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Research Article

Random Stability of an Additive-Quadratic-Quartic Functional Equation

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Using the fixed point method, we prove the generalized Hyers-Ulam stability of the following additive-quadratic-quartic functional equation $f(x+2y)+f(x-2y)=2f(x+y)+2f(-x-y)+2f(x-y)+2f(y-x)-4f(-x)-2f(x)+f(2y)+f(-2y)-4f(y)-4f(-y)$ in complete random normed spaces.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias [4] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or as *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Th. M. Rassias theorem was obtained by Găvruţa [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [6] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [7] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [8] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4, 9–26]).

In [27], Lee et al. considered the following quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{1.2}$$

It is easy to show that the function $f(x)=x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

Let X be a set. A function $d : X\times X \to [0,\infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,y)=0$ if and only if $x=y$,

(2) $d(x,y)=d(y,x)$ for all $x,y\in X$,

(3) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.1 ([28, 29]). *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x\in X$, either*

$$d\left(J^n x, J^{n+1}x\right)=\infty \tag{1.3}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x)<\infty$, *for all* $n\ge n_0$,

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J,

(3) y^* *is the unique fixed point of* J *in the set* $Y=\{y\in X \mid d(J^{n_0}x,y)<\infty\}$,

(4) $d(y,y^*)\le (1/(1-L))d(y,Jy)$ *for all* $y\in Y$.

In 1996, Isac and Th. M. Rassias [30] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [31–36]).

2. Preliminaries

In the sequel we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [37–41]. Throughout this paper, Δ^+ is the space of all probability distribution functions that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0,1]$, such that F is left-continuous, non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consising of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases} \tag{2.1}$$

Definition 2.1 ([40]). A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (briefly, a t-norm) if T satisfies the following conditions:

(a) T is commutative and associative;

(b) T is continuous;

(c) $T(a,1) = a$ for all $a \in [0,1]$;

(d) $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous t-norms are $T_P(a,b) = ab$, $T_M(a,b) = \min(a,b)$ and $T_L(a,b) = \max(a+b-1,0)$ (the Łukasiewicz t-norm).

Recall (see [42, 43]) that if T is a t-norm and $\{x_n\}$ is a given sequence of numbers in $[0,1]$, $T_{i=1}^n x_i$ is defined recurrently by $T_{i=1}^1 x_i = x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for $n \geq 2$. $T_{i=n}^\infty x_i$ is defined as $T_{i=1}^\infty x_{n+i}$.

It is known ([43]) that for the Łukasiewicz t-norm the following implication holds:

$$\lim_{n \to \infty} (T_L)_{i=1}^\infty x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^\infty (1 - x_n) < \infty. \tag{2.2}$$

Definition 2.2 ([41]). A *Random Normed space* (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that, the following conditions hold:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(RN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all $x \in X$, $\alpha \neq 0$;

(RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Definition 2.3. Let (X, μ, T) be a RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called *Cauchy* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $\mu_{x_n - x_m}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.

(3) A RN-space (X, μ, T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X. A complete RN-space is said to be random Banach space.

Theorem 2.4 ([40]). *If (X, μ, T) is a RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty}\mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

The theory of random normed spaces (RN-spaces) is important as a generalization of deterministic result of linear normed spaces and also in the study of random operator equations. The RN-spaces may also provide us the appropriate tools to study the geometry of nuclear physics and have important application in quantum particle physics. The generalized Hyers-Ulam stability of different functional equations in random normed spaces, RN-spaces and fuzzy normed spaces has been recently studied in, Alsina [44], Mirmostafaee, Mirzavaziri and Moslehian [33, 45–47], Miheţ and Radu [38, 39, 48, 49], Mihet, Saadati and Vaezpour [50, 51], Baktash et al. [52] and Saadati et al. [53].

