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Research Article

Applications of a Weighted Symmetrization Inequality to Elastic Membranes and Plates

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This paper is devoted to some applications of a weighted symmetrization inequality related to a second order boundary value problem. We first interpret the inequality in the context of elastic membranes, and observe that it lends itself to make a comparison between the deflection of a membrane with a varying density with that of a membrane with a uniform density. Some mathematical consequences of the inequality including a stability result are presented. Moreover, a similar inequality where the underlying differential equation is of fourth order is also discussed.

1. Introduction

In this paper we discuss some applications of a weighted symmetrization inequality related to a second-order boundary value problem. We begin by interpreting the inequality in the context of elastic membranes. Let us briefly describe the physical situation and its mathematical formulation that leads to the inequality we are interested in. An elastic membrane of varying density $a(x)$ is occupying a region Ω, a disk in the plane $\mathbb{R}^2$. The membrane is fixed at the boundary and is subject to a load $f(x)h(x)$. The governing equation in terms of the deflection function $u(x)$ is the elliptic boundary value problem

$$\begin{aligned} -\nabla\cdot(a(x)\nabla u) &= f(x)h(x), \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega \end{aligned} \tag{P}$$

On the other hand, the following boundary value problem models a membrane with uniform density:

$$\begin{aligned} -C\Delta v &= f_\mu^*(x), \quad \text{in } \Omega_\mu^*, \\ v &= 0, \quad \text{on } \partial\Omega_\mu^*, \end{aligned} \tag{S}$$

where C is a constant depending on $a(x)$ and $h(x)$, whereas Ω_μ^* and f_μ^* denote symmetrizations of Ω and f, with respect to the measure μ, respectively; see thefollowing section for precise notation and definitions. We call (S) the symmetrization of (P). In [1], see also [2], the following weighted symmetrization inequality is proved:

$$u_\mu^*(x) \le v(x), \quad x \in \Omega_\mu^*, \tag{1.1}$$

where u and v are solutions of (P) and (S), respectively. Physically, (1.1) implies that the deflection of the membrane with varying density, after symmetrization, is dominated by that of the membrane with uniform density.

The aim of the present paper is to point out some applications of (1.1). In particular, we prove the following inequality:

$$\int_\Omega a(x)|\nabla u|^2 dx \le C \int_{\Omega_\mu^*} |\nabla v|^2 dx. \tag{1.2}$$

We also address the case of equality in (1.2). In case $a(x) = 1$, the constant C in (1.2) is simply equal to 1; hence, (1.2) reduces to the well-known Pólya-Szegö inequality; see, for example, [3, 4]. Inequality (1.2) deserves to be added to the standard list of existing rearrangement inequalities since it can serve, mathematically, physical situations in which the object, whether it is a membrane, plate, or so forth, is made of several materials.

Once (1.2) is proved, we then present a stability result. Finally, the paper ends with a weighted rearrangement inequality related to a fourth-order boundary value problem. More precisely, we introduce

$$\begin{gathered} \nabla \cdot \left(a(x) \nabla \left(\frac{1}{h} \nabla \cdot (b(x) \nabla u) \right) \right) = f(x) h(x), \quad \text{in } \Omega, \\ u = \nabla \cdot (b(x) \nabla u) = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{PH}$$

and the symmetrization of (PH):

$$\begin{gathered} \Delta^2 v = f_\mu^*(x), \quad \text{in } \Omega_\mu^*, \\ v = \Delta v = 0, \quad \text{on } \partial\Omega_\mu^*. \end{gathered} \tag{SH}$$

We prove that

$$u_\mu^*(x) \le C v(x), \quad x \in \Omega_\mu^*, \tag{1.3}$$

where C is a constant depending on $a(x)$, $b(x)$, and $h(x)$.

2. Preliminaries

Henceforth $\Omega \subset \mathbb{R}^2$ denotes a disk centered at the origin. Suppose that (Ω, μ) is a measurable space. In the following three definitions we assume that $f : \Omega \to [0, \infty)$ is μ-measurable; see, for example, [5] for further reading.

