Research Article

Doob's Type Inequality and Strong Law of Large Numbers for Demimartingales

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We establish some maximal inequalities for demimartingales which generalize the result of Wang (2004). The maximal inequality for demimartingales is used as a key inequality to establish other results including Doob's type maximal inequality, strong law of large numbers, strong growth rate, and integrability of supremum for demimartingales, which generalize and improve partial results of Christofides (2000) and Prakasa Rao (2007).

1. Introduction

Definition 1.1. Let $S_1, S_2, \ldots$ be an L^1 sequence of random variables. Assume that for $j = 1, 2, \ldots$

$$E\{(S_{j+1} - S_j)f(S_1, \ldots, S_j)\} \geq 0 \tag{1.1}$$

for all coordinatewise nondecreasing functions f such that the expectation is defined. Then $\{S_j, j \geq 1\}$ is called a demimartingale. If in addition the function f is assumed to be nonnegative, then the sequence $\{S_j, j \geq 1\}$ is called a demisubmartingale.

Definition 1.2. A finite collection of random variables $X_1, X_2, \ldots, X_m$ is said to be associated if

$$\text{Cov}\{f(X_1, \ldots, X_m), g(X_1, \ldots, X_m)\} \geq 0 \tag{1.2}$$

for any two coordinatewise nondecreasing functions f, g on $\mathbb{R}^m$ such that the covariance is defined. An infinite sequence $\{X_n, n \geq 1\}$ is associated if every finite subcollection is associated.

Definition 1.3. A finite collection of random variables $X_1, X_2, \ldots, X_n$ is said to be strongly positive dependent if

$$P\left(\overline{X}_1 \in \Lambda_1; \overline{X}_2 \in \Lambda_2\right) \geq P\left(\overline{X}_1 \in \Lambda_1\right) P\left(\overline{X}_2 \in \Lambda_2\right) \tag{1.3}$$

for all Borel measurable and increasing (or decreasing) set pairs $(\Lambda_1, \Lambda_2) \subset R_1 \times R_2$ (A set Λ is said increasing (or decreasing) if $x \leq$ (or $\geq$)y implies $y \in \Lambda$ for any $x \in \Lambda$), where

$$\overline{X}_1 = (X_i, i \in I), \qquad \overline{X}_2 = (X_i, i \in I^c), \quad I \subset (1, 2, \ldots, n), I^c = (1, 2, \ldots, n) \setminus I,$$
$$R_1 = \mathbb{R}^{|I|}, \qquad R_2 = \mathbb{R}^{|I^c|} \quad (|I| \text{ stands for the base of } I). \tag{1.4}$$

An infinite sequence $\{X_n, n \geq 1\}$ is strongly positive dependent if every finite subcollection is strongly positive dependent.

Remark 1.4. Chow [1] proved a maximal inequality for submartingales. Newman and Wright [2] extended Doob's maximal inequality and upcrossing inequality to the case of demimartingales, and pointed out that the partial sum of a sequence of mean zero associated random variables is a demimartingale. Christofides [3] showed that the Chow's maximal inequality for (sub)martingales can be extended to the case of demi(sub)martingales. Wang [4] obtained Doob's type inequality for more general demimartingales. Hu et al. [5] gave a strong law of large numbers and growth rate for demimartingales. Prakasa Rao [6] established some maximal inequalities for demisubmartingales and N-demisuper-martingales.

It is easily seen that the partial sum of a sequence of mean zero strongly positive dependent random variables is also a demimartingale by the inequality (3) in Zheng [7], that is, for all $n \geq 1$,

$$E\{f(S_1, \ldots, S_n)(S_{n+1} - S_n)\} = E\{f(X_1, X_1 + X_2, \ldots, X_1 + X_2 + \cdots + X_n)X_{n+1}\} \geq 0 \tag{1.5}$$

for all coordinatewise nondecreasing functions f such that the expectation is defined. Therefore, the main results of this paper hold for the partial sums of sequences of mean zero associated random variables and strongly positive dependent random variables.

