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Research Article

Ordering Unicyclic Graphs in Terms of Their Smaller Least Eigenvalues

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Let G be a simple graph with n vertices, and let $\lambda_n(G)$ be the least eigenvalue of G. The connected graphs in which the number of edges equals the number of vertices are called unicyclic graphs. In this paper, the first five unicyclic graphs on order n in terms of their smaller least eigenvalues are determined.

1. Introduction

Let G be a simple graph with n vertices, and let A be the $(0,1)$-adjacency matrix of G. We call $\det(\lambda I - A)$ the characteristic polynomial of G, denoted by $P(G;\lambda)$, or abbreviated $P(G)$. Since A is symmetric, its eigenvalues $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$ are real, and we assume that $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. We call $\lambda_n(G)$ the least eigenvalue of G. Up to now, some good results on the least eigenvalues of simple graphs have been obtained.

(1) In [1], let G be a simple graph with n vertices, $G \neq K_n$, then

$$\lambda_n(G) \leq \lambda_n\left(K_{n-1}^1\right). \tag{1.1}$$

The equality holds if and only if $G \cong K_{n-1}^1$, where K_{n-1}^1 is the graph obtained from K_{n-1} by joining a vertex of K_{n-1} with K_1.

(2) In [2–4], let G be a simple graph with n vertices, then

$$\lambda_n(G) \geq -\sqrt{\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]}. \tag{1.2}$$

The equality holds if and only if $G \cong K_{[n/2],[(n+1)/2]}$.

(3) In [5], let G be a planar graph with $n \geq 3$ vertices, then

$$\lambda_n(G) \geq -\sqrt{2n-4}. \tag{1.3}$$

The equality holds if and only if $G \cong K_{2,n-2}$.

(4) In [6], the author surveyed the main results of the theory of graphs with least eigenvalue -2 starting from late 1950s.

Connected graphs in which the number of edges equals the number of vertices are called unicyclic graphs. Also, the least eigenvalues of unicyclic graphs have been studied in the past years. We now give some related works on it.

(1) In [7], let $\mathcal{U}_n$ denote the set of unicyclic graphs on order n. The authors characterized the unique graph with minimum least eigenvalue (also in [8, 9]) (resp., the unique graph with maximum spread) among all graphs in $\mathcal{U}_n$.

(2) In [10], let G be a unicyclic graph with n vertices, and let G^* be the graph obtained by joining each vertex of C_3 to a pendant vertex of $P_{k-1}, P_{k_1-1}, P_{k_2-1}$, respectively, where $k \geq k_1 \geq k_2 \geq 1$, $k - k_2 \leq 1$, and $k + k_1 + k_2 = n$. Then

$$\lambda_n(G) \leq \lambda_n(G^*). \tag{1.4}$$

The equality holds if and only if $G \cong G^*$.

In this paper, the first five unicyclic graphs on order n in terms of their smaller least eigenvalues are determined. The terminologies not defined here can be found in [11, 12].

2. Some Known Results on the Spectral Radii of Graphs

In this section, we will give some known results on the spectral radius of a forestry or an unicyclic graph. They will be useful in the proofs of the following results.

Firstly, we write $S(r, n-r-2)$ $(1 \leq r \leq n-r-2)$ denotes the tree of order n obtained from the star $K_{1,n-r-1}$ by joining a pendant vertex of $K_{1,n-r-1}$ with rK_1.

Lemma 2.1 (see [13]). *Let F be a forestry with n vertices. $F \neq K_{1,n-1}$, $S(1, n-3)$, $S(2, n-4)$. Then*

$$\lambda_1(F) < \lambda_1(S(2, n-4)) < \lambda_1(S(1, n-3)) < \lambda_1(K_{1,n-1}). \tag{2.1}$$

Now, we consider unicyclic graphs. For convenience, we write

$$\mathcal{U}_n = \{G \mid G \text{ is an unicyclic graph with } n \text{ vertices}\},$$
$$\mathcal{U}_n(k) = \{G \mid G \text{ is an unicyclic graph in } \mathcal{U}_n \text{ containing a circuit } C_k\}. \tag{2.2}$$

Also, we write C_k^{n-k} denotes the unicyclic graph obtained from C_k by joining a vertex of C_k with $(n-k)K_1$, and $C_k(n-k-1, 1)$ denotes the unicyclic graph obtained from C_k by joining two adjacent vertices of C_k with $(n-k-1)K_1$ and K_1, respectively. Then we have the following.

