Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 635720, 15 pages
doi:10.1155/2010/635720

Research Article

Stability of Quadratic Functional Equations via the Fixed Point and Direct Method

Eunyoung Son, Juri Lee, and Hark-Mahn Kim

Department of Mathematics, Chungnam National University, Daejeon 305-764, South Korea

Correspondence should be addressed to Hark-Mahn Kim, hmkim@math.cnu.ac.kr

Received 13 October 2009; Accepted 19 January 2010

Academic Editor: Yeol Je Cho

Cădariu and Radu applied the fixed point theorem to prove the stability theorem of Cauchy and Jensen functional equations. In this paper, we prove the generalized Hyers-Ulam stability via the fixed point method and investigate new theorems via direct method concerning the stability of a general quadratic functional equation.

1. Introduction

In 1940, Ulam [1] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Let G be a group and let G' be a metric group with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

The concept of stability for functional equations arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus we say that a functional equation $E_1(f) = E_2(f)$ is stable if any mapping g approximately satisfying the equation $d(E_1(g), E_2(g)) \le \varphi(x)$ is near to a true solution f such that $E_1(f) = E_2(f)$ and $d(f(x), g(x)) \le \Phi(x)$ for some function Φ depending on the given function φ. In 1941, the first result concerning the stability of functional equations for the case where G_1 and G_2 are Banach spaces was presented by Hyers [2]. In fact, he proved that each solution f of the inequality $\|f(x+y) - f(x) - f(y)\| \le \epsilon$ for all $x, y \in G_1$ can be approximated by a unique additive function $L : G_1 \to G_2$ defined by $L(x) = \lim_{n\to\infty}(f(2^n x)/2^n)$ such that $\|f(x) - L(x)\| \le \epsilon$ for every $x \in G_1$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in G_1$, then the function L is linear. And then Aoki [3], Bourgin [4], and Forti [5] have investigated the stability theorems of functional equations which generalize the Hyers' result. In 1978, Rassias [6] attempted to weaken the condition for the bound of Cauchy difference controlled by a sum of unbounded function $\varepsilon(\|x\|^p + \|y\|^p), 0 < p < 1$, and provided a generalization of Hyers' theorem. In 1991, Gajda [7] gave an affirmative solution to this question for $p > 1$ by following the same approach as in [6]. Rassias [8] established a similar stability theorem for the unbounded Cauchy difference controlled by a product of unbounded function $\varepsilon(\|x\|^p \cdot \|y\|^q), p+q \neq 1$. Găvruţa [9] provided a further generalization of Rassias' theorem by replacing the bound of Cauchy difference by a general control function. During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [10–15]).

Let E_1 and E_2 be real vector spaces. A function $f : E_1 \to E_2$ is called a quadratic function if and only if f is a solution function of the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

for all $x, y \in E_1$. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x,x)$ for all x, where the mapping B is given by $B(x,y) = (1/4)(f(x+y) - f(x-y))$. See [16, 17] for the details.

The Hyers-Ulam stability of the quadratic functional equation (1.1) was first proved by Skof [18] for functions $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa noticed that Skof's theorem is also valid if E_1 is replaced by an Abelian group. Czerwik [19] proved the generalized Hyers-Ulam stability of quadratic functional equation (1.1) in the spirit of Rassias approach. On the other hand, according to the theorem of Borelli and Forti [20], we know the following generalization of stability theorem for quadratic functional equation. Let G be a 2-divisible Abelian group and E a Banach space, and let $f : G \to E$ be a mapping with $f(0) = 0$ satisfying the inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \varphi(x,y) \tag{1.2}$$

for all $x, y \in G$. Assume that one of the series

$$\Phi(x,y) := \begin{cases} \displaystyle\sum_{k=0}^{\infty} \frac{1}{2^{2(k+1)}} \varphi(2^k x, 2^k y) < \infty, \\ \displaystyle\sum_{k=0}^{\infty} 2^{2k} \varphi\left(\frac{x}{2^{(k+1)}}, \frac{y}{2^{(k+1)}}\right) < \infty \end{cases} \tag{1.3}$$

holds for all $x, y \in G$, then there exists a unique quadratic function $Q : G \to E$ such that

$$\|f(x) - Q(x)\| \le \Phi(x,x) \tag{1.4}$$

for all $x \in G$. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem [21–27].

In 1996, Isac and Rassias [28] applied the stability theory of functional equations to prove fixed point theorems and study some new applications in nonlinear analysis.Radu [29], Cãdariu and Radu [30, 31] applied the fixed point theorem of alternative to the investigation of Cauchy and Jensen functional equations. Recently, Jung et al. [32], Jung [33, 34], Jung and Lee [35], Jung and Min [36], Jung and Rassias [37] have obtained the generalized Hyers-Ulam stability of functional equations via the fixed point method.

Now, we see that the norm defined by a real inner product space satisfies the following equality:

$$2\left\|\sum_{i=1}^{n}x_i\right\|^2+\sum_{i\neq j}\left\|x_i-x_j\right\|^2=2n\sum_{i=1}^{n}\left\|x_i\right\|^2 \tag{1.5}$$

for all vectors $x_1,\ldots,x_n$. Thus employing the last equality, we introduce to consider the following functional equation

$$2f\left(\sum_{i=1}^{n}x_i\right)+\sum_{i\neq j}f\left(x_i-x_j\right)=2n\sum_{i=1}^{n}f\left(x_i\right) \tag{1.6}$$

with several variables for any fixed $n\in\mathbb{N}$ with $n\geq 2$. It is obvious that if $n=2$ in (1.6), then the solution function is even and thus it reduces to (1.1). Conversely, we observe that the general solution of (1.6) in the class of all functions between vector spaces is exactly a quadratic function. In this paper, we are going to investigate the general solution of (1.6) and then we are to prove the generalized Hyers-Ulam stability of (1.6) for a large class of functions from vector spaces into complete β-normed spaces by using fixed point method, and direct method.