Let $\{X_n, n \geq 1\}$ and $\{S_n, n \geq 1\}$ be sequences of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$ and $I(A)$ the indicator function of the event A. Denote $S_0 = 0$, $X^+ = \max(0, X)$, $X^- = \max(0, -X)$, $\log x = \log_e x = \ln x$, $\log^+ x = \ln(\max(x, 1))$. The main results of this paper depend on the following lemmas.

Lemma 1.5 (see Wang [4, Theorem 2.1]). *Let $\{S_n, n \geq 1\}$ be a demimartingale and g a nonnegative convex function on $\mathbb{R}$ with $g(0) = 0$ and $g(S_i) \in L^1, i \geq 1$. Let $\{c_k, k \geq 1\}$ be a nonincreasing sequence of positive numbers. Then for any $\varepsilon > 0$,*

$$\varepsilon P\left[\max_{1 \leq k \leq n} c_k g(S_k) \geq \varepsilon\right] \leq \sum_{j=1}^{n} c_j E\left[(g(S_j) - g(S_{j-1}))I\left(\max_{1 \leq k \leq n} c_k g(S_k) \geq \varepsilon\right)\right]. \tag{1.6}$$

Lemma 1.6 (see Fazekas and Klesov [8, Theorem 2.1] and Hu et al. [5, Lemma 1.5]). *Let $\{X_n, n \geq 1\}$ be a random variable sequence and $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$. Let $b_1, b_2, \ldots$ be a nondecreasing unbounded sequence of positive numbers and $\alpha_1, \alpha_2, \ldots$ nonnegative numbers. Let p and C be fixed positive numbers. Assume that for each $n \geq 1$,*

$$E\left(\max_{1 \leq l \leq n} |S_l|\right)^p \leq C \sum_{l=1}^{n} \alpha_l, \qquad \sum_{l=1}^{\infty} \frac{\alpha_l}{b_l^p} < \infty, \tag{1.7}$$

then

$$\lim_{n \to \infty} \frac{S_n}{b_n} = 0 \quad a.s., \tag{1.8}$$

and with the growth rate

$$\frac{S_n}{b_n} = O\left(\frac{\beta_n}{b_n}\right) \quad a.s., \tag{1.9}$$

where

$$\beta_n = \max_{1 \leq k \leq n} b_k v_k^{\delta/p}, \quad \forall 0 < \delta < 1, \quad v_n = \sum_{k=n}^{\infty} \frac{\alpha_k}{b_k^p}, \quad \lim_{n \to \infty} \frac{\beta_n}{b_n} = 0. \tag{1.10}$$

In addition,

$$E\left(\max_{1 \leq l \leq n} \left|\frac{S_l}{b_l}\right|\right)^p \leq 4C \sum_{l=1}^{n} \frac{\alpha_l}{b_l^p} < \infty, \qquad E\left(\sup_{l \geq 1} \left|\frac{S_l}{b_l}\right|\right)^p \leq 4C \sum_{l=1}^{\infty} \frac{\alpha_l}{b_l^p} < \infty. \tag{1.11}$$

If further assumes one that $\alpha_n > 0$ for infinitely many n, then

$$E\left(\sup_{l \geq 1} \left|\frac{S_l}{\beta_l}\right|\right)^p \leq 4C \sum_{l=1}^{\infty} \frac{\alpha_l}{\beta_l^p} < \infty. \tag{1.12}$$

Lemma 1.7 (see Christofides [3, Lemma 2.1, Corollary 2.1]). (i) *If $\{S_n, n \geq 1\}$ is a demisubmartingale (or a demimartingale) and g is a nondecreasing convex function such that $g(S_i) \in L^1, i \geq 1$, then $\{g(S_n), n \geq 1\}$ is a demisubmartingale.*

(ii) *If $\{S_n, n \geq 1\}$ is a demimartingale, then $\{S_n^+, n \geq 1\}$ is a demisubmartingale and $\{S_n^-, n \geq 1\}$ is a demisubmartingale.*