Definition 2.1. The distribution function of f, with respect to μ, denoted as $\lambda_{f,\mu}$, is defined by

$$\lambda_{f,\mu}(\alpha) = \mu(\{x \in \Omega : f(x) \geq \alpha\}), \quad \alpha \in [0, \infty). \tag{2.1}$$

Definition 2.2. The decreasing rearrangement of f, with respect to μ, denoted as f_μ^Δ, is defined by

$$f_\mu^\Delta(s) = \inf\{\alpha : \lambda_{f,\mu}(\alpha) < s\}, \quad s \in [0, \mu(\Omega)]. \tag{2.2}$$

Definition 2.3. The decreasing radial symmetrization of f, with respect to μ, denoted f_μ^*, is defined by

$$f_\mu^*(x) = f_\mu^\Delta\left(\pi |x|^2\right), \quad x \in \Omega_\mu^*, \tag{2.3}$$

where Ω_μ^* is the ball centered at the origin with radius $(\mu(\Omega)/\pi)^{1/2}$.

In the following section we will use the following result which seems to have been overlooked in Theorem 7.1 in [1]. In the literature this result is usually referred to as the weighted Hardy-Littlewood inequality; see [5].

Lemma 2.4. *Let $f : \Omega \to [0, \infty)$ and $g : \Omega \to [0, \infty)$ be μ-measurable functions. Then*

$$\int_\Omega f g \, d\mu \leq \int_0^{\mu(\Omega)} f_\mu^\Delta(s) g_\mu^\Delta(s) ds, \tag{2.4}$$

provided the integrals converge.

Proof. See Theorem 1 in [3, 4]. □

An immediate consequence of (2.4) is the following.

Corollary 2.5. *Let $f : \Omega \to [0, \infty)$ and $g : \Omega \to [0, \infty)$ be μ-measurable functions. Then*

$$\int_\Omega f g \, d\mu \leq \int_{\Omega_\mu^*} f_\mu^*(x) g_\mu^*(x) dx, \tag{2.5}$$

provided the integrals converge.

Proof. From (2.4), we have

$$\int_{\Omega} fg\,d\mu \leq \int_0^{\mu(\Omega)} f_\mu^\Delta(s) g_\mu^\Delta(s)ds. \tag{2.6}$$

Hence, by changing the variable $s = \pi r^2$, we obtain

$$\int_{\Omega} fg\,d\mu \leq 2\pi \int_0^{(\mu(\Omega)/\pi)^{1/2}} f_\mu^\Delta\left(\pi r^2\right) g_\mu^\Delta\left(\pi r^2\right) r\,dr = \int_{\Omega_\mu^*} f_\mu^*(x) g_\mu^*(x)dx, \tag{2.7}$$

as desired. □

Definition 2.6. A pair $(h, a) \in C(\overline{\Omega}) \times C(\overline{\Omega})$ is called admissible if and only if the following conditions hold.

(i) $a(x) \geq a_0 > 0$, for some constant a_0.

(ii) h is almost radial in the sense that there exists a radial function $h_0 \geq 0$ such that

$$ch_0(x) \leq h(x) \leq h_0(x), \quad \text{in } \Omega, \tag{2.8}$$

for some $c \in (0, 1]$.

(iii) There exists $K > 0$ such that

$$s(r) \geq Kr\left(\frac{h_0(r)}{a(x)}\right)^{1/2}, \qquad \frac{ds}{dr} \geq K\left(\frac{h_0(r)}{a(x)}\right)^{1/2}, \tag{2.9}$$

where $r = |x|$, $x \in \Omega$. Here, $s(r)$ is the solution to the initial value problem

$$s\frac{ds}{dr} = rh_0(r), \qquad s(0) = 0, \tag{2.10}$$

in $(0, R)$, where R is the radius of the ball Ω.

The following result is a special case of Theorem 3.1 in [1].