3. Generalized Hyers-Ulam Stability of the Functional Equation $f(x+2y)+f(x-2y)=2f(x+y)+2f(-x-y)+2f(x-y)+2f(y-x)-4f(-x)-2f(x)+f(2y)+f(-2y)-4f(y)-4f(-y)$: An Odd Case

One can easily show that an odd mapping $f : X \to Y$ satisfies $f(x+2y)+f(x-2y) = 2f(x+y)+2f(-x-y)+2f(x-y)+2f(y-x)-4f(-x)-2f(x)+f(2y)+f(-2y)-4f(y)-4f(-y)$ if and only if the odd mapping mapping $f : X \to Y$ is an additive mapping, that is,

$$f(x+2y)+f(x-2y)=2f(x). \tag{3.1}$$

One can easily show that an even mapping $f : X \to Y$ satisfies $f(x+2y)+f(x-2y) = 2f(x+y)+2f(-x-y)+2f(x-y)+2f(y-x)-4f(-x)-2f(x)+f(2y)+f(-2y)-4f(y)-4f(-y)$ if and only if the even mapping $f : X \to Y$ is a quadratic-quartic mapping, that is,

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+2f(2y)-8f(y). \tag{3.2}$$

It was shown in [54, Lemma 2.1] that $g(x) := f(2x) - 4f(x)$ and $h(x) := f(2x) - 16f(x)$ are quartic and quadratic, respectively, and that $f(x) = (1/12)g(x) - (1/12)h(x)$.

For a given mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x,y) &:= f(x+2y)+f(x-2y)-2f(x+y)-2f(-x-y)-2f(x-y)-2f(y-x) \\ &\quad +4f(-x)+2f(x)-f(2y)-f(-2y)+4f(y)+4f(-y) \end{aligned} \tag{3.3}$$

for all $x, y \in X$.

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in complete RN-spaces: an odd case.

Theorem 3.1. *Let X be a linear space, (Y, μ, T_M) be a complete RN-space and Φ be a mapping from X^2 to $D^+(\Phi(x,y)$ is denoted by $(\Phi_{x,y}))$ such that, for some $0 < \alpha < 1/3$,*

$$\Phi_{3x,3y}(t) \leq \Phi_{x,y}(\alpha t) \quad (x, y \in X, t > 0) \tag{3.4}$$

Let $f : X \to Y$ be an odd mapping satisfying

$$\mu_{Df(x,y)}(t) \geq \Phi_{x,y}(t) \tag{3.5}$$

for all $x, y \in X$ and all $t > 0$. Then

$$A(x) := \lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right) \tag{3.6}$$

exists for each $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \geq \Phi_{x,x}\left(\frac{1-3\alpha}{\alpha}t\right) \tag{3.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $x = y$ in (3.5), we get

$$\mu_{f(3x)-3f(x)}(t) \geq \Phi_{x,x}(t) \tag{3.8}$$

for all $x \in X$ and all $t > 0$.

Consider the set

$$S := \{g : X \longrightarrow Y, \} \tag{3.9}$$

and introduce the generalized metric on S:

$$\mathrm{d}(g, h) = \inf\{u \in \mathbb{R}^+ : \mu_{g(x)-h(x)}(ut) \geq \Phi_{x,x}(t), \forall x \in X, \forall t > 0\}, \tag{3.10}$$

where, as usual, $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete. (See the proof of Lemma 2.1 in [38].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 3g\left(\frac{x}{3}\right) \tag{3.11}$$

for all $x \in X$ and we prove that J is a strictly contractive mapping with the Lipschitz constant 3α.

Let $g, h \in S$ be given such that $d(g,h) < \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq \Phi_{x,x}(t) \tag{3.12}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}
\mu_{Jg(x)-Jh(x)}(3\alpha\varepsilon t) &= \mu_{3g(x/3)-3h(x/3)}(3\alpha\varepsilon t) \\
&= \mu_{g(x/3)-h(x/3)}(\alpha\varepsilon t) \\
&\geq \Phi_{x/3,x/3}(\alpha t) \\
&\geq \Phi_{x,x}(t)
\end{aligned} \tag{3.13}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) < \varepsilon$ implies that $d(Jg, Jh) \leq 3\alpha\varepsilon$. This means that

$$d(Jg, Jh) \leq 3\alpha d(g,h) \tag{3.14}$$

for all $g, h \in S$.

It follows from (3.8) that

$$\mu_{f(x)-3f(x/3)}(\alpha t) \geq \Phi_{x,x}(t), \tag{3.15}$$

for all $x \in X$ and all $t > 0$. So

$$d(f, Jf) \leq \alpha < \frac{1}{3}. \tag{3.16}$$

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$A\left(\frac{x}{3}\right) = \frac{1}{3}A(x) \tag{3.17}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{h \in S : d(h,g) < \infty\}. \tag{3.18}$$

This implies that A is a unique mapping satisfying (3.17) such that there exists a $u \in (0, \infty)$ satisfying

$$\mu_{f(x)-A(x)}(ut) \geq \Phi_{x,x}(t), \tag{3.19}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right) = A(x) \tag{3.20}$$

for all $x \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is an odd mapping;

(3) $d(f, A) \le (1/(1-3\alpha))d(f, Jf)$ with $f \in M$, which implies the inequality

$$d(f, A) \le \frac{\alpha}{1-3\alpha}, \tag{3.21}$$

from which it follows

$$\mu_{f(x)-A(x)}\left(\frac{\alpha}{1-3\alpha}t\right) \ge \Phi_{x,x}(t). \tag{3.22}$$

This implies that the inequality (3.7) holds.