Lemma 1.8 (see Hu et al. [9, Theorem 2.1]). *Let $\{S_n, n \geq 1\}$ be a demimartingale and $\{c_k, k \geq 1\}$ be a nonincreasing sequence of positive numbers. Let $\nu \geq 1$ and $E|S_k|^\nu < \infty$ for each k, then for any $\varepsilon > 0$ and $1 \leq n \leq N$,*

$$P\left\{\max_{n \leq k \leq N} c_k |S_k| \geq \varepsilon\right\} \leq \frac{2}{\varepsilon^\nu}\left\{c_n^\nu E|S_n|^\nu + \sum_{k=n+1}^{N} c_k^\nu E(|S_k|^\nu - |S_{k-1}|^\nu)\right\}. \tag{1.13}$$

Lemma 1.9 (see Christofides [3, Corollary 2.4, Theorem 2.1]). (i) *Let $\{S_n, n \geq 1\}$ be a demisubmartingale. Then for any $\varepsilon > 0$,*

$$\varepsilon P\left(\max_{1 \leq k \leq n} S_k \geq \varepsilon\right) \leq \int_{(\max_{1 \leq k \leq n} S_k \geq \varepsilon)} S_n dP. \tag{1.14}$$

(ii) *Let $\{S_n, n \geq 1\}$ be a demisubmartingale and $\{c_k, k \geq 1\}$ a nonincreasing sequence of positive numbers. Then for any $\varepsilon > 0$,*

$$\varepsilon P\left(\max_{1 \leq k \leq n} c_k S_k \geq \varepsilon\right) \leq \sum_{j=1}^{n} c_j E\left(S_j^+ - S_{j-1}^+\right). \tag{1.15}$$

Using Lemma 1.5, Wang [4] obtained the following inequalities for demimartingales.

Theorem 1.10 (see Wang [4, Corollary 2.1]). *Let $\{S_n, n \geq 1\}$ be a demimartingale and $\{c_k, k \geq 1\}$ a nonincreasing sequence of positive numbers. Then*

$$E\left[\max_{1 \leq k \leq n} c_k |S_k|\right]^p \leq \left(\frac{p}{p-1}\right)^p E\left[\sum_{j=1}^{n} c_j(|S_j| - |S_{j-1}|)\right]^p, \quad p > 1, \tag{1.16}$$

$$E\left[\max_{1 \leq k \leq n} c_k |S_k|\right] \leq \frac{e}{e-1}\left\{1 + E\left[\left(\sum_{j=1}^{n} c_j(|S_j| - |S_{j-1}|)\right)\log^+\left(\sum_{j=1}^{n} c_j(|S_j| - |S_{j-1}|)\right)\right]\right\}. \tag{1.17}$$

We point out that there is a mistake in the proof of (1.17), *that is,*

$$E\left[\max_{1 \leq j \leq n} c_j|S_j|\right] - 1 \leq E\left[\left(\sum_{j=1}^{n} c_j(|S_j| - |S_{j-1}|)\right)\log\left(\max_{1 \leq j \leq n} c_j|S_j|\right)\right] \tag{1.18}$$

should be replaced by

$$E\left[\max_{1 \leq j \leq n} c_j|S_j|\right] - 1 \leq E\left[\left(\sum_{j=1}^{n} c_j(|S_j| - |S_{j-1}|)\right)\log^+\left(\max_{1 \leq j \leq n} c_j|S_j|\right)\right]. \tag{1.19}$$

In fact, by Lemma 1.5 and Fubini Theorem, we can see that

$$E\left[\max_{1\leq j\leq n}c_j|S_j|\right]-1\leq\int_1^\infty P\left(\max_{1\leq j\leq n}c_j|S_j|\geq x\right)dx$$

$$\leq\int_1^\infty\frac{1}{x}E\left[\sum_{j=1}^n c_j(|S_j|-|S_{j-1}|)I\left(\max_{1\leq j\leq n}c_j|S_j|\geq x\right)\right]dx \qquad (1.20)$$

$$\leq E\left[\left(\sum_{j=1}^n c_j(|S_j|-|S_{j-1}|)\right)\log^+\left(\max_{1\leq j\leq n}c_j|S_j|\right)\right].$$

The rest of the proof is similar to Corollary 2.1 in Wang [4].