Lemma 2.2 (see [14]). $\lambda_1(C_k^{n-k}) > \lambda_1(C_{k+1}^{n-k-1})$, $3 \leq k \leq n-1$.

Lemma 2.3 (see [15]). *Let* $G \in \mathcal{U}_n(k), G \neq C_k^{n-k}, C_k(n-k-1,1)$, *then*

$$\lambda_1(G) < \lambda_1(C_k(n-k-1,1)) < \lambda_1\left(C_k^{n-k}\right). \tag{2.3}$$

Lemma 2.4 (see [15]). *For* $n \geq 8$, *one has*

$$\lambda_1(C_4(n-5,1)) > \lambda_1\left(C_5^{n-5}\right). \tag{2.4}$$

3. The Least Eigenvalues of Unicyclic Graphs

Firstly, we give the following definitions of the order "$\preceq$ (or $\prec$)" between two graphs or two sets of graphs.

Definition 3.1. Let G, H be two simple graphs on order n, and let $\mathcal{G}$, $\mathcal{H}$ be two sets of simple graphs on order n.

(1) We say that "G is *majorized* (or strictly majorized) by H," denoted by $G \preceq H$ (or $G \prec H$) if $\lambda_n(G) \leq \lambda_n(H)$ (or $\lambda_n(G) < \lambda_n(H)$).

(2) We say that "$\mathcal{G}$ is *majorized* (or strictly majorized) by $\mathcal{H}$", denoted by $\mathcal{G} \preceq \mathcal{H}$ (or $\mathcal{G} \prec \mathcal{H}$) if $\lambda_n(G) \leq \lambda_n(H)$ (or $\lambda_n(G) < \lambda_n(H)$) for each $G \in \mathcal{G}$ and $H \in \mathcal{H}$.

The following lemmas will be useful in the proofs of the main results.

Lemma 3.2 (see [16]). *Let G be a simple graph with vertex set* $V(G)$ *and* $u \in V(G)$, *then*

$$P(G) = \lambda P(G-u) - \sum_{v} P(G-u-v) - 2\sum_{Z \in C(u)} P(G-V(Z)), \tag{3.1}$$

where the first summation goes through all vertices v *adjacent to* u, *and the second summation goes through all circuits* Z *belonging to* $C(u)$, $C(u)$ *denotes the set of all circuits containing the vertex* u.

Lemma 3.3 (see [12]). *Let* V_1 *be a subset of vertices of a graph G and* $|V(G)| = n, |V_1| = k$, *then*

$$\lambda_i(G) \geq \lambda_i(G-V_1) \geq \lambda_{i+k}(G), \quad (1 \leq i \leq n-k). \tag{3.2}$$

Lemma 3.4 (see [2]). *Let G be a bipartite graph with* n *vertices, then*

$$\lambda_i(G) = -\lambda_{n-i+1}(G), \quad \left(1 \leq i \leq \left[\frac{n}{2}\right]\right). \tag{3.3}$$

Lemma 3.5 (see [3]). *Let G be a simple graph with* n *vertices. Then there exist a spanning subgraph* G' *of G such that* G' *is a bipartite graph and* $\lambda_n(G) \geq \lambda_n(G')$.

Now, we consider the least eigenvalues of unicyclic graphs. For the graphs in $\mathcal{U}_n(3)$, we have the following results.