2. Stability of (1.6) by Fixed Point Method

For notational convenience, given a mapping $f:X\to Y$, we define the difference operator $Df:X^n\to Y$ of (1.6) by

$$Df(x_1,\ldots,x_n):=2f\left(\sum_{i=1}^{n}x_i\right)+\sum_{i\neq j}f\left(x_i-x_j\right)-2n\sum_{i=1}^{n}f\left(x_i\right),\quad n\geq 2 \tag{2.1}$$

for all $x_1,\ldots,x_n\in X$, which is called the approximate remainder of the functional equation (1.6) and acts as a perturbation of the equation.

We now introduce a fundamental result of fixed point theory. We refer to [38] for the proof of it. For an extensive theory of fixed point theorems and other nonlinear methods, the reader is referred to the book of Hyers et al. [39].

Theorem 2.1. *Let (Ω,d) be a generalized complete metric space (i.e., d may assume infinite values). Assume that $\Lambda:\Omega\to\Omega$ is a strictly contractive operator, that is, there exists a Lipschitz constant L with $0<L<1$ such that $d(\Lambda(x),\Lambda(y))\leq Ld(x,y)$ for all $x,y\in\Omega$. Then for a given element $x\in\Omega$*

one of the following assertions is true:

(A_1) $d(\Lambda^{k+1}x, \Lambda^k x) = \infty$ *for all* $k \geq 0$;

(A_2) *there exists a nonnegative integer* n_0 *such that*

$(A_{2.1})$ $d(\Lambda^{n+1}x, \Lambda^n x) < \infty$ *for all* $n \geq n_0$;

$(A_{2.2})$ *the sequence* $\{\Lambda^n x\}$ *converges to a fixed point* x^* *of* Λ;

$(A_{2.3})$ x^* *is the unique fixed point of* Λ *in* $\Delta = \{y \in \Omega : d(\Lambda^{n_0}x, y) < \infty\}$;

$(A_{2.4})$ $d(y, x^*) \leq (1/1-L)d(y, \Lambda y)$ *for all* $y \in \Delta$.

Throughout this paper, we consider a β-Banach space. Let β be a real number with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either real field $\mathbb{R}$ or complex field $\mathbb{C}$. Suppose E is a vector space over $\mathbb{K}$. A function $\|\cdot\|_\beta : E \to [0, \infty)$ is called a β-norm if and only if it satisfies

(N_1) $\|x\|_\beta = 0$, if and only if $x = 0$;

(N_2) $\|\lambda x\|_\beta = |\lambda|^\beta \|x\|_\beta$, for all $\lambda \in \mathbb{K}$ and all $x \in E$;

(N_3) $\|x + y\|_\beta \leq \|x\|_\beta + \|y\|_\beta$, for all $x, y \in E$.

A β-Banach space is a β-normed space which is complete with respect to the β-norm. Now we are ready to investigate the generalized Hyers-Ulam stability problem for the functional equation (1.6) using the fixed point method. From now on, let X be a linear space and let Y be a β-Banach space over $\mathbb{K}$ unless we give any specific reference where β is a fixed real number with $0 < \beta \leq 1$.

Theorem 2.2. *Let* $f : X \to Y$ *be a function with* $f(0) = 0$ *for which there exists a function* $\varphi : X^n \to [0, \infty)$ *such that there exists a constant* $L, 0 < L < 1$, *satisfying the inequalities*

$$\left\| Df(x_1, \ldots, x_n) \right\|_\beta \leq \varphi(x_1, \ldots, x_n), \tag{2.2}$$

$$\varphi(nx_1, \ldots, nx_n) \leq n^{2\beta} L \varphi(x_1, \ldots, x_n) \tag{2.3}$$

for all $x_1, \ldots, x_n \in X$. *Then there exists a unique quadratic function* $Q : X \to Y$ *defined by* $\lim_{k\to\infty}(f(n^k x)/n^{2k}) = Q(x)$ *such that*

$$\left\| f(x) - Q(x) \right\|_\beta \leq \frac{1}{2^\beta n^{2\beta}(1-L)} \varphi(x, \ldots, x) \tag{2.4}$$

for all $x \in X$.

Proof. Let us define Ω to be the set of all functions $g : X \to Y$ and introduce a generalized metric d on Ω as follows:

$$d(g, h) = \inf\left\{ C \in [0, \infty] : \left\| g(x) - h(x) \right\|_\beta \leq C\varphi(x, \ldots, x), \quad \forall x \in X \right\}. \tag{2.5}$$

Then it is easy to show that (Ω, d) is complete (see the proof of Theorem 3.1 of [35]). Now we define an operator $\Lambda : \Omega \to \Omega$ by

$$\Lambda g(x) = \frac{g(nx)}{n^2}, \quad g \in \Omega \tag{2.6}$$

for all $x \in X$. First, we assert that Λ is strictly contractive with constant L on Ω. Given $g, h \in \Omega$, let $C \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le C$, that is, $\|g(x) - h(x)\|_\beta \le C\varphi(x, \dots, x)$. Then it follows from (2.3) that

$$\begin{aligned} \|\Lambda g(x) - \Lambda h(x)\|_\beta &= \frac{1}{n^{2\beta}}\|g(nx) - h(nx)\|_\beta \le \frac{1}{n^{2\beta}} C\varphi(nx, \dots, nx) \\ &\le LC\varphi(x, \dots, x) \end{aligned} \tag{2.7}$$

for all $x \in X$, that is, $d(\Lambda g, \Lambda h) \le LC$. Thus we see that $d(\Lambda g, \Lambda h) \le Ld(g, h)$ for any $g, h \in \Omega$ and so Λ is strictly contractive with constant L on Ω.

