, V^{j,s}, V^{j,t}\right),\\
W_5 &= \left(e_h, V^{i,s}, V^{i,t}, V^{i,u}, V^{j,r}, V^{j,s}, V^{j,t}, V^{j,u}\right),\\
W_6 &= \left(e_h, V^{i,t}, V^{i,u}, V^{j,r}, V^{j,s}, V^{j,t}, V^{j,u}, V^{k,r}\right),\\
W_7 &= \left(e_h, V^{i,u}, V^{j,r}, V^{j,s}, V^{j,t}, V^{j,u}, V^{k,r}, V^{k,s}\right),\\
W_8 &= \left(e_h, V^{j,r}, V^{j,s}, V^{j,t}, V^{j,u}, V^{k,r}, V^{k,s}, V^{k,t}\right),\\
W_9 &= \left(e_h, V^{j,s}, V^{j,t}, V^{j,u}, V^{k,r}, V^{k,s}, V^{k,t}, V^{k,u}\right),\\
W_{10} &= \left(e_h, V^{j,t}, V^{j,u}, V^{k,r}, V^{k,s}, V^{k,t}, V^{k,u}, V^{h,r}\right),\\
W_{11} &= \left(e_h, V^{j,u}, V^{k,r}, V^{k,s}, V^{k,t}, V^{k,u}, V^{h,r}, V^{h,s}\right),\\
W_{12} &= \left(e_h, V^{k,r}, V^{k,s}, V^{k,t}, V^{k,u}, V^{h,r}, V^{h,s}, V^{h,t}\right),\\
W_{13} &= \left(e_h, V^{k,s}, V^{k,t}, V^{k,u}, V^{h,r}, V^{h,s}, V^{h,t}, V^{h,u}\right).
\end{aligned} \tag{2.8}$$

(f) *If* $m = 6$, *then* α *has exactly six negative entries:* $a_{1,i+1}, a_{1,j+1}, a_{1,k+1}, a_{1,h+1}, a_{1,f+1}, a_{1,g+1}$ *and* $\{r, s\} = \{1, 2, \dots, 8\} \setminus \{i, j, k, h, f, g\}$, *then* A *is copositive if and only if* $W_1^T BW_1, \dots, W_{11}^T BW_{11}$

are all copositive; A is strictly copositive if and only if $W_1^T B W_1, \ldots, W_{11}^T B W_{11}$ *are all strictly copositive and* $a_{11} > 0$ *and* A_2 *is strictly copositive, where*

$$\begin{aligned}
W_1 &= \left(e_i, e_j, e_k, e_h, e_f, e_g, V^{i,r}, V^{i,s}\right),\\
W_2 &= \left(e_j, e_k, e_h, e_f, e_g, V^{i,r}, V^{i,s}, V^{j,r}\right),\\
W_3 &= \left(e_k, e_h, e_f, e_g, V^{i,r}, V^{i,s}, V^{j,r}, V^{j,s}\right),\\
W_4 &= \left(e_h, e_f, e_g, V^{i,r}, V^{i,s}, V^{j,r}, V^{j,s}, V^{k,r}\right),\\
W_5 &= \left(e_f, e_g, V^{i,r}, V^{i,s}, V^{j,r}, V^{j,s}, V^{k,r}, V^{k,s}\right),\\
W_6 &= \left(e_g, V^{i,r}, V^{i,s}, V^{j,r}, V^{j,s}, V^{k,r}, V^{k,s}, V^{h,r}\right),\\
W_7 &= \left(e_g, V^{i,s}, V^{j,r}, V^{j,s}, V^{k,r}, V^{k,s}, V^{h,r}, V^{h,s}\right),\\
W_8 &= \left(e_g, V^{j,r}, V^{j,s}, V^{k,r}, V^{k,s}, V^{h,r}, V^{h,s}, V^{f,r}\right),\\
W_9 &= \left(e_g, V^{j,s}, V^{k,r}, V^{k,s}, V^{h,r}, V^{h,s}, V^{f,r}, V^{f,s}\right),\\
W_{10} &= \left(e_g, V^{k,r}, V^{k,s}, V^{h,r}, V^{h,s}, V^{f,r}, V^{f,s}, V^{g,r}\right),\\
W_{11} &= \left(e_f, V^{k,s}, V^{h,r}, V^{h,s}, V^{f,r}, V^{f,s}, V^{g,r}, V^{h,s}\right).
\end{aligned} \tag{2.9}$$