Now, we have,

$$\mu_{3^n Df(x/3^n, y/3^n)}(t) = \mu_{Df(x/3^n, y/3^n)}\left(\frac{t}{3^n}\right) \ge \Phi_{x/3^n, y/3^n}\left(\frac{t}{3^n}\right) \tag{3.23}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$.

So, we obtain by (3.4)

$$\mu_{3^n Df(x/3^n, y/3^n)}(t) \ge \Phi_{x,y}\left(\frac{t}{(3\alpha)^n}\right) \tag{3.24}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$.

Since $\lim_{n\to\infty}\Phi_{x,y}(t/(3\alpha)^n) = 1$ for all $x, y \in X$ and all $t > 0$, by Theorem 2.4, we deduce that

$$\mu_{DA(x,y)}(t) = 1 \tag{3.25}$$

for all $x, y \in X$ and all $t > 0$. Thus the mapping $A : X \to Y$ is additive, as desired. □

Corollary 3.2. *Let $\theta \ge 0$ and let p be a real number with $p > 3$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\mu_{Df(x,y)}(t) \ge \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{3.26}$$

for all $x, y \in X$ and all $t > 0$. Then

$$A(x) := \lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right) \tag{3.27}$$

exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \geq \frac{(3^p - 3)t}{(3^p - 3)t + 2.3^p\theta\|x\|^p} \tag{3.28}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.1 by taking

$$\Phi_{x,y}(t) := \frac{t}{t + \theta\left(\|x\|^p + \|y\|^p\right)} \tag{3.29}$$

for all $x, y \in X$. Then we can choose $\alpha = 3^{-p}$ and we get the desired result. □

Similarly, we can obtain the following. We will omit the proof.

Theorem 3.3. *Let X be a linear space, (Y, μ, T_M) be a complete RN-space and Φ be a mapping from X^2 to D^+ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$)such that, for some $0 < \alpha < 3$,*

$$\Phi_{x/3,y/3}(t) \leq \Phi_{x,y}(\alpha t) \quad \left(x, y \in X, t > 0\right). \tag{3.30}$$

Let $f : X \to Y$ be an odd mapping satisfying (3.5). Then

$$A(x) := \lim_{n\to\infty} \frac{1}{3^n} f(3^n x) \tag{3.31}$$

exists for each $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \geq \Phi_{x,x}((3 - \alpha)t) \tag{3.32}$$

for all $x \in X$ and all $t > 0$.

Corollary 3.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying (3.26). Then*

$$A(x) := \lim_{n\to\infty} 3^{-n} f(3^n x) \tag{3.33}$$

exists for each $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \geq \frac{(3 - 3^p)t}{(3 - 3^p)t + 2\theta\|x\|^p} \tag{3.34}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.3 by taking

$$\mu_{Df(x,y)}(t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{3.35}$$

for all $x, y \in X$. Then we can choose $\alpha = 3^p$ and we get the desired result. □

4. Generalized Hyers-Ulam Stability of the Functional Equation $f(x + 2y) + f(x - 2y) = 2f(x + y) + 2f(-x - y) + 2f(x - y) + 2f(y - x) - 4f(-x) - 2f(x) + f(2y) + f(-2y) - 4f(y) - 4f(-y)$: An Even Case

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in random Banach spaces: an even case.