The same problem exists in Shiryaev [10, page 495, in the proof of Theorem 2] and Krishna and Soumendra [11, page 414, in the proof of Theorem 13.2.13]. For example, the following inequality

$$E\left(\max_{0\leq j\leq n}|X_j|\right)-1\leq E\left(X_n\int_0^{\max_{0\leq j\leq n}|X_j|-1}\frac{dt}{1+t}\right)=E\left[X_n\log\left(\max_{0\leq j\leq n}|X_j|\right)\right] \qquad (1.21)$$

in Shiryaev [10, page 495] should be revised as

$$E\left(\max_{0\leq j\leq n}|X_j|\right)-1\leq E\left[X_n I\left(\max_{0\leq j\leq n}|X_j|\geq 1\right)\int_0^{\max_{0\leq j\leq n}|X_j|-1}\frac{dt}{1+t}\right]$$

$$=E\left[X_n\log^+\left(\max_{0\leq j\leq n}|X_j|\right)\right]. \qquad (1.22)$$

2. Main Results and Their Proofs

Theorem 2.1. *Let $\{S_n, n\geq 1\}$ be a demimartingale and g a nonnegative convex function on $\mathbb{R}$ with $g(0)=0$. Let $\{c_k, k\geq 1\}$ be a nonincreasing sequence of positive numbers. $p>1$. Suppose that $E(g(S_k))^p<\infty$ for each $k\geq 1$, then for every $n\geq 1$,*

$$E\left[\max_{1\leq k\leq n}c_k g(S_k)\right]^p\leq\left(\frac{p}{p-1}\right)^p E\left[\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\right]^p, \qquad (2.1)$$

$$E\left[\max_{1\leq k\leq n}c_k g(S_k)\right]\leq\frac{e}{e-1}\left\{1+E\left[\left(\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\right)\log^+\left(\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\right)\right]\right\}. \qquad (2.2)$$

Proof. By Lemma 1.5 and Hölder's inequality, we have

$$
\begin{aligned}
E\left[\max_{1\leq k\leq n} c_k g(S_k)\right]^p &= p\int_0^\infty x^{p-1}P\left[\max_{1\leq k\leq n} c_k g(S_k) \geq x\right]dx \\
&\leq p\int_0^\infty x^{p-2}E\left[\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))I\left(\max_{1\leq k\leq n} c_k g(S_k)\geq x\right)\right]dx \\
&= \frac{p}{p-1}E\left\{\left[\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\right]\left(\max_{1\leq k\leq n} c_k g(S_k)\right)^{p-1}\right\} \\
&\leq \frac{p}{p-1}\left\{E\left[\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\right]^p\right\}^{1/p}\left\{E\left[\max_{1\leq k\leq n} c_k g(S_k)\right]^p\right\}^{1/q},
\end{aligned}
\tag{2.3}
$$

where q is a real number and satisfies $1/p + 1/q = 1$. Since $E(g(S_k))^p < \infty$ for each $k \geq 1$, we can obtain

$$
\left\{E\left[\max_{1\leq k\leq n} c_k g(S_k)\right]^p\right\}^{1/p} \leq \frac{p}{p-1}\left\{E\left[\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\right]^p\right\}^{1/p},
\tag{2.4}
$$

therefore,

$$
E\left[\max_{1\leq k\leq n} c_k g(S_k)\right]^p \leq \left(\frac{p}{p-1}\right)^p E\left[\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\right]^p.
\tag{2.5}
$$

Similar to the proof of (2.3) and using Lemma 1.5 again, we can see that

$$
E\left[\max_{1\leq k\leq n} c_k g(S_k)\right] \leq 1 + E\left[\left(\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\right)\log^+\left(\max_{1\leq k\leq n} c_k g(S_k)\right)\right].
\tag{2.6}
$$