Theorem 2.7. *Suppose that $(h, a) \in C(\overline{\Omega}) \times C(\overline{\Omega})$ is admissible. Suppose that $f \in C(\overline{\Omega})$ is a nonnegative function, $d\mu = h(x)dx$, and $C := Kc^2$, where K and c are the constants in Definition 2.6, corresponding to the pair (h, a). Let $u \in W_0^{1,2}(\Omega)$ and $v \in W_0^{1,2}(\Omega_\mu^*)$ be solutions of (P) and (S), respectively. Then*

$$u_\mu^*(x) \leq v(x), \tag{2.11}$$

for $x \in \Omega_\mu^*$.

Remark 2.8. In case $h(x) = 1$, in Theorem 2.7, that is, $d\mu$ coincides with the usual Lebesgue measure, (2.11) reduces to the classical symmetrization inequality; see, for example, [6, 7].

3. Main Results

Our first main result is the following.

Theorem 3.1. *Suppose that* $(h, a) \in C(\overline{\Omega}) \times C(\overline{\Omega})$ *is admissible,* $f \in C(\overline{\Omega})$ *is non-negative, and* $d\mu = h(x)dx$. *Suppose that* $u \in W_0^{1,2}(\Omega)$ *satisfies*

$$\begin{gathered} -\nabla \cdot (a(x)\nabla u) = fh, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{3.1}$$

Suppose that $v \in W_0^{1,2}(\Omega_\mu^*)$ *satisfies*

$$\begin{gathered} -C\Delta v = f_\mu^*, \quad \text{in } \Omega_\mu^*, \\ v = 0, \quad \text{on } \partial\Omega_\mu^*, \end{gathered} \tag{3.2}$$

where $C := Kc^2$. *Then*

$$\int_\Omega a(x)|\nabla u|^2 dx \le C \int_{\Omega_\mu^*} |\nabla v|^2 dx. \tag{3.3}$$

In addition, if equality holds in (3.3), *then*

$$u_\mu^*(x) = v(x), \quad x \in \Omega_\mu^*. \tag{3.4}$$

Proof. Multiplying the differential equation in (3.1) by u and integrating over Ω yield

$$\int_\Omega a(x)|\nabla u|^2 d\mu = \int_\Omega fu\, d\mu. \tag{3.5}$$

Now we can apply Corollary 2.5 to the right-hand side of the above equation to deduce

$$\int_\Omega a(x)|\nabla u|^2 d\mu \le \int_{\Omega_\mu^*} f_\mu^*(x) u_\mu^*(x) dx. \tag{3.6}$$

Hence, by (2.11), we obtain

$$\int_\Omega a(x)|\nabla u|^2 d\mu \le \int_{\Omega_\mu^*} f_\mu^*(x) v(x) dx. \tag{3.7}$$

Next, we multiply the differential equation in (3.2) by v and integrate over Ω_μ^* to obtain

$$C\int_{\Omega_\mu^*}|\nabla v|^2dx = \int_{\Omega_\mu^*} f_\mu^*(x)v(x)dx. \tag{3.8}$$

From (3.7) and (3.8), we obtain (3.3).

Now we assumes equality holds in (3.3). This, in conjunction with (3.6) and (3.7), yield that

$$\int_{\Omega_\mu^*} f_\mu^*(x)u_\mu^*(x)dx = \int_{\Omega_\mu^*} f_\mu^*(x)v(x)dx. \tag{3.9}$$

Hence

$$\int_{\Omega_\mu^*} f_\mu^*(x)\Big(v(x)-u_\mu^*(x)\Big)dx = 0. \tag{3.10}$$

Since $v(x)-u_\mu^*(x)\ge 0$, thanks to (2.11), we infer that $v(x)=u_\mu^*(x)$, over the set $\{x\in\Omega_\mu^* : f_\mu^*(x)>0\}$. In particular, it follows that $v(0)=u_\mu^*(0)$. At this point, we recall the function

$$\xi(t)=\frac{1}{4\pi C}\Big(u_\mu^\Delta(t)\Big)^{-1}\Big(-u_\mu^\Delta(t)\Big)'\int_{\{x\in\Omega_\mu^*:u_\mu^*(x)>t\}} f_\mu^*(y)dy, \tag{3.11}$$

which was implicitly used in the proof of Theorem 3.1 in [1]. This function satisfies

(a) $\xi(t)\ge 1$, for almost every $t\in[0,u_\mu^*(0)]$,

(b) $\int_0^{u_\mu^*(x)}\xi(t)dt = v(x)$, for every $x\in\Omega_\mu^*$.