Lemma 3.6. *$K_{1,n-1} \prec C_3^{n-3} \prec S(1, n-3)$, $(n \geq 6)$.*

Proof. By Lemma 3.2, we have

$$P\left(C_3^{n-3}\right) = \lambda^{n-4}\left[\lambda^4 - n\lambda^2 - 2\lambda + (n-3)\right], \tag{3.4}$$

and by Lemma 3.5, there exist a spanning subgraph G' of C_3^{n-3} such that G' is a bipartite graph and $\lambda_n(C_3^{n-3}) \geq \lambda_n(G')$. Obviously, G' is a forestry. So, by Lemma 2.1, we have $\lambda_n(G') \geq \lambda_n(K_{1,n-1})$. But $P(C_3^{n-3}; \lambda_n(K_{1,n-1})) \neq 0$. Thus, $\lambda_n(K_{1,n-1}) < \lambda_n(C_3^{n-3})$.

Also, by Lemma 3.2, we have

$$P(S(1, n-3)) = \lambda^{n-4}\left[\lambda^4 - (n-1)\lambda^2 + (n-3)\right]. \tag{3.5}$$

Then $P(C_3^{n-3}) - P(S(1, n-3)) = -\lambda^{n-3}(\lambda+2)$. From the table of connected graphs on six vertices in [17], we know that

$$\lambda_6(S(1,3)) < -2. \tag{3.6}$$

So, by Lemma 3.3, we have

$$\lambda_n(S(1, n-3)) \leq \lambda_6(S(1,3)) < -2. \tag{3.7}$$

Thus, $P(C_3^{n-3}; \lambda_n(S(1, n-3))) = (-1)^{n-1} q_n$, where $q_n > 0$. Also, by Lemma 3.3, we have

$$\lambda_{n-1}\left(C_3^{n-3}\right) \geq \lambda_{n-1}(K_{1,n-2}) \geq \lambda_n(S(1, n-3)). \tag{3.8}$$

So, $\lambda_n(C_3^{n-3}) < \lambda_n(S(1, n-3))$. Hence the result holds. □

Lemma 3.7. *For $n \geq 9$, one has*

$$S(1, n-3) \prec C_3^{n-4}(1) \prec C_3(n-4, 1) \prec S(2, n-4), \tag{3.9}$$

where $C_3^{n-4}(1)$ denotes the graph obtained from C_3^{n-4} by joining a pendant vertex of C_3^{n-4} with K_1.

Proof. By Lemma 3.2, we have

$$\begin{aligned} P\left(C_3^{n-4}(1)\right) &= \lambda^{n-6}\left(\lambda^2 - 1\right)\left[\lambda^4 - (n-1)\lambda^2 - 2\lambda + (n-5)\right], \\ P(S(1, n-3)) &= \lambda^{n-4}\left[\lambda^4 - (n-1)\lambda^2 + (n-3)\right]. \end{aligned} \tag{3.10}$$

And by Lemma 3.5, there exist a spanning subgraph G' of $C_3^{n-4}(1)$ such that G' is a bipartite graph and $\lambda_n(C_3^{n-4}(1)) \geq \lambda_n(G')$. Obviously, G' is a forestry and $G' \neq K_{1,n-1}$ for $n \geq 5$. So, by Lemma 3.6, we have $\lambda_n(G') \geq \lambda_n(S(1, n-3))$. But $P(C_3^{n-4}(1); \lambda_n(S(1, n-3))) \neq 0$. Thus

$$S(1, n-3) \prec C_3^{n-4}(1). \tag{3.11}$$

Also, by Lemma 3.2, we have

$$P(C_3(n-4, 1)) = \lambda^{n-4}\left[\lambda^4 - n\lambda^2 - 2\lambda + (2n-7)\right], \tag{3.12}$$

So

$$P\left(C_3^{n-4}(1)\right) - P(C_3(n-4, 1)) = \lambda^{n-6}\left[\lambda^2 + 2\lambda - (n-5)\right]. \tag{3.13}$$

The least root of $\lambda^2 + 2\lambda - (n-5) = 0$ is $-1 - \sqrt{n-4}$. Let $f_n(\lambda) = \lambda^4 - n\lambda^2 - 2\lambda + (2n-7)$, then we have

$$f_n\left(-1 - \sqrt{n-4}\right) = 9n - 12 + 2(n-5)\sqrt{n-4} > 0, \quad (n \geq 5). \tag{3.14}$$

