Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 842715, 15 pages
doi:10.1155/2010/842715

Research Article

Kuhn-Tucker Optimality Conditions for Vector Equilibrium Problems

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Received 20 January 2010; Accepted 17 April 2010

Academic Editor: Radu Precup

By using the concept of Fréchet differentiability of mapping, we present the Kuhn-Tucker optimality conditions for weakly efficient solution, Henig efficient solution, superefficient solution, and globally efficient solution to the vector equilibrium problems with constraints.

1. Introduction

Recently, some authors have studied the optimality conditions for vector variational inequalities. Giannessi et al. [1] turned the vector variational inequalities with constraints into vector variational inequalities without constraints and gave sufficient conditions for the efficient solutions and the weakly efficient solutions to the vector variational inequalities in R^n. By using the concept of subdifferential of the function, Morgan and Romaniello [2] gave the scalarization and Kuhn-Tucker-like conditions for the weak vector generalized quasivariational inequalities in Hilbert space. Yang and Zheng [3] gave the optimality conditions for approximate solutions of vector variational inequalities in Banach space. On the other hand, some authors have derived the optimality conditions for weakly efficient solutions to vector optimization problems (see [4–19]).

Vector variational inequality problems and vector optimization problems, as well as several other problems, are special realizations of vector equilibrium problems (see [20, 21]); therefore, it is important to give the optimality conditions for the solution to the vector equilibrium problems for in this way we can turn the vector equilibrium problem with constraints to a corresponding scalar optimization problem without constraints, and we can then determine if the solution of the scalar optimization problem is a solution of the original vector equilibrium problem. Under the assumption of convexity, Gong [22] investigated optimality conditions for weakly efficient solutions, Henig solutions, superefficient solutions, and globally efficient solutions to vector equilibrium problems with constraints and obtained that the weakly efficient solutions, Henig efficient solutions, globally efficient solutions, and superefficient solutions to vector equilibrium problems with constraints are equivalent to solution of corresponding scalar optimization problems without constraints, respectively. Qiu [23] presented the necessary and sufficient conditions for globally efficient solution under generalized cone-subconvexlikeness. Gong and Xiong [24] weakened the convexity assumptions in [22] and obtained necessary and sufficient conditions for weakly efficient solution, too.

In this paper, by using the concept of Fréchet differentiability of mapping, we study the optimality conditions for weakly efficient solutions, Henig solutions, superefficient solutions, and globally efficient solutions to the vector equilibrium problems. We give Kuhn-Tucker necessary conditions to the vector equilibrium problems without convexity conditions and Kuhn-Tucker sufficient conditions with convexity conditions.

2. Preliminaries and Definitions

Throughout this paper, let $(X, \|\cdot\|_X)$ be a real normed space, let $(Y, C_Y, \|\cdot\|_Y)$ and $(Z, C_Z, \|\cdot\|_Z)$ be real partially ordered normed spaces, and let C_Y and C_Z be closed convex pointed cones with $\operatorname{int} C_Z \neq \emptyset$, where $\operatorname{int} C_Z$ denotes the interior of the set C_Z.

Let Y^* and Z^* be the topological dual spaces of Y and Z, respectively. Let

$$\begin{aligned} C_Y^* &= \{y^* \in Y^* : y^*(y) \geq 0,\ \forall y \in C_Y\}, \\ C_Z^* &= \{z^* \in Z^* : z^*(z) \geq 0,\ \forall z \in C_Z\} \end{aligned} \tag{2.1}$$

be the dual cones of C_Y and C_Z, respectively. Denote the quasi-interior of C_Y^* by $C_Y^\#$, that is

$$C_Y^\# = \{y^* \in Y^* : y^*(y) > 0,\ \forall y \in C_Y \setminus \{0\}\}. \tag{2.2}$$

Let D be a nonempty subset of Y. The cone hull of D is defined as

$$\operatorname{cone}(D) = \{td : t \geq 0,\ d \in D\}. \tag{2.3}$$

Denote the closure of D by $\operatorname{cl} D$ and interior of D by $\operatorname{int} D$.