Next, if we put $(x_1, \dots, x_n) := (x, \dots, x)$ in (2.2) and we divide both sides by $(2n^2)^\beta$, then we get

$$\begin{aligned} \left\|\frac{f(nx)}{n^2} - f(x)\right\|_\beta &= \frac{1}{(2n^2)^\beta}\left\|2f(nx) - 2n^2 f(x)\right\|_\beta \\ &\le \frac{1}{(2n^2)^\beta}\varphi(x, \dots, x) \end{aligned} \tag{2.8}$$

for all $x \in X$, which implies $d(\Lambda f, f) \le 1/(2n^2)^\beta < \infty$.

Thus applying Theorem 2.1 to the complete generalized metric space (Ω, d) with contractive constant L, we see from Theorem 2.1$(A_{2.2})$ that there exists a function $Q : X \to Y$ which is a fixed point of Λ, that is, $Q(x) = \Lambda Q(x) = Q(nx)/n^2$, such that $d(\Lambda^k f, Q) \to 0$ as $k \to \infty$. By mathematical induction we know that

$$\Lambda^k Q(x) = \frac{Q(n^k x)}{n^{2k}} = Q(x) \tag{2.9}$$

for all $k \in \mathbb{N}$. Since $d(\Lambda^k f, Q) \to 0$ as $k \to \infty$, there exists a sequence $\{C_k\}$ such that $C_k \to 0$ as $k \to \infty$, and $d(\Lambda^k f, Q) \le C_k$ for every $k \in \mathbb{N}$. Hence, it follows from the definition of d that

$$\left\|\Lambda^k f(x) - Q(x)\right\|_\beta \le C_k\varphi(x, \dots, x) \tag{2.10}$$

for all $x \in X$. This implies

$$\lim_{k\to\infty}\left\|\Lambda^k f(x) - Q(x)\right\|_\beta = 0, \quad \text{that is,} \quad \lim_{k\to\infty}\frac{f(n^k x)}{n^{2k}} = Q(x) \tag{2.11}$$

for all $x \in X$. By Theorem 2.1$(A_{2.4})$, we obtain

$$d(f, Q) \le \frac{1}{1 - L} d(\Lambda f, f) \le \frac{1}{(2n^2)^\beta (1 - L)}, \tag{2.12}$$

which yields inequality (2.4).

In turn, it follows from (2.2) and (2.3) that

$$\begin{aligned}\|DQ(x_1,\ldots,x_n)\|_\beta &= \lim_{k\to\infty}\frac{1}{n^{2k\beta}}\left\|Df\left(n^k x_1,\ldots,n^k x_n\right)\right\|_\beta \\ &\le \lim_{k\to\infty}\frac{1}{n^{2k\beta}}\varphi\left(n^k x_1,\ldots,n^k x_n\right) \le \lim_{k\to\infty} L^k\varphi(x_1,\ldots,x_n) \\ &= 0\end{aligned} \tag{2.13}$$

for all $x_1,\ldots,x_n \in X$, which implies that Q is a solution of (1.6) and so the mapping Q is quadratic.

To prove the uniqueness of Q, assume now that $Q_1 : X \to Y$ is another quadratic mapping satisfying inequality (2.4). Then Q_1 is a fixed point of Λ and $Q_1 \in \Delta = \{g \in \Omega : d(f,g) < \infty\}$. Since the mapping Q is a unique fixed point of Λ in the set $\Delta = \{g \in \Omega : d(f,g) < \infty\}$, we see that $Q = Q_1$ by Theorem 2.1($A_{2.3}$). The proof is complete. □

The following theorem is an alternative result of Theorem 2.2.

Theorem 2.3. *Let $f : X \to Y$ be a function with $f(0) = 0$ for which there exists a function $\varphi : X^n \to [0,\infty)$ such that there exists a constant $L, 0 < L < 1$, satisfying the inequalities*

$$\left\|Df(x_1,\ldots,x_n)\right\|_\beta \le \varphi(x_1,\ldots,x_n), \tag{2.14}$$

$$\varphi\left(\frac{x_1}{n},\ldots,\frac{x_n}{n}\right) \le \frac{L}{n^{2\beta}}\varphi(x_1,\ldots,x_n) \tag{2.15}$$

for all $x_1,\ldots,x_n \in X$. Then there exists a unique quadratic function $Q : X \to Y$ defined by $\lim_{k\to\infty} n^{2k} f(x/n^k) = Q(x)$ such that

$$\left\|f(x) - Q(x)\right\|_\beta \le \frac{L}{2^\beta n^{2\beta}(1-L)}\varphi(x,\ldots,x) \tag{2.16}$$

for all $x \in X$.

Proof. We use the same notations for Ω and d as in the proof of Theorem 2.2. Thus (Ω, d) is a complete generalized metric space. Let us define an operator $\Lambda : \Omega \to \Omega$ by

$$\Lambda g(x) = n^2 g\left(\frac{x}{n}\right), \qquad g \in \Omega \tag{2.17}$$

for all $x \in X$.

Then it follows from (2.15) that

$$\begin{aligned}\left\|\Lambda g(x) - \Lambda h(x)\right\|_\beta &= n^{2\beta}\left\|g\left(\frac{x}{n}\right) - h\left(\frac{x}{n}\right)\right\|_\beta \le n^{2\beta}C\varphi\left(\frac{x}{n},\ldots,\frac{x}{n}\right) \\ &\le LC\varphi(x,\ldots,x)\end{aligned} \tag{2.18}$$

for all $x \in X$, that is, $d(\Lambda g, \Lambda h) \leq LC$. Thus we see that $d(\Lambda g, \Lambda h) \leq Ld(g,h)$ for any $g, h \in \Omega$ and so Λ is strictly contractive with constant L on Ω.