As mentioned in [1], we have made six MATLAB functions: Cha3(A), Cha4(A), Cha5(A), Cha6(A), Cha7(A) and Cha(n, A), for determining the copositivity of symmetric matrices. Now we have made two more MATLAB functions of these type: Cha8(A) and Cha9(A) based on the two algorithms given by Theorems 2.1 and 2.2. The input of the functions is any 8×8 or 9×9 symmetric matrix A and there are four possible return values: $y = 0, 1, 2, 3$ meaning "not copositive", "copositive (not strictly)" and "strictly copositive", "cannot determined," respectively.

Main steps of Function $y = Cha8(A)$

(1) Find out if A has any 7×7 principal submatrix which is not copositive. If so, then return with "$y = 0$" (Theorem 2.1(a)). Otherwise go to next step.

(2) Calculate the number m of the negative entries of the first row of A.

When $m = 0$ $(\alpha \geq 0)$ use Theorem 2.1(b) to determine copositivity of A and return.

When $m = 7$ $(\alpha \leq 0)$ use Theorem 2.1(c) to determine copositivity of A and return.

When $m = 1$ use Theorem 2.1(d) to determine copositivity of A and return.

When $m = 2$ use Theorem 2.1(e) to determine copositivity of A and return.

When $m = 3$ use Theorem 2.1(f) to determine copositivity of A and return.

When $m = 4$ use Theorem 2.1(g) to determine copositivity of A and return.

Table 1

n	#strico	%	#notstri	%	#noncopo	%	#undeter	%
8	32	3.2	2	0.2	966	96.6	0	0
9	5	0.5	0	0	995	99.5	0	0
10	0	0	0	0	1000	100	0	0

When $m = 5$ use Theorem 2.1(h) to determine copositivity of A and return.

When $m = 6$ use Theorem 2.1(i) to determine copositivity of A and return.

Main steps of Function $y = Cha9(A)$

(1) Find out if A has any 8×8 principal submatrix which is not copositive. If so, then return with "$y = 0$" (Theorem 2.2(a)). Otherwise A must have some row containing exactly m ($m = 0, 2, 4, 6, 8$) negative entries and go to the next step.

(2) Find out if A has any row which has exactly $m(m = 0, 2, 4, 6, 8)$ negative entries. If the ith row does, then interchange the ith row and column of A with the first row and column.

When $m = 0$ use Theorem 2.2(b) to determine copositivity of A and return.

When $m = 8$ use Theorem 2.2(c) to determine copositivity of A and return.

When $m = 2$ use Theorem 2.2(d) to determine copositivity of A and return.

When $m = 4$ use Theorem 2.2(e) to determine copositivity of A and return.

When $m = 6$ use Theorem 2.2(f) to determine copositivity of A and return.

3. Numerical Experiments and Discussion

Having all these eight functions: Cha3(A), Cha4$(A), \ldots,$ Cha8(A), Cha9(A), and Cha(n, A) we have performed the following experiments. Firstly, we use these functions to determine the copositivity of the $n \times n$ symmetric matrix M studied in [3], where $M = (m_{ij})$ satisfies $|m_{ij}| = 1$; $m_{ij} = -1$ only if $|i - j| = 1$ and $|i - j| = n - 1$. When $n = 9$, the experimental results obtained by old Cha(n, A) together with Cha3(A), Cha4$(A), \ldots,$ Cha7(A) are "$y = 3$" meaning "cannot be determined" and the experimental results by Cha9(A) are "$y = 1$" meaning "copositive but not strictly", which are the same results as obtained in [3]. Secondly we generate 1000 symmetric random matrices of order $n \in \{8, 9, 10\}$ with unit diagonal and with positive entries all being less than or equal to 1 and negative entries all being greater than or equal to -1, and then use our MATLAB functions to determine the copositivity of these matrices. The main numerical result of the experiments is given in Table 1, where #strico, #notstri, #noncopo, #undeter denote the number of strictly copositive matrices, the number of copositive (but not strictly) matrices, the number of noncopositive matrices, and the number of the remained matrices whose copositivity could not be determined by our algorithms, respectively.