A nonempty convex subset B of the convex cone C_Y is called a base of C_Y, if $C_Y = \operatorname{cone}(B)$ and $0 \notin \operatorname{cl}(B)$. It is easy to see that $C_Y^\# \neq \emptyset$ if and only if C_Y has a base.

Let B be a base of C_Y. Set

$$C^\Delta(B) = \{y^* \in C_Y^* : \text{there exists } t > 0 \text{ such that } y^*(b) \geq t,\ \forall b \in B\}. \tag{2.4}$$

By the separation theorem of convex sets, we know $C^\Delta(B) \neq \emptyset$.

Denote the closed unit ball of Y by U. Suppose that C_Y has a base B. Let $\delta = \inf\{\|b\| : b \in B\}$. It is clear that $\delta > 0$. The δ will be used for the rest of the paper. For any $0 < \varepsilon < \delta$, denote $C_\varepsilon(B) = \operatorname{cone}(B + \varepsilon U)$, then $\operatorname{cl}(C_\varepsilon(B))$ is a closed convex pointed cone, and $C_Y \setminus \{0\} \subset \operatorname{int} C_\varepsilon(B)$, for all $0 < \varepsilon < \delta$ (see [25]).

Let $S_1 \subset X$ be a nonempty open convex subset, and let $F : S_1 \times S_1 \to Y$, $g : S_1 \to Z$ be mappings.

We define the constraint set

$$S = \{x \in S_1 : g(x) \in -C_Z\} \tag{2.5}$$

and consider the vector equilibrium problems with constraints (for short, VEPC): find $x \in S$ such that

$$F(x,y) \notin -P \setminus \{0\}, \quad \forall y \in S, \tag{2.6}$$

where P is a convex cone in Y.

Definition 2.1. If $\operatorname{int} C_Y \neq \emptyset$, a vector $x \in S$ satisfying

$$F(x,y) \notin -\operatorname{int} C_Y, \quad \forall y \in S \tag{2.7}$$

is called a weakly efficient solution to the VEPC.

For each $x \in S$, we denote

$$F(x,S) = \bigcup_{y \in S} F(x,y). \tag{2.8}$$

Definition 2.2 (see [22]). Let C_Y have a base B. A vector $x \in S$ is called a Henig efficient solution to the VEPC if there exists some $0 < \varepsilon < \delta$ such that

$$F(x,S) \cap (-\operatorname{int} C_\varepsilon(B)) = \emptyset. \tag{2.9}$$

Definition 2.3 (see [22]). A vector $x \in S$ is called a globally efficient solution to the VEPC if there exists a point convex cone $H \subset Y$ with $C \setminus \{0\} \subset \operatorname{int} H$ such that

$$F(x,S) \cap ((-H) \setminus \{0\}) = \emptyset. \tag{2.10}$$

Definition 2.4 (see [22]). A vector $x \in S$ is called a superefficient solution to the VEPC if there exists $M > 0$ such that

$$\operatorname{cone}(F(x,S)) \cap (U - C_Y) \subset MU, \tag{2.11}$$

where U is closed unit ball of Y.

Definition 2.5. Let X be a real linear space, and let Y be a real topological linear space. Let S_2 be a nonempty subset of X, and let a mapping $f : S_2 \to Y$ and some $\overline{x} \in S_2$ be given. If for some $h \in X$ the limit

$$f'(\overline{x})(h) = \lim_{\lambda \to 0} \frac{1}{\lambda}\left(f(\overline{x} + \lambda h) - f(\overline{x})\right) \tag{2.12}$$

exists, then $f'(\overline{x})(h)$ is called the Gâteaux derivative of f at $\overline{x}$ in the direction h. If this limit exists for each direction h, the mapping f is called Gâteaux differentiable at $\overline{x}$.