Moreover, by Lemma 3.3, we know

$$\lambda_{n-1}(C_3(n-4, 1)) \geq \lambda_{n-1}(K_{1,n-3} \cup K_1) = -\sqrt{n-3} > -1 - \sqrt{n-4}, \quad (n \geq 5). \tag{3.15}$$

So,

$$\lambda_n(C_3(n-4, 1)) > -1 - \sqrt{n-4}. \tag{3.16}$$

Thus,

$$P\left(C_3^{n-4}(1); \lambda_n(C_3(n-4, 1))\right) = (-1)^{n+1} q_n, \quad q_n > 0. \tag{3.17}$$

Then, $C_3^{n-4}(1) \prec C_3(n-4, 1)$.

By Lemma 3.2, we have

$$P(S(2, n-4)) = \lambda^{n-4}\left[\lambda^4 - (n-1)\lambda^2 + 2(n-4)\right], \tag{3.18}$$

and

$$\lambda_n(S(2, n-4)) = -\left[\frac{1}{2}\left(n - 1 + \sqrt{(n-5)^2 + 8}\right)\right]^{1/2}. \tag{3.19}$$

So,

$$P(C_3(n-4,1)) - P(S(2,n-4)) = -\lambda^{n-4}\left(\lambda^2 + 2\lambda - 1\right). \tag{3.20}$$

Thus, when $n \geq 9$, it is not difficult to know that

$$P(C_3(n-4,1); \lambda_n(S(2,n-4))) = (-1)^{n+1} q_n, \quad q_n > 0, \tag{3.21}$$

then $C_3(n-4,1) \prec S(2,n-4)$. □

Lemma 3.8. *Let $G \in \mathcal{U}_n(3)$, $G \neq C_3^{n-3}$, $C_3^{n-4}(1)$, $C_3(n-4,1)$, then, for $n \geq 6$, one has*

$$S(2,n-4) \preceq G. \tag{3.22}$$

Proof. Let $G \neq C_3^{n-3}, C_3^{n-4}(1), C_3(n-4,1)$. Then, by Lemma 3.5, there exist a spanning subgraph G' such that G' is a bipartite graph and $\lambda_n(G) \geq \lambda_n(G')$. Obviously, G' is a forestry and $G' \neq K_{1,n-1}, S(1,n-3)$ for $n \geq 6$. So, by Lemma 2.1, we have

$$\lambda_n(G') \geq \lambda_n(S(2,n-4)), \quad (n \geq 6). \tag{3.23}$$

Thus

$$S(2,n-4) \preceq G, \quad (n \geq 6). \tag{3.24}$$

□

Now, we consider the graphs in $\mathcal{U}_n(4)$, we have the following results.

Lemma 3.9. $K_{1,n-1} \prec C_4^{n-4} \prec S(1,n-3)$, $(n \geq 4)$.

Proof. By Lemma 3.2, we have

$$\begin{aligned} P(S(1,n-3)) &= \lambda^{n-4}\left[\lambda^4 - (n-1)\lambda^2 + (n-3)\right], \\ P\left(C_4^{n-4}\right) &= \lambda^{n-4}\left[\lambda^4 - n\lambda^2 + 2(n-4)\right]. \end{aligned} \tag{3.25}$$

We can easily to know that

$$\begin{aligned} \lambda_n\left(C_4^{n-4}\right) &= -\left[\frac{1}{2}\left(n + \sqrt{(n-4)^2 + 16}\right)\right]^{1/2}, \\ \lambda_n(S(1,n-3)) &= -\left[\frac{1}{2}\left(n - 1 + \sqrt{(n-3)^2 + 4}\right)\right]^{1/2}. \end{aligned} \tag{3.26}$$

Moreover, $\lambda_n(K_{1,n-1}) = -\sqrt{n-1}$. So,

$$\lambda_n(K_{1,n-1}) < \lambda_n\left(C_4^{n-4}\right) < \lambda_n(S(1,n-3)), \quad (n \geq 4). \tag{3.27}$$