For constants $a \geq 0$ and $b > 0$, it follows that

$$
a\log^+ b \leq a\log^+ a + be^{-1}.
\tag{2.7}
$$

Combining (2.6) and (2.7), we have

$$
E\left[\max_{1\leq k\leq n} c_k g(S_k)\right] \leq 1 + E\left[\left(\sum_{j=1}^{n} c_j\big(g(S_j) - g(S_{j-1})\big)\right)\log^+\left(\sum_{j=1}^{n} c_j\big(g(S_j) - g(S_{j-1})\big)\right)\right]
$$
$$
+ e^{-1} E\left[\max_{1\leq k\leq n} c_k g(S_k)\right].
$$

(2.8)

Thus, (2.2) follows from (2.8) immediately. The proof is complete. $\square$

Remark 2.2. If we take $g(x) = |x|$ in Theorem 2.1, then Theorem 2.1 implies Corollary 2.1 in Wang [4].

Corollary 2.3. *Let the conditions of Theorem 2.1 be satisfied with $c_k \equiv 1$ for each $k \geq 1$. Then for every $n \geq 1$,*

$$
E\left[\max_{1\leq k\leq n} g(S_k)\right]^p \leq \left(\frac{p}{p-1}\right)^p E\big(g(S_n)\big)^p,
$$

(2.9)

$$
E\left[\max_{1\leq k\leq n} g(S_k)\right] \leq \frac{e}{e-1}\big\{1 + E\big[g(S_n)\log^+ g(S_n)\big]\big\}.
$$

(2.10)

Corollary 2.4 (Doob's type maximal inequality for demimartingales). *Let $p > 1$ and $\{S_n, n \geq 1\}$ be a demimartingale. Suppose that $E|S_k|^p < \infty$ for each $k \geq 1$, then for every $n \geq 1$,*

$$
E\left[\max_{1\leq k\leq n} |S_k|\right]^p \leq \left(\frac{p}{p-1}\right)^p E|S_n|^p,
$$
$$
E\left[\max_{1\leq k\leq n} |S_k|\right] \leq \frac{e}{e-1}\big[1 + E\big(|S_n|\log^+ |S_n|\big)\big].
$$

(2.11)

Theorem 2.5. *Let $\{S_n, n \geq 1\}$ be a demimartingale and g a nonnegative convex function on $\mathbb{R}$ with $g(0) = 0$. Let $\{b_n, n \geq 1\}$ be a nondecreasing unbounded sequence of positive numbers. $p > 1$. Suppose that $E(g(S_{k-1}))^p \leq E(g(S_k))^p < \infty$ for each $k \geq 1$ and*

$$
\sum_{n=1}^{\infty} \frac{E(g(S_n))^p - E(g(S_{n-1}))^p}{b_n^p} < \infty,
$$

(2.12)

then $\lim_{n\to\infty}(g(S_n)/b_n) = 0$ a.s., and (1.9)-(1.10) hold (S_n is replaced by $g(S_n)$), where

$$
\alpha_n = \left(\frac{p}{p-1}\right)^p \big[E(g(S_n))^p - E(g(S_{n-1}))^p\big], \quad n \geq 1.
$$

(2.13)

In addition,

$$E\left(\max_{1\le k\le n}\left|\frac{g(S_k)}{b_k}\right|^p\right)\le 4\sum_{k=1}^n\frac{\alpha_k}{b_k^p}<\infty,\quad E\left(\sup_{n\ge 1}\left|\frac{g(S_n)}{b_n}\right|^p\right)\le 4\sum_{n=1}^\infty\frac{\alpha_n}{b_n^p}<\infty. \qquad (2.14)$$

If further one assumes that $\alpha_k>0$ for infinitely many k, then

$$E\left(\sup_{n\ge 1}\left|\frac{g(S_n)}{\beta_n}\right|^p\right)\le 4\sum_{n=1}^\infty\frac{\alpha_n}{\beta_n^p}<\infty. \qquad (2.15)$$