Theorem 4.1. *Let X be a linear space, let (Y, μ, T_M) be a complete RN-space and Φ be a mapping from X^2 to D^+ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that, for some $0 < \alpha < 1/16$,*

$$\Phi_{x,y}(\alpha t) \geq \Phi_{2x,2y}(t) \quad (x, y \in X, t > 0). \tag{4.1}$$

Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.5). Then

$$Q(x) := \lim_{n\to\infty} 16^n \left(f\left(\frac{x}{2^{n-1}}\right) - 4f\left(\frac{x}{2^n}\right)\right) \tag{4.2}$$

exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$\mu_{f(2x)-4f(x)-Q(x)}(t) \geq T_M\left(\Phi_{x,x}\left(\frac{1-16\alpha}{5\alpha}t\right), \Phi_{2x,x}\left(\frac{1-16\alpha}{5\alpha}t\right)\right) \tag{4.3}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $x = y$ in (3.5), we get

$$\mu_{f(3y)-6f(2y)+15f(y)}(t) \geq \Phi_{y,y}(t) \tag{4.4}$$

for all $y \in X$ and all $t > 0$.

Replacing x by $2y$ in (3.5), we get

$$\mu_{f(4y)-4f(3y)+4f(2y)+4f(y)}(t) \geq \Phi_{2y,y}(t) \tag{4.5}$$

for all $y \in X$ and all $t > 0$.

By (4.4) and (4.5),

$$\begin{aligned}
&\mu_{f(4x)-20f(2x)+64f(x)}(5t) \\
&\quad\geq T_M\left(\mu_{4(f(3x)-6f(2x)+15f(x))}(4t), \mu_{f(4x)-4f(3x)+4f(2x)+4f(x)}(t)\right) \\
&\quad\geq T_M\left(\Phi_{x,x}(t), \Phi_{2x,x}(t)\right)
\end{aligned} \tag{4.6}$$

for all $x \in X$ and all $t > 0$. Letting $g(x) := f(2x) - 4f(x)$ for all $x \in X$, we get

$$\mu_{g(x)-16g(x/2)}(5t) \geq T_M\left(\Phi_{x/2,x/2}(t), \Phi_{x,x/2}(t)\right) \tag{4.7}$$

for all $x \in X$ and all $t > 0$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 3.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jh(x) := 16h\left(\frac{x}{2}\right) \tag{4.8}$$

for all $x \in X$. It is easy to see that J is a strictly contractive self-mapping on S with the Lipschitz constant 16α.

It follows from (4.7) that

$$\mu_{g(x)-16g(x/2)}(5\alpha t) \geq T_M\left(\Phi_{x,x}(t), \Phi_{2x,x}(t)\right) \tag{4.9}$$

for all $x \in X$ and all $t > 0$. So

$$d(g, Jg) \leq 5\alpha \leq \frac{5}{16} < \infty. \tag{4.10}$$

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, that is,

$$Q\left(\frac{x}{2}\right) = \frac{1}{16}Q(x) \tag{4.11}$$

for all $x \in X$. Since $g : X \to Y$ is even with $g(0) = 0$, $Q : X \to Y$ is an even mapping with $Q(0) = 0$. The mapping Q is a unique fixed point of J in the set

$$M = \{h \in S : d(h, g) < \infty\}. \tag{4.12}$$

This implies that Q is a unique mapping satisfying (4.11) such that there exists a $u \in (0, \infty)$ satisfying

$$\mu_{g(x)-Q(x)}(ut) \geq T_M\left(\Phi_{x,x}(t), \Phi_{2x,x}(t)\right) \tag{4.13}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n g, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 16^n g\left(\frac{x}{2^n}\right) = Q(x) \tag{4.14}$$

for all $x \in X$;

(3) $d(h, Q) \le 1/(1-16\alpha) d(h, Jh)$ for every $h \in M$, which implies the inequality

$$d(g, Q) \le \frac{5\alpha}{1-16\alpha}. \tag{4.15}$$

This implies that the inequality (4.3) holds.

Proceeding as in the proof of Theorem 3.1, we obtain that the mapping $Q : X \to Y$ satisfies $f(x+2y) + f(x-2y) = 2f(x+y) + 2f(-x-y) + 2f(x-y) + 2f(y-x) - 4f(-x) - 2f(x) + f(2y) + f(-2y) - 4f(y) - 4f(-y)$.