And then,

$$K_{1,n-1} \prec C_4^{n-4} \prec S(1,n-3), \quad (n \geq 4). \tag{3.28}$$

□

Lemma 3.10. *For $n \geq 9$, one has*

$$S(1,n-3) \prec C_4(n-5,1) \prec S(2,n-4). \tag{3.29}$$

Proof. By Lemma 3.2, we get

$$\begin{aligned} P(C_4(n-5,1)) &= \lambda^{n-6}\left[\lambda^6 - n\lambda^4 + (3n-13)\lambda^2 - (n-5)\right], \\ P(S(1,n-3)) &= \lambda^{n-6}\left[\lambda^6 - (n-1)\lambda^4 + (n-3)\lambda^2\right]. \end{aligned} \tag{3.30}$$

So,

$$P(C_4(n-5,1)) - P(S(1,n-3)) = \lambda^{n-6}\left[-\lambda^4 + 2(n-5)\lambda^2 - (n-5)\right]. \tag{3.31}$$

Since

$$\lambda_n(S(1,n-3)) = -\left[\frac{1}{2}\left(n-1+\sqrt{(n-3)^2+4}\right)\right]^{1/2}. \tag{3.32}$$

So,

$$P(C_4(n-5,1);\lambda_n(S(1,n-3))) = \frac{1}{2}[\lambda_n(S(1,n-3))]^{n-6}\left[(n-5)^2 + (n-9)\sqrt{(n-3)^2+4} - 12\right]. \tag{3.33}$$

It is not difficult to know that $(n-5)^2 + (n-9)\sqrt{(n-3)^2+4} - 12 > 0$ for $n \geq 9$. Thus,

$$P(C_4(n-5,1);\lambda_n(S(1,n-3))) = (-1)^n q_n, \quad q_n > 0. \tag{3.34}$$

Furthermore, by Lemma 3.3, we have

$$\lambda_n(S(1,n-3)) \leq \lambda_{n-1}(S(1,n-4)) \leq \lambda_{n-1}(C_4(n-5,1)). \tag{3.35}$$

So $\lambda_n(C_4(n-5,1)) > \lambda_n(S(1,n-3))$ for $n \geq 9$. It means that $S(1,n-3) \prec C_4(n-5,1)$ for $n \geq 9$.

Since $S(2,n-4)$ is a spanning subgraph of $C_4(n-5,1)$. So $\lambda_n(C_4(n-5,1)) < \lambda_n(S(2,n-4))$ for $n \geq 6$. It means that $C_4(n-5,1) \prec S(2,n-4)$ for $n \geq 6$. □

Lemma 3.11. *Let $G \in \mathcal{U}_n(k), k \geq 4, G \neq C_4^{n-4}, C_4(n-5,1)$. Then $C_4(n-5,1) \prec G$.*

Proof. When $G \in \mathcal{U}_n(4)$, $G \neq C_4^{n-4}$, $C_4(n-5,1)$, by Lemma 2.3, we have

$$\lambda_1(G) < \lambda_1(C_4(n-5,1)). \tag{3.36}$$

Then, by Lemma 2.2, we have

$$\lambda_n(G) > \lambda_n(C_4(n-5,1)). \tag{3.37}$$

When $G \in \mathcal{U}_n(k), k \geq 5$, by Lemmas 2.2 and 2.4, we have

$$\lambda_1(G) \leq \lambda_1\left(C_5^{n-5}\right) < \lambda_1(C_4(n-5,1)). \tag{3.38}$$

So,

$$\lambda_n(G) \geq -\lambda_1(G) > \lambda_n(C_4(n-5,1)). \tag{3.39}$$

Thus the result holds. □

Lemma 3.12. *$C_4^{n-4} \prec C_3^{n-3}$ for $4 \leq n \leq 11$ and $C_3^{n-3} \prec C_4^{n-4}$ for $n > 11$.*

Proof. By the proof of Lemma 3.6, we have

$$\begin{aligned} P\left(C_3^{n-3}\right) &= \lambda^{n-4}\left[\lambda^4 - n\lambda^2 - 2\lambda + (n-3)\right], \\ \lambda_n\left(C_4^{n-4}\right) &= -\left[\frac{1}{2}\left(n + \sqrt{(n-4)^2+16}\right)\right]^{1/2}. \end{aligned} \tag{3.40}$$