Kaplan [4, Theorem 3.1] proved that a symmetric matrix A is copositive if and only if the minimum principal submatrix A_1 of A which shares the maximum positive diagonal entries with A is copositive and the matrix which is constructed from A by replacing each entry of A_1 by 0 is nonnegative. To answer the third open problem of [4, 5], we proved that a symmetric matrix A with unit diagonal is copositive if and only if the matrix constructed from A by replacing each off-diagonal entry a_{ij} by $\min\{a_{ij}, 1\}$ is copositive. These two results make it reasonable that for determining copositivity we can restrict our attention only to symmetric matrices with unit diagonal and with positive entries all being less than or equal to 1, and our experimental matrices are all of this type. Furthermore, each of the test matrices is required that every of its principal 2×2 submatrix is copositive (Note that for a matrix with $n > 9$ the chance that every principal 2×2 submatrix is copositive is much less). In addition, the last line of Table 1 also holds for $n \geq 10$ because of the fact that a symmetric matrix is not copositive if any of its principal submatrix is not copositive. Table 1 does give us some noticeable information as follows.

Remark 3.1. For $n \geq 10$ almost always no random matrix is copositive, in other words, there is almost always no matrix remaining undetermined by our algorithms including the new ones developed in this paper. Therefore, the algorithms for $n = 10, 11$ and so forth. which might be established by our method are not practically needed.

We surely believe that algorithm for $n \geq 10$ will be tedious to describe and take more time to run because of its recurrent property.

Since there is almost always no symmetric copositive matrix of order larger than 9, the interest of researchers may concentrate on sufficient conditions for copositive matrices of larger orders, or of general order n. For instance, [3] proved the matrix M mentioned at the beginning of this section is copositive (but not strictly) for any n. Here we give another interesting example as follows.

Proposition 3.2. *Let $A = (a_{ij})$ be a symmetric matrix of any order, n; $r(i)$ $(i = 1, \ldots, n)$ be the sum of all the negative entries of the ith row of A. Then A is copositive if $(a_{11}, \ldots, a_{nn}) \geq (-r(1), \ldots, -r(n))$; A is strictly copositive if $(a_{11}, \ldots, a_{nn}) > (-r(1), \ldots, -r(n))$; (A_-) is irreducible and $(a_{11}, \ldots, a_{nn}) \geq (\neq)(-r(1), \ldots, -r(n))$.*

Proof. Write $A = \mathrm{diag}(A) - A_- + A_+$, where $\mathrm{diag}(A) = \mathrm{diag}(a_{11}, \ldots, a_{nn})$ and $A_+(A_-)$ is the $n \times n$ nonnegative matrix which shares all the negative (nonnegative) entries with A and has the remained entries all being zero. Then $\mathrm{diag}(A) - A_- = mI - P$, where $m = \max\{a_{11}, \ldots, a_{nn}\}$ and P is a nonnegative matrix whose spectral radius $\rho(P) \leq m(< m)$ if $(a_{11}, \ldots, a_{nn}) \geq (>)(-r(1), \ldots, -r(n))$. Therefore, $\mathrm{diag}(A)-A_-$ is an M-matrix if $(a_{11}, \ldots, a_{nn}) \geq (-r(1), \ldots, -r(n))$; a nonsingular M-matrix if $(a_{11}, \ldots, a_{nn}) > (-r(1), \ldots, -r(n))$ or (A_-) is irreducible and $(a_{11}, \ldots, a_{nn}) \geq (\neq)(-r(1), \ldots, -r(n))$, whence it is copositive, strictly copositive, respectively by [1, Theorem 2.1]. Finally A (as the sum of two copositive matrices) is copositive. □

Acknowledgments

This work was supported by the NNSF China no. 10871230, NSF Zhejiang no. y607480, and Innovation Group Foundation of Anhui University

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