We claim that $\xi(t)=1$. To derive a contradiction, let us assume that the assertion in the claim is false, that is, there is a set of positive measure upon which $\xi(t)>1$. In this case, by (a), we obtain $\int_0^{u_\mu^*(0)}\xi(t)dt > u_\mu^*(0)$. However, by (b), $\int_0^{u_\mu^*(0)}\xi(t)dt = v(0)$; hence $u_\mu^*(0)<v(0)$, which is a contradiction. Finally, since $\xi(t)=1$, we can apply (b) again to deduce $u_\mu^*(x)=v(x)$, for $x\in\Omega_\mu^*$, as desired. □

As mentioned in the introduction, we prove a stability result.

Theorem 3.2. *Let $(h_n,a)\in C(\overline{\Omega})\times C(\overline{\Omega})$, $n\in\mathbb{N}$, be admissible. Suppose that $C_n := K_n^2c_n$ converges to, say, $C>0$. In addition, suppose that the sequence $\{h_n\}$ is decreasing and pointwise convergent to $h\in C(\overline{\Omega})$. Suppose that $f\in C(\overline{\Omega})$ is a non-negative function, and $d\mu_n = h_n(x)dx$. Let $u_n\in W_0^{1,2}(\Omega)$ satisfy*

$$\begin{gathered}-\nabla\cdot(a(x)\nabla u_n) = fh_n, \quad \text{in } \Omega,\\ u_n = 0, \quad \text{on } \partial\Omega,\end{gathered} \tag{3.12}$$

and let $v_n \in W_0^{1,2}(\Omega^*_{\mu_n})$ *satisfy*

$$\begin{gathered} -C_n \Delta v_n = f^*_{\mu_n}, \quad in\ \Omega^*_{\mu_n}, \\ v_n = 0, \quad on\ \partial\Omega^*_{\mu_n}. \end{gathered} \tag{3.13}$$

Then, there exist $\widehat{u} \in W_0^{1,2}(\Omega)$ *and* $\widehat{v} \in W_0^{1,2}(\Omega^*_\mu)$ *such that*

$$\begin{gathered} -\nabla \cdot (a(x)\nabla \widehat{u}) = fh, \quad in\ \Omega, \\ \widehat{u} = 0, \quad on\ \partial\Omega, \end{gathered} \tag{3.14}$$

$$\begin{gathered} -C\Delta \widehat{v} = f^*_\mu, \quad in\ \Omega^*_\mu, \\ \widehat{v} = 0, \quad on\ \partial\Omega^*_\mu, \end{gathered} \tag{3.15}$$

where $d\mu := h(x)dx$. *Moreover,*

$$\widehat{u}^*_\mu(x) \le \widehat{v}(x), \tag{3.16}$$

for $x \in \Omega^*_\mu$.

Proof. Since $\{h_n\}$ is decreasing, we can apply the Maximum Principle, see, for example, [8], to deduce that $\{u_n\}$ is also decreasing. On the other hand, it is easy to show that $\{u_n\}$ is a Cauchy sequence in $W_0^{1,2}(\Omega)$; hence there exists $\widehat{u} \in W_0^{1,2}(\Omega)$ such that $u_n \to \widehat{u}$, in $W_0^{1,2}(\Omega)$. Multiplying the differential equation in (3.12) by an arbitrary $u \in W_0^{1,2}(\Omega)$ and integrating over Ω yield

$$\int_\Omega a(x)\nabla u_n \cdot \nabla u\, dx = \int_\Omega f h_n u\, dx. \tag{3.17}$$

Hence, taking the limit as $n \to \infty$, keeping in mind that $h_n \to h$ and $\nabla u_n \to \nabla \widehat{u}$, in $L^2(\Omega)$, we obtain

$$\int_\Omega a(x)\nabla \widehat{u} \cdot \nabla u\, dx = \int_\Omega f h u\, dx. \tag{3.18}$$

Thus, since u is arbitrary, $\widehat{u}$ verifies (3.14), as desired.