So

$$P\left(C_3^{n-3}; \lambda_n\left(C_4^{n-4}\right)\right) = \left[\lambda_n\left(C_4^{n-4}\right)\right]^{n-4}\left\{-n+5+\left[2\left(n+\sqrt{(n-4)^2+16}\right)\right]^{1/2}\right\}. \tag{3.41}$$

Let $f_n = -n+5+[2(n+\sqrt{(n-4)^2+16})]^{1/2}$. It is not difficult to know that $f_n > 0$ for $4 \leq n \leq 11$ and $f_n < 0$ for $n > 11$. Furthermore, by Lemma 3.3, we have

$$\lambda_n\left(C_4^{n-4}\right) \leq \lambda_{n-1}(K_{1,n-2}) \leq \lambda_{n-1}\left(C_3^{n-3}\right). \tag{3.42}$$

So, by the sign of $P(C_3^{n-3}; \lambda_n(C_4^{n-4}))$, we know that $\lambda_n(C_3^{n-3}) > \lambda_n(C_4^{n-4})$ for $4 \leq n \leq 11$ and $\lambda_n(C_3^{n-3}) < \lambda_n(C_4^{n-4})$ for $n > 11$. Thus the result holds. □

Lemma 3.13. *$C_3(n-4,1) \prec C_4(n-5,1)$ for $n \geq 6$.*

Proof. By the proofs of Lemmas 3.7 and 3.10, we have

$$P(C_3(n-4,1)) = \lambda^{n-4}\left[\lambda^4 - n\lambda^2 - 2\lambda + (2n-7)\right],$$
$$P(C_4(n-5,1)) = \lambda^{n-6}\left[\lambda^6 - n\lambda^4 + (3n-13)\lambda^2 - (n-5)\right], \quad (3.43)$$

so

$$P(C_3(n-4,1)) - P(C_4(n-5,1)) = -\lambda^{n-6}\left[2\lambda^3 + (n-6)\lambda^2 - (n-5)\right], \quad (3.44)$$

since

$$\lambda_n(C_4(n-5,1)) > \lambda_n(K_{1,n-1}) = -\sqrt{n-1} \quad (n \geq 8). \quad (3.45)$$

And by Lemma 3.3, we know that

$$\lambda_n(C_4(n-5,1)) \leq \lambda_{n-2}(K_{1,n-3}) = -\sqrt{n-3}. \quad (3.46)$$

Now, let

$$f_n(\lambda) = 2\lambda^3 + (n-6)\lambda^2 - (n-5) = \lambda^2[2\lambda + (n-6)] - (n-5), \quad (3.47)$$

then

$$f_n(\lambda_n(C_4(n-5,1))) > (n-3)\left[-2\sqrt{n-1} + (n-6)\right] - (n-5). \quad (3.48)$$

It is easy to know that $f_n(\lambda_n(C_4(n-5,1))) > 0$ for $n \geq 15$. Thus,

$$P(C_3(n-4,1); \lambda_n(C_4(n-5,1))) = (-1)^{n+1} q_n, \quad q_n > 0. \quad (3.49)$$

Hence $C_3(n-4,1) \prec C_4(n-5,1)$ for $n \geq 15$.

When $6 \leq n \leq 14$, by immediate calculation, we know the result holds too. This completes the proof. □

4. Main Results

Now, we give the main result of this paper.

Theorem 4.1. *Let* $G \in \mathcal{U}_n$, $G \neq C_3^{n-3}, C_4^{n-4}, C_3^{n-4}(1), C_3(n-4,1), C_4(n-5,1)$, *then*

(1) $C_4^{n-4} \prec C_3^{n-3} \prec C_3^{n-4}(1) \prec C_3(n-4,1) \prec C_4(n-5,1) \prec G$ *for* $9 \le n \le 11$;

(2) $C_3^{n-3} \prec C_4^{n-4} \prec C_3^{n-4}(1) \prec C_3(n-4,1) \prec C_4(n-5,1) \prec G$ *for* $n > 11$.

Proof. By the Lemmas 3.6–3.13, we know that the result holds. □

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