Proof. By the condition of the theorem, we can see that $\alpha_n\ge 0$ for all $n\ge 1$. Thus,

$$E\left[\max_{1\le k\le n}g(S_k)\right]^p\le\left(\frac{p}{p-1}\right)^p E(g(S_n))^p=\sum_{k=1}^n\alpha_k \qquad (2.16)$$

follows from (2.9) for each $n\ge 1$. By (2.12), we have

$$\sum_{n=1}^\infty\frac{\alpha_n}{b_n^p}=\left(\frac{p}{p-1}\right)^p\sum_{n=1}^\infty\frac{E(g(S_n))^p-E(g(S_{n-1}))^p}{b_n^p}<\infty. \qquad (2.17)$$

Therefore, $\lim_{n\to\infty}(g(S_n)/b_n)=0$ *a.s.* follows from **Lemma 1.6**, (2.16), and (2.17); (1.9), (1.10), (2.14), (2.15) hold. This completes the proof of the theorem. $\qquad\square$

In Theorem 2.5, if we assume that $g(x)$ is a nonnegative and nondecreasing convex function on $\mathbb{R}$ with $g(0)=0$, then the condition "$E(g(S_{k-1}))^p\le E(g(S_k))^p$ for each $k\ge 1$" is satisfied.

Remark 2.6. Theorem 2.5 generalizes and improves the results of Theorem 2.2 in Christofides [3] and Theorem 2.7 in Prakasa Rao [6].

Theorem 2.7. *Let $p>1$ and $\{S_n,n\ge 1\}$ a demimartingale with $E|S_k|^p<\infty$ for each $k\ge 1$. Let $\{b_n,n\ge 1\}$ be a nondecreasing sequence of positive numbers. If*

$$\sum_{k=1}^\infty\frac{E|S_k|^p-E|S_{k-1}|^p}{b_k^p}<\infty, \qquad (2.18)$$

then for any $0<r<p$,

$$E\left(\sup_{k\ge 1}\left|\frac{S_k}{b_k}\right|^r\right)\le 1+\frac{2r}{p-r}\sum_{k=1}^\infty\frac{E|S_k|^p-E|S_{k-1}|^p}{b_k^p}<\infty. \qquad (2.19)$$

Proof. Taking $n = 1$, $v = p$ and $c_k = 1/b_k$ in Lemma 1.8, we have

$$P\left\{\max_{1\le k\le N}\left|\frac{S_k}{b_k}\right|\ge\varepsilon\right\}\le\frac{2}{\varepsilon^p}\sum_{k=1}^{N}\frac{E|S_k|^p-E|S_{k-1}|^p}{b_k^p}. \tag{2.20}$$

Thus, by (2.20) and (2.18), we can get

$$E\left(\sup_{k\ge 1}\left|\frac{S_k}{b_k}\right|^r\right)\le 1+\int_1^\infty P\left(\sup_{k\ge 1}\left|\frac{S_k}{b_k}\right|^r>t\right)dt=1+\int_1^\infty\lim_{N\to\infty}P\left(\max_{1\le k\le N}\left|\frac{S_k}{b_k}\right|>t^{1/r}\right)dt$$

$$\le 1+\frac{2r}{p-r}\sum_{k=1}^\infty\frac{E|S_k|^p-E|S_{k-1}|^p}{b_k^p}<\infty. \tag{2.21}$$

$\square$

Theorem 2.8. *Let $\{S_n, n\ge 1\}$ be a demisubmartingale and g a nondecreasing and nonnegative convex function on $\mathbb{R}$ with $g(0) = 0$ and $g(S_i)\in L^1, i\ge 1$. Let $\{c_k, k\ge 1\}$ be a nonincreasing sequence of positive numbers. Then for all $0 < p < 1$ and each $n\ge 1$,*

$$E\left[\max_{1\le k\le n}c_k g(S_k)\right]^p\le(c_n Eg(S_n))^{p-1}\left[c_n Eg(S_n)+\frac{p}{1-p}\sum_{j=1}^n c_j E(g(S_j)-g(S_{j-1}))\right]. \tag{2.22}$$