Next we prove existence of $\widehat{v}$ such that $v_n \to \widehat{v}$, in $W_0^{1,2}(\Omega^*_\mu)$, and verify (3.15). We proceed in this direction by first showing that

$$f^*_{\mu_n}(x) \longrightarrow f^*_\mu(x) \tag{3.19}$$

for $x \in \Omega_\mu^*$. Indeed, since $\{h_n\}$ is decreasing, the sequence $\{\lambda_{f,\mu_n}\}$ is also decreasing. This, in turn, implies that $\{f_{\mu_n}^\Delta\}$ is decreasing. Moreover, by the Lebesgue Dominated Convergence Theorem, we have

$$\lambda_{f,\mu_n}(\alpha) = \int_{\{x\in\Omega: f(x)\geq\alpha\}} h_n(x)dx \longrightarrow \int_{\{x\in\Omega: f(x)\geq\alpha\}} h(x)dx, \quad \text{as } n \longrightarrow \infty. \tag{3.20}$$

Since $\lambda_{f,\mu_n}(\alpha) \geq \lambda_{f,\mu}(\alpha)$, we can apply Definition 2.3 to infer that $f_\mu^\Delta(s) \leq f_{\mu_n}^\Delta(s)$, $s \in [0, \mu(\Omega)]$. Now, fix $s \in [0, \mu(\Omega)]$, and consider an arbitrary $\eta > 0$. Then, $f_\mu^\Delta(s) + \eta > \alpha$, for some α satisfying $\lambda_{f,\mu}(\alpha) < s$. Since $\lim_{n\to\infty}\lambda_{f,\mu_n}(\alpha) = \lambda_{f,\mu}(\alpha)$, it follows that $\lambda_{f,\mu}(\alpha) \leq \lambda_{f,\mu_n}(\alpha) < s$, for $n \geq n_0$, for some $n_0 \in \mathbb{N}$. Therefore, again from Definition 2.3, we deduce $f_{\mu_n}^\Delta(s) \leq \alpha$, for $n \geq n_0$. In conclusion, we have

$$f_{\mu_n}^\Delta(s) - \eta \leq f_\mu^\Delta(s) \leq f_{\mu_n}^\Delta(s), \quad n \geq n_0. \tag{3.21}$$

This implies that $|f_{\mu_n}^\Delta(s) - f_\mu^\Delta(s)| < \eta$, $n \geq n_0$. Since η is arbitrary, we deduce $\lim_{n\to\infty} f_{\mu_n}^\Delta(s) = f_\mu^\Delta(s)$, that is, (3.19) is verified. By taking the zero extensions of v_n and $f_{\mu_n}^*$ outside $\Omega_{\mu_n}^*$, we can apply (3.19), keeping in mind that $C_n \to C$, to deduce that $\{v_n\}$ is a Cauchy sequence in $W_0^{1,2}(\Omega_{\mu_1}^*)$. Hence, there exists $\widehat{v} \in W_0^{1,2}(\Omega_{\mu_1}^*)$ such that $v_n \to \widehat{v}$, in $W_0^{1,2}(\Omega_{\mu_1}^*)$. Next, for an arbitrary $v \in W_0^{1,2}(\Omega_\mu^*)$, extended to all of $\Omega_{\mu_1}^*$ by setting $v = 0$ in $\Omega_{\mu_1}^* \setminus \Omega_\mu^*$, we derive

$$C\int_{\Omega_{\mu_1}^*} \nabla\widehat{v} \cdot \nabla v \, dx = \int_{\Omega_{\mu_1}^*} f_\mu^* v \, dx. \tag{3.22}$$