Definition 2.6. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real normed spaces, and let D be a nonempty open subset of X. Moreover, let a mapping $f : D \to Y$ and some $\overline{x} \in D$ be given. If there exists a continuous linear mapping $f'(\overline{x}) : X \to Y$ with the property

$$\lim_{\|h\|_X \to 0} \frac{\left\| f(\overline{x} + h) - f(\overline{x}) - f'(\overline{x})(h) \right\|_Y}{\|h\|_X} = 0, \tag{2.13}$$

then $f'(\overline{x})$ is called the Fréchet derivative of f at $\overline{x}$ and f is called Fréchet differentiable at $\overline{x}$.

Remark 2.7. By [26, Lemma 2.18], we can see that if f is Fréchet differentiable at $\overline{x}$, then f is Gâteaux differentiable at $\overline{x}$ and the Fréchet derivative of f at $\overline{x}$ is equal to the Gâteaux derivative of f at $\overline{x}$ in each direction h.

Definition 2.8. Let X and Y be real linear spaces, let C_Y be a pointed convex cone in Y, and let A be a nonempty convex subset of X. A mapping $f : A \to Y$ is called C_Y-convex, if for all $x, y \in A$ and all $\lambda \in [0, 1]$

$$\lambda f(x) + (1 - \lambda) f(y) - f(\lambda x + (1 - \lambda) y) \in C_Y. \tag{2.14}$$

Lemma 2.9 (see [18]). *Assume that pointed convex cone C_Y has a base B; then one has the following.*

(i) *For any $0 < \varepsilon < \delta$, $(C_\varepsilon(B))^* \setminus \{0_{Y^*}\} \subset C^\Delta(B)$.*

(ii) *For any $f \in C^\Delta(B)$, there exists some $0 < \varepsilon_0 < \delta$ with $f \in (C_{\varepsilon_0}(B))^* \setminus \{0_{Y^*}\}$.*

(iii) $\operatorname{int} C_Y^* \subset C^\Delta(B)$, *and when B is bounded and closed, then* $\operatorname{int} C_Y^* = C^\Delta(B)$, *where* $\operatorname{int} C_Y^*$ *is the interior of C_Y^* in Y^* with respect to the norm of Y^*.*

3. Optimality Condition

In this section, we give the Kuhn-Tucker necessary conditions and Kuhn-Tucker sufficient conditions for weakly efficient solution, Henig efficient solution, globally efficient solution, and superefficient solution to the vector equilibrium problems with constraints.

Let $\overline{x} \in S_1$ be given. Denote the mapping $F_{\overline{x}} : S_1 \to Y$ by

$$F_{\overline{x}}(y) = F(\overline{x}, y), \quad y \in S_1. \tag{3.1}$$

By the proof of Theorem 2.20 in [26] or from the definition, we have the following lemma.

Lemma 3.1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real normed spaces, let A be a nonempty open convex subset of X, and let C_Y be closed convex pointed cone in Y. Assume that $f : A \to Y$ is C_Y-convex and f is Gâteaux differentiable at $\overline{x} \in A$. Then*

$$f(x) - f(\overline{x}) - f'(\overline{x})(x - \overline{x}) \in C_Y, \quad \forall x \in A. \tag{3.2}$$

Theorem 3.2. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, and $(Z, \|\cdot\|_Z)$ be real normed spaces, and let C_Y and C_Z be closed convex pointed cones in Y and Z with $\operatorname{int} C_Y \neq \emptyset$ and $\operatorname{int} C_Z \neq \emptyset$, respectively. Let $F(\overline{x}, \overline{x}) = 0$. Assume that $F_{\overline{x}}(\cdot)$ and $g(\cdot)$ are Fréchet differentiable at $\overline{x} \in S$. Furthermore, assume that there exists $x_1 \in S_1$ such that $g(\overline{x}) + g'(\overline{x})(x_1 - \overline{x}) \in -\operatorname{int} C_Z$. If $\overline{x} \in S$ is a weakly efficient solution to the VEPC, then there exist $v \in C_Y^* \setminus \{0_{Y^*}\}$, $u \in C_Z^*$, such that*

$$\begin{gathered}\left(v \circ F'_{\overline{x}}(\overline{x}) + u \circ g'(\overline{x})\right)(x - \overline{x}) \geq 0, \quad \forall x \in S_1, \\ u \circ g(\overline{x}) = 0.\end{gathered} \tag{3.3}$$