Next, if we put $(x_1, \ldots, x_n) := (x/n, \ldots, x/n)$ in (2.14) and we multiply both sides by $1/2^\beta$, then we get by virtue of (2.15)

$$\begin{aligned} \left\| f(x) - n^2 f\left(\frac{x}{n}\right) \right\|_\beta &= \frac{1}{2^\beta} \varphi\left(\frac{x}{n}, \ldots, \frac{x}{n}\right) \\ &\leq \frac{L}{2^\beta n^{2\beta}} \varphi(x, \ldots, x) \end{aligned} \tag{2.19}$$

for all $x \in X$, which implies $d(f, \Lambda f) \leq L/2^\beta n^{2\beta} < \infty$.

Thus according to $(A_{2.2})$ of Theorem 2.1, there exists a function $Q : X \to Y$ which is a fixed point of Λ, that is, $Q(x) = \Lambda Q(x) = n^2 Q(x/n)$, such that

$$\lim_{k \to \infty} d\left(\Lambda^k f, Q\right) = 0, \quad \text{that is, } \lim_{k \to \infty} n^{2k} f\left(\frac{x}{n^k}\right) = Q(x), \quad x \in X. \tag{2.20}$$

By Theorem 2.1$(A_{2.4})$, we obtain

$$d(f, Q) \leq \frac{1}{1-L} d(\Lambda f, f) \leq \frac{L}{2^\beta n^{2\beta}(1-L)}, \tag{2.21}$$

which yields the inequality (2.16).

Replacing x/n^k instead of $n^k x$ in the last part of Theorem 2.2, we can prove that $Q : X \to Y$ is a unique quadratic function satisfying (2.16) for all $x \in X$. □

As applications, one has the following corollaries concerning the stability of (1.6).

Corollary 2.4. *Let ε be a real number with $\varepsilon \geq 0$. Assume that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\| Df(x_1, \ldots, x_n) \|_\beta \leq \varepsilon \tag{2.22}$$

for all $x_1, \ldots, x_n \in X$. Then there exists a unique quadratic function $Q : X \to Y$ given by $Q(x) = \lim_{k \to \infty}(f(n^k x)/n^{2k})$, which satisfies the inequality

$$\| f(x) - Q(x) \|_\beta \leq \frac{\varepsilon}{2^\beta (n^{2\beta} - 1)} \tag{2.23}$$

for all $x \in X$.

Proof. Letting $\varphi(x_1, \ldots, x_n) := \varepsilon$ and then applying Theorem 2.2 with contractive constant $1/n^{2\beta}$, we obtain easily the result. □

Corollary 2.5. *Let X be an α-normed space with $0 < \alpha \leq 1$ and Y a β-Banach space, respectively. Let $\{\theta_i\}_{i=1}^n$ be real numbers such that $\theta_i \geq 0$ for all $i = 1, \ldots, n$ and let $\{p_i\}_{i=1}^n$ be real numbers such that* *either $\alpha(\max\{p_i\}) < 2\beta$, or $\alpha(\min\{p_i\}) > 2\beta$. Assume that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\left\|Df(x_1,\ldots,x_n)\right\|_\beta \le \sum_{i=1}^{n}\theta_i\|x_i\|_\alpha^{p_i} \tag{2.24}$$

for all $x_1,\ldots,x_n \in X$ and $X\setminus\{0\}$ if $p_i < 0$. Then there exists a unique quadratic function $Q : X \to Y$ which satisfies the inequality

$$\left\|f(x) - Q(x)\right\|_\beta \le \begin{cases} \dfrac{\sum_{i=1}^{n}\theta_i\|x\|_\alpha^{p_i}}{2^\beta\left(n^{2\beta} - n^{\alpha(\max\{p_i\})}\right)} & \text{if } \alpha\left(\max\{p_i\}\right) < 2\beta, \\ \dfrac{\sum_{i=1}^{n}\theta_i\|x\|_\alpha^{p_i}}{2^\beta\left(n^{\alpha(\min\{p_i\})} - n^{2\beta}\right)} & \text{if } \alpha\left(\min\{p_i\}\right) > 2\beta, \end{cases} \tag{2.25}$$

for all $x \in X$ and $X\setminus\{0\}$ if $p < 0$. The function Q is given by

$$Q(x) = \begin{cases} \lim\limits_{k\to\infty}\dfrac{f\left(n^k x\right)}{n^{2k}} & \text{if } \alpha\left(\max\{p_i\}\right) < 2\beta, \\ \lim\limits_{k\to\infty} n^{2k} f\left(\dfrac{x}{n^k}\right) & \text{if } \alpha\left(\min\{p_i\}\right) > 2\beta, \end{cases} \tag{2.26}$$

for all $x \in X$.

Proof. Letting $\varphi(x_1,\ldots,x_n) := \sum_{i=1}^{n}\theta_i\|x_i\|_\alpha^{p_i}$ for all $x_1,\ldots,x_n \in X$ and then applying Theorem 2.2 with contractive constant $n^{\alpha(\max\{p_i\})}/n^{2\beta}$ and Theorem 2.3 with contractive constant $n^{2\beta}/n^{\alpha(\min\{p_i\})}$, we obtain easily the results. □

3. Stability of (1.6) by Direct Method

In the next two theorems, let $\varphi : X^n \to [0,\infty)$ be a mapping satisfying one of the conditions

$$\Phi_1(x_1,\ldots,x_n) := \sum_{l=0}^{\infty}\frac{1}{2^\beta n^{2\beta(l+1)}}\varphi\left(n^l x_1,\ldots,n^l x_n\right) < \infty, \tag{3.1}$$

$$\Phi_2(x_1,\ldots,x_n) := \sum_{l=0}^{\infty}\frac{1}{2^\beta}n^{2\beta l}\varphi\left(\frac{x_1}{n^{l+1}},\ldots,\frac{x_n}{n^{l+1}}\right) < \infty \tag{3.2}$$

for all $x_1,\ldots,x_n \in X$.