Proof. By Fubini theorem, it is easy to check that

$$E\left[\max_{1\le k\le n}c_k g(S_k)\right]^p=p\int_0^\infty x^{p-1}P\left(\max_{1\le k\le n}c_k g(S_k)\ge x\right)dx$$

$$\le p\int_0^{c_n Eg(S_n)}x^{p-1}dx+p\int_{c_n Eg(S_n)}^\infty x^{p-1}P\left(\max_{1\le k\le n}c_k g(S_k)\ge x\right)dx \tag{2.23}$$

$$=(c_n Eg(S_n))^p+p\int_{c_n Eg(S_n)}^\infty x^{p-1}P\left(\max_{1\le k\le n}c_k g(S_k)\ge x\right)dx.$$

It follows from Lemma 1.7(i) and Lemma 1.9(ii) that

$$p\int_{c_n Eg(S_n)}^\infty x^{p-1}P\left(\max_{1\le k\le n}c_k g(S_k)\ge x\right)dx\le p\int_{c_n Eg(S_n)}^\infty x^{p-2}\sum_{j=1}^n c_j E(g(S_j)-g(S_{j-1}))dx$$

$$=\frac{p}{1-p}(c_n Eg(S_n))^{p-1}\sum_{j=1}^n c_j E(g(S_j)-g(S_{j-1})). \tag{2.24}$$

Therefore, (2.22) follows from the above statements immediately. $\square$

Corollary 2.9. *Let the conditions of Theorem 2.8 be satisfied with $c_k \equiv 1$ for each $k \geq 1$. Then for all $0 < p < 1$ and each $n \geq 1$,*

$$E\left[\max_{1 \leq k \leq n} g(S_k)\right]^p \leq \frac{1}{1-p}(Eg(S_n))^p. \tag{2.25}$$

By Corollary 2.9, we can get the following theorem.

Theorem 2.10. *Let $\{S_n, n \geq 1\}$ be a demisubmartingale and g a nondecreasing and nonnegative convex function on $\mathbb{R}$ with $g(0) = 0$ and $g(S_i) \in L^1, i \geq 1$. Let $\{b_n, n \geq 1\}$ be a nondecreasing unbounded sequence of positive numbers. If there exists some $0 < p < 1$ such that*

$$\sum_{n=1}^{\infty} \frac{(Eg(S_n))^p - (Eg(S_{n-1}))^p}{b_n^p} < \infty, \tag{2.26}$$

then $\lim_{n \to \infty}(g(S_n)/b_n) = 0$ a.s., and (1.9)-(1.10) hold (S_n is replaced by $g(S_n)$), where

$$\alpha_n = \frac{1}{1-p}\left[(Eg(S_n))^p - (Eg(S_{n-1}))^p\right], \quad n \geq 1. \tag{2.27}$$

In addition,

$$E\left(\max_{1 \leq k \leq n}\left|\frac{g(S_k)}{b_k}\right|^p\right) \leq 4\sum_{k=1}^{n}\frac{\alpha_k}{b_k^p} < \infty, \qquad E\left(\sup_{n \geq 1}\left|\frac{g(S_n)}{b_n}\right|^p\right) \leq 4\sum_{n=1}^{\infty}\frac{\alpha_n}{b_n^p} < \infty. \tag{2.28}$$

If further one assumes that $\alpha_k > 0$ for infinitely many k, then

$$E\left(\sup_{n \geq 1}\left|\frac{g(S_n)}{\beta_n}\right|^p\right) \leq 4\sum_{n=1}^{\infty}\frac{\alpha_n}{\beta_n^p} < \infty. \tag{2.29}$$

Similar to the proof of Theorem 2.8 and using Lemma 1.9(i), we can get the following.

Theorem 2.11. *Let $\{S_n, n \geq 1\}$ be a nonnegative demisubmartingale. Then for all $0 < p < 1$, $E[\max_{1 \leq k \leq n} S_k]^p \leq (1/(1-p))(ES_n)^p$.*

Acknowledgments

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