Since $v_n = 0$ on $\Omega_{\mu_1}^* \setminus \Omega_{\mu_n}^*$, it is clear that $\widehat{v} = 0$ on $\partial\Omega_\mu^*$. This, coupled with (3.22), implies that $\widehat{v}$ satisfy (3.15). If (3.15) were the symmetrization of (3.14), then (3.16) would follow from (2.11). However, this is not known to us a priori. Therefore, in order to derive (3.16), we first apply Theorem 2.7 to (3.12) and (3.13) to obtain

$$(u_n)_{\mu_n}^*(x) \leq v_n(x), \quad x \in \Omega_{\mu_n}^*. \tag{3.23}$$

Since $\{u_n\}$ and $\{h_n\}$ are decreasing, and, in addition, $u_n \to \widehat{u}$, $h_n \to h$, pointwise; after passing to a subsequence, if necessary, we can use similar arguments to those used in the proof of (3.19) to show that

$$\lim_{n\to\infty} (u_n)_{\mu_n}^*(x) = (\widehat{u})_\mu^*(x), \quad x \in \Omega_\mu^*. \tag{3.24}$$

Therefore, by taking the limit $n \to \infty$, in (3.23), we derive (3.16), as desired. □

Our next result concerns problems (PH) and (SH).

Theorem 3.3. *Suppose that $(h, a) \in C(\overline{\Omega}) \times C(\overline{\Omega})$ and $(h, b) \in C(\overline{\Omega}) \times C(\overline{\Omega})$ are admissible; in addition, $h(x) > 0$. Suppose that $f \in C(\overline{\Omega})$ is non-negative. Suppose that u and v satisfy (PH) and (SH), respectively, where $d\mu = h(x)dx$. Then*

$$u_\mu^*(x) \le Cv(x), \quad x \in \Omega_\mu^*, \tag{3.25}$$

where C is a constant depending on $a(x)$, $b(x)$, and $h(x)$.

Proof. We begin by setting $U := -(1/h)\nabla \cdot (b(x)\nabla u)$. Then, we obtain

$$\begin{gathered} -\nabla \cdot (b(x)\nabla u) = hU, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{3.26}$$

and, by (PH),

$$\begin{gathered} -\nabla \cdot (a(x)\nabla U) = hf, \quad \text{in } \Omega, \\ U = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{3.27}$$

Since (h, a) is admissible, we can apply Theorem 2.7 to (3.27), and obtain

$$U_\mu^*(x) \le w(x), \quad x \in \Omega_\mu^*, \tag{3.28}$$

where w satisfies

$$\begin{gathered} -C_1 \Delta w = f_\mu^*, \quad \text{in } \ \Omega_\mu^*, \\ w = 0, \quad \text{on } \ \partial\Omega_\mu^*, \end{gathered} \tag{3.29}$$

for $C_1 := K_1^2 c$, where K_1 and c are the constants in Definition 2.6, corresponding to the pair (h, a). Similarly, since (h, b) is admissible, another application of Theorem 2.7, to (3.26), yields

$$u_\mu^*(x) \le \mathcal{J}(x), \quad x \in \Omega_\mu^*, \tag{3.30}$$

where $\mathcal{J}$ satisfies

$$\begin{gathered} -C_2 \Delta \mathcal{J} = U_\mu^*, \quad \text{in } \Omega_\mu^*, \\ \mathcal{J} = 0, \quad \text{on } \partial\Omega_\mu^*, \end{gathered} \tag{3.31}$$

for $C_2 := K_2 c$, where K_2 and c are the constants in Definition 2.6, corresponding to the pair (h, b). From (3.28) and (3.31), we deduce $-C_2 \Delta \mathcal{J} \le w$, in Ω_μ^*. On the other hand, we know that $C_1 w = -\Delta v$, where v is the solution of (SH). Thus, $-C_1 C_2 \Delta \mathcal{J} \le -\Delta v$, in Ω_μ^*. Since $\mathcal{J} = v = 0$, on $\partial\Omega_\mu^*$, we can apply the Maximum Principle to deduce $C_1 C_2 \mathcal{J} \le v$, in Ω_μ^*. The latter inequality, coupled with (3.30), implies that $u_\mu^* \leq (1/C_1C_2)v$, in Ω_μ^*. Setting $C := 1/C_1C_2$, we derive (3.25), as desired. □

Remark 3.4. The result in Theorem 3.3 can be interpreted in the context of plates with hinged boundaries. The inequality (3.25) implies that the deflection of a plate, with varying density, hinged at the boundary, is dominated by the deflection of another plate, similarly hinged at the boundary, with uniform density. See [9, 10] for similar results.