Theorem 3.1. *Assume that a function $f : X \to Y$ satisfies*

$$\left\|Df(x_1,\ldots,x_n)\right\|_\beta \le \varphi(x_1,\ldots,x_n) \tag{3.3}$$

for all $x_1, \ldots, x_n \in X$ and φ satisfies the condition (3.1)*. Then there exists a unique quadratic function $Q : X \to Y$ satisfying*

$$\left\| f(x) - \frac{nf(0)}{2(n+1)} - Q(x) \right\|_\beta \le \Phi_1(x, \ldots, x) \tag{3.4}$$

for all $x \in X$, where $\|f(0)\|_\beta \le \varphi(0, \ldots, 0)/(n-1)^\beta(n+2)^\beta$. The function Q is given by

$$Q(x) = \lim_{k\to\infty} \frac{f(n^k x)}{n^{2k}} \tag{3.5}$$

for all $x \in X$.

Proof. Putting $x_1 =, \ldots, = x_n = 0$ in (3.3), we get $\|f(0)\|_\beta \le \varphi(0, \ldots, 0)/(n-1)^\beta(n+2)^\beta$. Putting $x_1 =, \ldots, = x_n = x$ in (3.3), we obtain

$$\left\| 2f(nx) + n(n-1)f(0) - 2n^2 f(x) \right\|_\beta \le \varphi(x, \ldots, x) \tag{3.6}$$

for $x \in X$. Dividing (3.6) by $2^\beta n^{2\beta}$, we get

$$\left\| \frac{1}{n^2} \overline{f}(nx) - \overline{f}(x) \right\|_\beta \le \frac{1}{2^\beta n^{2\beta}} \varphi(x, \ldots, x), \tag{3.7}$$

where $\overline{f}(x) = f(x) - (n/2(n+1))f(0)$ for any $x \in X$. Thus it follows from formula (3.7) and triangle inequality that

$$\left\| \frac{1}{n^{2k}} \overline{f}(n^k x) - \overline{f}(x) \right\|_\beta \le \sum_{l=0}^{k-1} \frac{1}{2^\beta n^{2\beta(l+1)}} \varphi\left(n^l x, \ldots n^l x\right) \tag{3.8}$$

for all $x \in X$ and all $k \in \mathbb{N}$, which is verified by induction. Therefore we prove from inequality (3.8) that for any integers m, k with $m > k \ge 0$

$$\begin{aligned} \left\| \frac{1}{n^{2m}} \overline{f}(n^m x) - \frac{1}{n^{2k}} \overline{f}\left(n^k x\right) \right\|_\beta &\le \frac{1}{n^{2\beta k}} \left\| \frac{1}{n^{2(m-k)}} \overline{f}\left(n^{m-k} n^k x\right) - \overline{f}\left(n^k x\right) \right\|_\beta \\ &= \sum_{l=0}^{m-k-1} \frac{1}{2^\beta n^{2\beta(l+k+1)}} \varphi\left(n^{l+k} x, \ldots, n^{l+k} x\right) \\ &= \sum_{l=k}^{m-1} \frac{1}{2^\beta n^{2\beta(l+1)}} \varphi\left(n^l x, \ldots, n^l x\right) \end{aligned} \tag{3.9}$$

for all $x \in X$. Since the right-hand side of (3.9) tends to zero as $k \to \infty$, the sequence $\{(1/n^{2k})\overline{f}(n^k x)\}$ is a Cauchy sequence for all $x \in X$ and thus converges by the completeness of Y. Define $Q : X \to Y$ by

$$Q(x) = \lim_{k\to\infty} \frac{1}{n^{2k}} \left(f\left(n^k x\right) - \frac{n}{2(n+1)} f(0) \right) \\ = \lim_{k\to\infty} \frac{f\left(n^k x\right)}{n^{2k}}, \quad x \in X. \tag{3.10}$$

Taking the limit in (3.8) as $k \to \infty$, we obtain that

$$\left\| f(x) - \frac{n}{2(n+1)} f(0) - Q(x) \right\|_\beta \le \Phi_1(x, \dots, x) \tag{3.11}$$

for all $x \in X$. Letting $x_i := n^k x_i$ for all $i = 1, \dots, n$ in (3.3), respectively, and dividing both sides by $n^{2\beta k}$ and after then taking the limit in the resulting inequality, we have

$$\left\| 2Q\left(\sum_{i=1}^n x_i\right) + \sum_{i \neq j} Q(x_i - x_j) - 2n \sum_{i=1}^n Q(x_i) \right\|_\beta \\ = \lim_{k\to\infty} \frac{1}{n^{2\beta k}} \left\| Df\left(n^k x_1, \dots, n^k x_n\right) \right\|_\beta \\ \le \lim_{k\to\infty} \frac{1}{n^{2\beta k}} \varphi\left(n^k x_1, \dots, n^k x_n\right) = 0, \tag{3.12}$$

so the function Q is quadratic.