Proof. Assume that $\overline{x} \in S$ is a weakly efficient solution to the VEPC. Define the set

$$\begin{aligned} M = \{(y, z) \in Y \times Z : &\text{ there exists } x \in S_1 \text{ such that } y - F'_{\overline{x}}(\overline{x})(x - \overline{x}) \in \operatorname{int} C_Y, \\ &z - \left(g(\overline{x}) + g'(\overline{x})(x - \overline{x})\right) \in \operatorname{int} C_Z\}. \end{aligned} \tag{3.4}$$

Since $F'_{\overline{x}}(\overline{x})$ and $g'(\overline{x})$ are linear operators, we can see that M is a nonempty open convex set. We claim that $(0, 0) \notin M$. If not, then there exists $x_0 \in S_1$ such that

$$F'_{\overline{x}}(\overline{x})(x_0 - \overline{x}) \in -\operatorname{int} C_Y, \qquad g(\overline{x}) + g'(\overline{x})(x_0 - \overline{x}) \in -\operatorname{int} C_Z. \tag{3.5}$$

From Remark 2.7, we obtain

$$\begin{gathered} F'_{\overline{x}}(\overline{x})(x_0 - \overline{x}) = \lim_{\lambda \to 0} \frac{1}{\lambda}\left(F_{\overline{x}}(\overline{x} + \lambda(x_0 - \overline{x})) - F_{\overline{x}}(\overline{x})\right) \in -\operatorname{int} C_Y, \\ g(\overline{x}) + g'(\overline{x})(x_0 - \overline{x}) = g(\overline{x}) + \lim_{\lambda \to 0} \frac{1}{\lambda}\left(g(\overline{x} + \lambda(x_0 - \overline{x})) - g(\overline{x})\right) \in -\operatorname{int} C_Z. \end{gathered} \tag{3.6}$$

Since $-\operatorname{int} C_Y$ and $-\operatorname{int} C_Z$ are open sets, there exists some $0 < \lambda_0 < 1$ such that

$$\begin{gathered} \frac{1}{\lambda_0}\left(F_{\overline{x}}(\overline{x} + \lambda_0(x_0 - \overline{x})) - F_{\overline{x}}(\overline{x})\right) \in -\operatorname{int} C_Y, \\ g(\overline{x}) + \frac{1}{\lambda_0}\left(g(\overline{x} + \lambda_0(x_0 - \overline{x})) - g(\overline{x})\right) \in -\operatorname{int} C_Z. \end{gathered} \tag{3.7}$$

From $g(\overline{x}) \in -C_Z$, $F(\overline{x}, \overline{x}) = 0$, and $1/\lambda_0 > 1$, we can see that

$$F_{\overline{x}}(\overline{x} + \lambda_0(x_0 - \overline{x})) \in -\operatorname{int} C_Y, \qquad g(\overline{x} + \lambda_0(x_0 - \overline{x})) \in -\operatorname{int} C_Z. \tag{3.8}$$

Since S_1 is a convex set, $\overline{x} + \lambda_0(x_0 - \overline{x}) \in S_1$. Thus, we get

$$\overline{x} + \lambda_0(x_0 - \overline{x}) \in S, \qquad F(\overline{x}, \overline{x} + \lambda_0(x_0 - \overline{x})) \in -\operatorname{int} C_Y. \tag{3.9}$$