The last result of this paper is somewhat similar to the result of Theorem 3.3, but the reader should take note that the underlying differential equation in the next result is *different* from that in Theorem 3.3.

Theorem 3.5. *Suppose that $(h, 1) \in C(\overline{\Omega}) \times C(\overline{\Omega})$ is admissible. Suppose that $h(x) \geq 1$ in $\overline{\Omega}$, and $f \in C(\overline{\Omega})$ is non-negative. Let u and v satisfy*

$$\Delta^2 u = fh, \quad \text{in } \Omega, \qquad u = \Delta u = 0, \quad \text{on } \partial\Omega, \tag{3.32}$$

$$\Delta^2 v = f_\mu^*, \quad \text{in } \Omega_\mu^*, \qquad v = \Delta v = 0, \quad \text{on } \partial\Omega_\mu^*, \tag{3.33}$$

respectively. Then

$$u_e^*(x) \leq Cv(x), \quad x \in \Omega_\mu^*, \tag{3.34}$$

where C is a constant depending on h_0. Here u_e^* denotes the decreasing radial symmetrization of u, with respect to the Lebesgue measure, extended to Ω_μ^* by setting $u_e^*(x) = 0$ for $x \in \Omega_\mu^* \setminus \Omega^*$, where Ω^* is the symmetrization of Ω with respect to the Lebesgue measure, that is, $\Omega^* = \Omega$.

Proof. As in the proof of Theorem 3.3, we set $U = -\Delta u$. Then, by (3.32), we obtain

$$-\Delta u = U, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega, \tag{3.35}$$

$$-\Delta U = fh, \quad \text{in } \Omega, \qquad U = 0, \quad \text{on } \partial\Omega. \tag{3.36}$$

Since $(h, 1)$ is admissible, we can apply Theorem 2.7 to (3.36), and obtain

$$U_\mu^*(x) \leq w(x), \quad x \in \Omega_\mu^*, \tag{3.37}$$

where w satisfies

$$-C_1 \Delta w = f_\mu^*, \quad \text{in } \Omega_\mu^*,$$
$$w = 0, \quad \text{on } \partial\Omega_\mu^*, \tag{3.38}$$

where C_1 is a constant related to admissibility of $(h, 1)$. On the other hand, applying Theorem 2.7 to (3.35), with $d\mu = dx$, yields

$$u^*(x) \leq \mathcal{I}(x), \quad x \in \Omega^* = \Omega, \tag{3.39}$$

where $\mathcal{I}$ satisfies

$$-\Delta\mathcal{I} = U^*, \quad \text{in } \Omega,$$
$$\mathcal{I} = 0, \quad \text{on } \partial\Omega. \tag{3.40}$$

Since $h(x) \geq 1$, it readily follows that $U_\mu^*(x) \geq U^*(x)$, for $x \in \Omega$. This, in conjunction with (3.37) and (3.40), implies that

$$-\Delta\mathcal{I}(x) \leq U_\mu^*(x) \leq w(x), \quad x \in \Omega. \tag{3.41}$$

Note that, from (3.33) and (3.34), we deduce $C_1 w = -\Delta v$ in Ω_μ^*. So, because $\Omega \subseteq \Omega_\mu^*$, it follows that $-\Delta\mathcal{I} \leq -(1/C_1)\Delta v$, in Ω. In addition, on $\partial\Omega$, $\mathcal{I} = 0$, while v is positive, as a consequence of the Maximum Principle. Thus, by another application of the Maximum Principle, we infer that $\mathcal{I} \leq (1/C_1)v$, in Ω. This, coupled with (3.39), implies that $u^* \leq (1/C_1)v$, in Ω. Since $v > 0$ in Ω_μ^*, it follows that $u_e^* \leq Cv$, in Ω_μ^*, where $C := 1/C_1$, as desired. □

Remark 3.6. All results presented in this paper can easily be extended to higher dimensions; only simple technical adjustments are required.

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