To prove the uniqueness of the quadratic function Q subject to (3.4), let us assume that there exists a quadratic function $Q' : X \to Y$ which satisfies (1.6) and inequality (3.4). Obviously, we obtain that

$$Q(x) = n^{-2k} Q\left(n^k x\right), \quad Q'(x) = n^{-2k} Q'\left(n^k x\right) \tag{3.13}$$

for all $x \in X$. Hence it follows from (3.4) that

$$\|Q(x) - Q'(x)\|_\beta = \frac{1}{n^{2\beta k}} \left\| Q\left(n^k x\right) - Q'\left(n^k x\right) \right\|_\beta \\ \le \frac{2}{n^{2\beta k}} \sum_{l=0}^\infty \frac{1}{n^{2\beta(l+1)}} \varphi\left(n^{l+k} x, \dots, n^{l+k} x\right) \\ = 2 \sum_{l=k}^\infty \frac{1}{n^{2\beta(l+1)}} \varphi\left(n^l x, \dots, n^l x\right), \quad x \in X \tag{3.14}$$

for all $k \in \mathbb{N}$. Therefore letting $k \to \infty$, one has $Q(x) - Q'(x) = 0$ for all $x \in X$, completing the proof of uniqueness. □

Theorem 3.2. *Assume that a function $f : X \to Y$ satisfies*

$$\left\|Df(x_1,\dots,x_n)\right\|_\beta \le \varphi(x_1,\dots,x_n) \tag{3.15}$$

for all $x_1,\dots,x_n \in X$ and φ satisfies condition (3.2). *Then there exists a unique quadratic function $Q : X \to Y$ satisfying*

$$\left\|f(x) - Q(x)\right\|_\beta \le \Phi_2(x,\dots,x) \tag{3.16}$$

for all $x \in X$. The function Q is given by

$$Q(x) = \lim_{k\to\infty} n^{2k} f\left(\frac{x}{n^k}\right) \tag{3.17}$$

for all $x \in X$.

Proof. In this case, $f(0) = 0$ since $\sum_{l=0}^{\infty} 1/2^\beta n^{2\beta l}\varphi(0,\dots,0) < \infty$ and so $\varphi(0,\dots,0) = 0$ by assumption. Replacing x by x/n in (3.6), we obtain

$$\left\|f(x) - n^2 f\left(\frac{x}{n}\right)\right\|_\beta \le \frac{1}{2^\beta}\varphi\left(\frac{x}{n},\dots,\frac{x}{n}\right) \tag{3.18}$$

for $x \in X$.

Therefore we prove from inequality (3.18) that for any integers m, k with $m > k \ge 0$

$$\begin{aligned}\left\|\mathrm{n}^{2m} f\left(\frac{x}{n^m}\right) - n^{2k} f\left(\frac{x}{n^k}\right)\right\|_\beta &= n^{2\beta k}\left\|n^{2(m-k)} f\left(\frac{x}{n^m}\right) - f\left(\frac{x}{n^k}\right)\right\|_\beta \\ &\le n^{2\beta k}\sum_{l=0}^{m-k-1}\frac{1}{2^\beta}n^{2\beta l}\varphi\left(\frac{x}{n^{l+k+1}},\dots,\frac{x}{n^{l+k+1}}\right) \\ &= \sum_{l=k}^{m-1}\frac{1}{2^\beta}n^{2\beta l}\varphi\left(\frac{x}{n^{l+1}},\dots,\frac{x}{n^{l+1}}\right)\end{aligned} \tag{3.19}$$

for all $x \in X$. Since the right-hand side of (3.19) tends to zero as $k \to \infty$, the sequence $\{n^{2k} f(x/n^k)\}$ is a Cauchy sequence for all $x \in X$, and thus converges by the completeness of Y. Define $Q : X \to Y$ by

$$Q(x) = \lim_{k\to\infty} n^{2k} f\left(\frac{x}{n^k}\right) \tag{3.20}$$

for all $x \in X$. Taking the limit in (3.19) with $k = 0$ as $m \to \infty$, we obtain that

$$\left\|f(x) - Q(x)\right\|_\beta \le \Phi_2(x,\dots,x), \qquad x \in X. \tag{3.21}$$

Replacing $(x_1,\dots,x_n)$ in (3.3) by $(x_1/n^k,\dots,x_n/n^k)$, multiplying both sides by $n^{2\beta k}$, and then taking the limit as $k \to \infty$ in the resulting inequality, we have

$$2Q\left(\sum_{i=1}^{n}x_i\right)+\sum_{i\neq j}Q(x_i-x_j)-2n\sum_{i=1}^{n}Q(x_i)=0 \tag{3.22}$$

for all $x_1,\dots,x_n \in X$. Therefore the function Q is quadratic.

To prove the uniqueness, let Q' be another quadratic function satisfying (3.16). Then it is easy to see that the following identities $Q(x)=n^{2k}Q(x/n^k)$ and $Q'(x)=n^{2k}Q'(x/n^k)$ hold for all $x \in X$. Thus we have

$$\begin{aligned}\left\|Q(x)-Q'(x)\right\|_\beta &= n^{2\beta k}\left\|Q\left(\frac{x}{n^k}\right)-Q'\left(\frac{x}{n^k}\right)\right\|_\beta \\ &\le \frac{2}{2^\beta}\sum_{l=0}^{\infty}n^{2\beta(l+k)}\varphi\left(\frac{x}{n^{l+k+1}},\dots,\frac{x}{n^{l+k+1}}\right) \\ &\le \frac{2}{2^\beta}\sum_{l=k}^{\infty}n^{2\beta l}\varphi\left(\frac{x}{n^{l+1}},\dots,\frac{x}{n^{l+1}}\right)\end{aligned} \tag{3.23}$$

for all $x \in X$ and all $k \in \mathbb{N}$. Therefore letting $k \to \infty$, one has $Q(x)-Q'(x)=0$ for all $x \in X$. This completes the proof. □

In the following corollary, we have a stability result of (1.6) with difference operator Df bounded by the sum of powers of α-norms.