Now, we have

$$Q(2x)-16Q(x)=\lim_{n\to\infty}\left[16^n g\left(\frac{x}{2^{n-1}}\right)-16^{n+1}g\left(\frac{x}{2^n}\right)\right]=16\lim_{n\to\infty}\left[16^{n-1}g\left(\frac{x}{2^{n-1}}\right)-16^n g\left(\frac{x}{2^n}\right)\right]=0 \tag{4.16}$$

for every $x \in X$. Since the mapping $x \to Q(2x) - 4Q(x)$ is quartic (see [54, Lemma 2.1]), we get that the mapping $Q : X \to Y$ is quartic. □

Corollary 4.2. *Let $\theta \ge 0$ and let p be a real number with $p > 4$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.26). Then*

$$Q(x) := \lim_{n\to\infty} 16^n \left(f\left(\frac{x}{2^{n-1}}\right) - 4f\left(\frac{x}{2^n}\right)\right) \tag{4.17}$$

exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$\mu_{f(2x)-4f(x)-Q(x)}(t) \ge \frac{(2^p-16)t}{(2^p-16)t + 5(1+2^p)\theta\|x\|^p} \tag{4.18}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.1 by taking

$$\mu_{Df(x,y)}(t) \ge \frac{t}{t + \theta(\|x\|^{\mathrm{p}} + \|y\|^p)} \tag{4.19}$$

for all $x, y \in X$. Then we can choose $\alpha = 2^{-p}$ and we get the desired result. □

Similarly, we can obtain the following. We will omit the proof.

Theorem 4.3. *Let X be a linear space, (Y, μ, T_M) be a complete RN-space and Φ be a mapping from X^2 to D^+ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that, for some $0 < \alpha < 16$,*

$$\Phi_{x,y}(\alpha t) \geq \Phi_{x/2,y/2}(t) \quad (x, y \in X, t > 0). \tag{4.20}$$

Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.5). Then

$$Q(x) := \lim_{n\to\infty} \frac{1}{16^n}\left(f\left(2^{n+1}x\right) - 4f(2^n x)\right) \tag{4.21}$$

exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$\mu_{f(2x)-4f(x)-Q(x)}(t) \geq T_M\left(\Phi_{x,x}\left(\frac{16-\alpha}{5}t\right), \Phi_{2x,x}\left(\frac{16-\alpha}{5}t\right)\right) \tag{4.22}$$

for all $x \in X$ and all $t > 0$.

Corollary 4.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 4$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.26). Then*

$$Q(x) := \lim_{n\to\infty} \frac{1}{16^n}\left(f\left(2^{n+1}x\right) - 4f(2^n x)\right) \tag{4.23}$$

exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$\mu_{f(2x)-4f(x)-Q(x)}(t) \geq \frac{(16-2^p)t}{(16-2^p)t + 5(1+2^p)\theta\|x\|^p} \tag{4.24}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.3 by taking

$$\mu_{Df(x,y)}(t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{4.25}$$

for all $x, y \in X$. Then we can choose $\alpha = 2^p$ and we get the desired result. □

Theorem 4.5. *Let X be a linear space, (Y, μ, T_M) be a complete RN-space and Φ be a mapping from X^2 to D^+ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$)such that, for some $0 < \alpha < 1/4$,*

$$\Phi_{x,y}(\alpha t) \geq \Phi_{2x,2y}(t) \quad (x, y \in X, t > 0). \tag{4.26}$$

Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.5). Then

$$T(x) := \lim_{n\to\infty} 4^n \left(f\left(\frac{x}{2^{n-1}}\right) - 16 f\left(\frac{x}{2^n}\right)\right) \tag{4.27}$$

exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that

$$\mu_{f(2x)-16f(x)-T(x)}(t) \geq T_M\left(\Phi_{x,x}\left(\frac{1-4\alpha}{5\alpha}t\right), \Phi_{2x,x}\left(\frac{1-4\alpha}{5\alpha}t\right)\right) \tag{4.28}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 4.1.

Letting $g(x) := f(2x) - 16f(x)$ for all $x \in X$ in (4.6), we get

$$\mu_{g(x)-4g(x/2)}(5t) \geq T_M(\Phi_{x/2,x/2}(t), \Phi_{x,x/2}(t)) \tag{4.29}$$

for all $x \in X$ and all $t > 0$.

It is easy to see that the linear mapping $J : S \to S$ such that

$$Jh(x) := 4h\left(\frac{x}{2}\right), \tag{4.30}$$

for all $x \in X$, is a strictly contractive self-mapping with the Lipschitz constant 4α.

It follows from (4.29) that

$$\mu_{g(x)-4g(x/2)}(5\alpha t) \geq T_M(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{4.31}$$

for all $x \in X$ and all $t > 0$. So

$$d(g, Jg) \leq 5\alpha < \infty. \tag{4.32}$$

By Theorem 1.1, there exists a mapping $T : X \to Y$ satisfying the following.