Corollary 3.3. *Let X be an α-normed space with $0<\alpha\le 1$ and Y a β-Banach space, respectively. Let $\{\theta_i\}_{i=1}^n$ be real numbers with $\theta_i \ge 0$ for all i, and let $\varepsilon, \{p_i\}_{i=1}^n$ be real numbers such that either $\alpha(\max\{p_i\})<2\beta, \varepsilon\ge 0$ or $\alpha(\min\{p_i\})>2\beta, \varepsilon=0$. Assume that a function $f: X\to Y$ satisfies the inequality*

$$\left\|Df(x_1,\dots,x_n)\right\|_\beta \le \varepsilon+\sum_{i=1}^{n}\theta_i\|x_i\|_\alpha^{p_i} \tag{3.24}$$

for all $x_1,\dots,x_n \in X$ and $X\setminus\{0\}$ if $p_i<0$. Then there exists a unique quadratic function $Q: X\to Y$ which satisfies the inequality

$$\left\|f(x)-\frac{nf(0)}{2(n+1)}-Q(x)\right\|_\beta \le \begin{cases}\dfrac{\varepsilon}{2^\beta(n^{2\beta}-1)}+\displaystyle\sum_{i=1}^{n}\frac{\theta_i\|x\|_\alpha^{p_i}}{2^\beta(n^{2\beta}-n^{\alpha p_i})} & \text{if } \alpha(\max\{p_i\})<2\beta,\quad \varepsilon\ge 0, \\ \displaystyle\sum_{i=1}^{n}\frac{\theta_i\|x\|_\alpha^{p_i}}{2^\beta(n^{\alpha p_i}-n^{2\beta})} & \text{if } \alpha(\min\{p_i\})>2\beta,\quad \varepsilon=0\end{cases} \tag{3.25}$$

for all $x \in X$ and $X \setminus \{0\}$ if $p_i < 0$. The function Q is given by

$$Q(x) = \begin{cases} \lim_{k\to\infty} \dfrac{f(n^k x)}{n^{2k}}, & \text{if } \alpha(\max\{p_i\}) < 2\beta,\ \varepsilon \ge 0 \\ \lim_{k\to\infty} n^{2k} f\left(\dfrac{x}{n^k}\right), & \text{if } \alpha(\min\{p_i\}) > 2\beta,\ \varepsilon = 0. \end{cases} \tag{3.26}$$

for all $x \in X$.

Proof. Letting $\varphi(x_1,\ldots,x_n) := \varepsilon + \sum_{i=1}^{n} \theta_i \|x_i\|_\alpha^{p_i}$ for all $x_1,\ldots,x_n \in X$ and then applying Theorems 3.1 and 3.2, we obtain easily the results. □

We observe that if $f(0) = 0$ and $\varepsilon = 0$ in Corollary 3.3, then the stability result obtained by the fixed point method in Corollary 2.5 is somewhat different from the stability result obtained by direct method in Corollary 3.3. The stability result in Corollary 3.3 is sharper than that of Corollary 2.5.

In the next corollary, we get a stability result of (1.6) with difference operator Df bounded by the product of powers of α-norms.

Corollary 3.4. *Let X be an α-normed space with $0 < \alpha \le 1$ and Y a β-Banach space, respectively, and let $\theta, \{p_i\}_{i=1}^n$ be real numbers such that $\theta \ge 0$ and $\alpha p \ne 2\beta$, where $p := \sum_{i=1}^{n} p_i$. Suppose that a function $f : X \to Y$ satisfies*

$$\left\| Df(x_1,\ldots,x_n) \right\|_\beta \le \theta \prod_{i=1}^{n} \|x_i\|_\alpha^{p_i} \tag{3.27}$$

for all $x_1,\ldots,x_n \in X$ and $X \setminus \{0\}$ if $p_i < 0$. Then there exists a unique quadratic function $Q : X \to Y$ which satisfies the inequality

$$\left\| f(x) - \frac{n}{2(n+1)} f(0) - Q(x) \right\|_\beta \le \frac{\theta \|x\|^p}{2^\beta\left(\left|n^{2\beta} - n^{\alpha p}\right|\right)} \tag{3.28}$$

for all $x \in X$, and for all $X \setminus \{0\}$ if $p < 0$, where $f(0) = 0$ if $p > 0$.

Proof. We remark that $\varphi(x_1,\ldots,x_n) := \theta \prod_{i=1}^{n} \|x_i\|_\alpha^{p_i}$ satisfies condition (3.1) for the case $\alpha p < 2\beta$ or condition (3.2) for the case $\alpha p > 2\beta$. By Theorems 3.1 and 3.2, we get the results. □

We observe that if $f(0) = 0$ in Corollary 3.4, then the stability result obtained by the fixed point method with contractive constants $n^{\alpha p}/n^{2\beta}(\alpha p < 2\beta), n^{2\beta}/n^{\alpha p}(\alpha p > 2\beta)$, respectively, coincides with the stability result (3.28) obtained by direct method.

Acknowledgment

This work was supported by the National Research Foundation of Korea Grant funded by the Korean Government (no. 2009-0070940).

References

[1] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience, New York, NY, USA, 1960.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[3] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.

[4] D. G. Bourgin, "Classes of transformations and bordering transformations," *Bulletin of the American Mathematical Society*, vol. 57, pp. 223–237, 1951.

[5] G. L. Forti, "An existence and stability theorem for a class of functional equations," *Stochastica*, vol. 4, no. 1, pp. 23–30, 1980.

[6] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[7] Z. Gajda, "On stability of additive mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.

[8] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Bulletin des Sciences Mathématiques*, vol. 108, no. 4, pp. 445–446, 1984.