(1) T is a fixed point of J, that is,

$$T\left(\frac{x}{2}\right) = \frac{1}{4}T(x) \tag{4.33}$$

for all $x \in X$. Since $g : X \to Y$ is even with $g(0) = 0$, $T : X \to Y$ is an even mapping with $T(0) = 0$. The mapping T is a unique fixed point of J in the set

$$M = \{h \in S : d(h, g) < \infty\}. \tag{4.34}$$

This implies that T is a unique mapping satisfying (4.33) such that there exists a $u \in (0, \infty)$ satisfying

$$\mu_{g(x)-T(x)}(ut) \geq T_M(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{4.35}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n g, T) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 4^n g\left(\frac{x}{2^n}\right) = T(x) \tag{4.36}$$

for all $x \in X$;

(3) $d(h, T) \leq \dfrac{1}{1-4\alpha} d(h, Jh)$ for each $h \in M$, which implies the inequality

$$d(g, T) \leq \frac{5\alpha}{1-4\alpha}. \tag{4.37}$$

This implies that the inequality (4.28) holds.

Proceeding as in the proof of Theorem 4.1, we obtain that the mapping $T : X \to Y$ satisfies $f(x+2y) + f(x-2y) = 2f(x+y) + 2f(-x-y) + 2f(x-y) + 2f(y-x) - 4f(-x) - 2f(x) + f(2y) + f(-2y) - 4f(y) - 4f(-y)$.

Now, we have

$$T(2x) - 4T(x) = \lim_{n\to\infty}\left[4^n g\left(\frac{x}{2^{n-1}}\right) - 4^{n+1} g\left(\frac{x}{2^n}\right)\right] = 4\lim_{n\to\infty}\left[4^{n-1} g\left(\frac{x}{2^{n-1}}\right) - 4^n g\left(\frac{x}{2^n}\right)\right] = 0 \tag{4.38}$$

for every $x \in X$. Since the mapping $x \to T(2x) - 16\, T(x)$ is quadratic (see [54, Lemma 2.1]), we get that the mapping $T : X \to Y$ is quadratic. □

Corollary 4.6. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.26). Then*

$$T(x) := \lim_{n\to\infty} 4^n \left(f\left(\frac{x}{2^{n-1}}\right) - 16 f\left(\frac{x}{2^n}\right)\right) \tag{4.39}$$

exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that

$$\mu_{f(2x)-16f(x)-T(x)}(t) \geq \frac{(2^p-4)t}{(2^p-4)t + 5(1+2^p)\theta\|x\|^p} \tag{4.40}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 4.5 by taking

$$\Phi_{x,y}(t) := \frac{t}{t+\theta(\|x\|^p + \|y\|^p)} \tag{4.41}$$

for all $x, y \in X$. Then we can choose $\alpha = 2^{-p}$ and we get the desired result. □

Similarly, we can obtain the following. We will omit the proof.

Theorem 4.7. *Let X be a linear space, (Y, μ, T_M) be a complete RN-space and Φ be a mapping from X^2 to D^+ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that, for some $0 < \alpha < 4$,*

$$\Phi_{x,y}(\alpha t) \geq \Phi_{x/2,y/2}(t) \quad (x, y \in X, t > 0). \tag{4.42}$$

Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.5). Then

$$T(x) := \lim_{n\to\infty} \frac{1}{4^n}\left(f\left(2^{n+1}x\right) - 16f(2^n x)\right) \tag{4.43}$$

exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that

$$\mu_{f(2x)-16f(x)-T(x)}(t) \geq T_M\left(\Phi_{x,x}\left(\frac{4-\alpha}{5}t\right), \Phi_{2x,x}\left(\frac{4-\alpha}{5}t\right)\right) \tag{4.44}$$

for all $x \in X$ and all $t > 0$.

Corollary 4.8. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.26). Then*

$$T(x) := \lim_{n\to\infty} \frac{1}{4^n}\left(f\left(2^{n+1}x\right) - 16f(2^n x)\right) \tag{4.45}$$

exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that

$$\mu_{f(2x)-16f(x)-T(x)}(t) \geq \frac{(4-2^p)t}{(4-2^p)t + 5(1+2^p)\theta\|x\|^p} \tag{4.46}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 4.7 by taking

$$\Phi_{x,y}(t) := \frac{t}{t+\theta(\|x\|^p + \|y\|^p)} \tag{4.47}$$

for all $x, y \in X$. Then we can choose $\alpha = 2^p$ and we get the desired result. □

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