[9] P. Găvruţa, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.

[10] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, USA, 2002.

[11] M. Eshaghi Gordji, T. Karimi, and S. Kaboli Gharetapeh, "Approximately n-Jordan homomorphisms on Banach algebras," *Journal of Inequalities and Applications*, vol. 2009, Article ID 870843, 8 pages, 2009.

[12] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and Their Applications, 34, Birkhäuser, Boston, Mass, USA, 1998.

[13] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.

[14] A. Najati and C. Park, "Fixed points and stability of a generalized quadratic functional equation," *Journal of Inequalities and Applications*, vol. 2009, Article ID 193035, 19 pages, 2009.

[15] C. Park, J. S. An, and F. Moradlou, "Additive functional inequalities in Banach modules," *Journal of Inequalities and Applications*, vol. 2008, Article ID 592504, 10 pages, 2008.

[16] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, vol. 31 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 1989.

[17] D. H. Hyers and Th. M. Rassias, "Approximate homomorphisms," *Aequationes Mathematicae*, vol. 44, no. 2-3, pp. 125–153, 1992.

[18] F. Skof, "Local properties and approximation of operators," *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 53, pp. 113–129, 1983.

[19] S. Czerwik, "On the stability of the quadratic mapping in normed spaces," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 62, pp. 59–64, 1992.

[20] C. Borelli and G. L. Forti, "On a general Hyers-Ulam stability result," *International Journal of Mathematics and Mathematical Sciences*, vol. 18, no. 2, pp. 229–236, 1995.

[21] B. Bouikhalene, E. Elqorachi, and J. M. Rassias, "The superstability of d'Alembert's functional equation on the Heisenberg group," *Applied Mathematics Letters*, vol. 23, no. 1, pp. 105–109, 2010.

[22] H.-X. Cao, J.-R. Lv, and J. M. Rassias, "Superstability for generalized module left derivations and generalized module derivations on a Banach module. I," *Journal of Inequalities and Applications*, vol. 2009, Article ID 718020, 10 pages, 2009.

[23] A. Gilányi, "On the stability of monomial functional equations," *Publicationes Mathematicae Debrecen*, vol. 56, no. 1-2, pp. 201–212, 2000.

[24] M. Eshaghi Gordji, S. Abbaszadeh, and C. Park, "On the stability of a generalized quadratic and quartic type functional equation in quasi-Banach spaces," *Journal of Inequalities and Applications*, vol. 2009, Article ID 153084, 26 pages, 2009.

[25] H.-M. Kim, J. M. Rassias, and Y.-S. Cho, "Stability problem of Ulam for Euler-Lagrange quadratic mappings," *Journal of Inequalities and Applications*, vol. 2007, Article ID 10725, 15 pages, 2007.

[26] J. M. Rassias and H.-M. Kim, "Generalized Hyers-Ulam stability for general additive functional equations in quasi-β-normed spaces," *Journal of Mathematical Analysis and Applications*, vol. 356, no. 1, pp. 302–309, 2009.

[27] M. B. Savadkouhi, M. Eshaghi Gordji, J. M. Rassias, and N. Ghobadipour, "Approximate ternary Jordan derivations on Banach ternary algebras," *Journal of Mathematical Physics*, vol. 50, no. 4, Article ID 042303, 9 pages, 2009.

[28] G. Isac and Th. M. Rassias, "Stability of ψ-additive mappings: applications to nonlinear analysis," *International Journal of Mathematics and Mathematical Sciences*, vol. 19, no. 2, pp. 219–228, 1996.

[29] V. Radu, "The fixed point alternative and the stability of functional equations," *Fixed Point Theory*, vol. 4, no. 1, pp. 91–96, 2003.

[30] L. Cădariu and V. Radu, "Fixed points and the stability of Jensen's functional equation," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 1, article 4, 7 pages, 2003.

[31] L. Cădariu and V. Radu, "Fixed point methods for the generalized stability of functional equations in a single variable," *Fixed Point Theory and Applications*, vol. 2008, Article ID 749392, 15 pages, 2008.

[32] S.-M. Jung, T.-S. Kim, and K.-S. Lee, "A fixed point approach to the stability of quadratic functional equation," *Bulletin of the Korean Mathematical Society*, vol. 43, no. 3, pp. 531–541, 2006.

[33] S.-M. Jung, "A fixed point approach to the stability of isometries," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 2, pp. 879–890, 2007.

[34] S.-M. Jung, "A fixed point approach to the stability of a Volterra integral equation," *Fixed Point Theory and Applications*, vol. 2007, Article ID 57064, 9 pages, 2007.

[35] S.-M. Jung and Z.-H. Lee, "A fixed point approach to the stability of quadratic functional equation with involution," *Fixed Point Theory and Applications*, vol. 2008, Article ID 732086, 11 pages, 2008.

[36] S.-M. Jung and S. Min, "A fixed point approach to the stability of the functional equation $f(x+y) = F[f(x), f(y)]$," *Fixed Point Theory and Applications*, vol. 2009, Article ID 912046, 8 pages, 2009.

[37] S.-M. Jung and J. M. Rassias, "A fixed point approach to the stability of a functional equation of the spiral of Theodorus," *Fixed Point Theory and Applications*, vol. 2008, Article ID 945010, 7 pages, 2008.

[38] J. B. Diaz and B. Margolis, "A fixed point theorem of the alternative, for contractions on a generalized complete metric space," *Bulletin of the American Mathematical Society*, vol. 74, no. 2, pp. 305–309, 1968.

[39] D. H. Hyers, G. Isac, and Th. M. Rassias, *Topics in Nonlinear Analysis and Applications*, World Scientific, River Edge, NJ, USA, 1997.