This contradicts that $\overline{x} \in S$ is a weakly efficient solution to the VEPC. Thus $(0,0) \notin M$. Noting that M is an open set, by the separation theorem of convex sets (see [26]), there exists $(0,0) \neq (v,u) \in (Y \times Z)^* = Y^* \times Z^*$ such that

$$v(y) + u(z) > 0, \quad \forall (y,z) \in M. \tag{3.10}$$

Let $(y,z) \in M$. Then there exists $x \in S_1$ such that

$$y - F'_{\overline{x}}(\overline{x})(x - \overline{x}) \in \operatorname{int} C_Y, \qquad z - \left(g(\overline{x}) + g'(\overline{x})(x - \overline{x})\right) \in \operatorname{int} C_Z. \tag{3.11}$$

It is clear that for every $c \in \operatorname{int} C_Y$, $k \in \operatorname{int} C_Z$, $t' > 0$, $t'' > 0$, we have $(y + t'c, z) \in M$ and $(y, z + t''k) \in M$. By (3.10), we have

$$v(y + t'c) + u(z) > 0, \quad \forall c \in \operatorname{int} C_Y,\ t' > 0. \tag{3.12}$$

We can get

$$v(c) \geq 0, \quad \forall c \in \operatorname{int} C_Y. \tag{3.13}$$

Since C_Y is a closed convex cone, $C_Y = \operatorname{cl}(\operatorname{int} C_Y)$. By the continuity of v, we can see that $v(c) \geq 0$ for all $c \in C_Y$. That is, $v \in C_Y^*$. Similarly, we can show that $u \in C_Z^*$. We also have $v \neq 0_{Y^*}$. In fact, if $v = 0_{Y^*}$, from (3.10) we get

$$u(z) > 0, \quad \forall (y,z) \in M. \tag{3.14}$$

By assumption, there exists $x_1 \in S_1$ such that $g(\overline{x}) + g'(\overline{x})(x_1 - \overline{x}) \in -\operatorname{int} C_Z$; thus, we have

$$\left(F'_{\overline{x}}(\overline{x})(x_1 - \overline{x}) + c, g(\overline{x}) + g'(\overline{x})(x_1 - \overline{x}) + k\right) \in M, \quad \forall c \in \operatorname{int} C_Y,\ k \in \operatorname{int} C_Z. \tag{3.15}$$

Hence

$$u\left(g(\overline{x}) + g'(\overline{x})(x_1 - \overline{x}) + k\right) > 0, \quad \forall k \in \operatorname{int} C_Z. \tag{3.16}$$

In particular, we have $-(g(\overline{x}) + g'(\overline{x})(x_1 - \overline{x})) \in \operatorname{int} C_Z$, and we get $u(0) = 0 > 0$. This is a contradiction. Thus, $v \neq 0_{Y^*}$. It is clear that

$$\left(F'_{\overline{x}}(\overline{x})(x - \overline{x}) + c, g(\overline{x}) + g'(\overline{x})(x - \overline{x}) + k\right) \in M \tag{3.17}$$

for all $x \in S_1$, $c \in \operatorname{int} C_Y$, $k \in \operatorname{int} C_Z$. By (3.10), we obtain

$$v\left(F'_{\overline{x}}(\overline{x})(x-\overline{x})+c\right)+u\left(g(\overline{x})+g'(\overline{x})(x-\overline{x})+k\right)>0 \tag{3.18}$$

for all $x \in S_1$, $c \in \operatorname{int} C_Y$, $k \in \operatorname{int} C_Z$.

Letting $c \to 0$, $k \to 0$, we get

$$v\left(F'_{\overline{x}}(\overline{x})(x-\overline{x})\right)+u\left(g(\overline{x})+g'(\overline{x})(x-\overline{x})\right)\geq 0, \quad \forall x \in S_1. \tag{3.19}$$

It is clear that

$$\left(F'_{\overline{x}}(\overline{x})(\overline{x}-\overline{x})+t'c, g(\overline{x})+g'(\overline{x})(\overline{x}-\overline{x})+t'k\right)\in M \tag{3.20}$$

for all $c \in \operatorname{int} C_Y$, $k \in \operatorname{int} C_Z$, $t' > 0$. By (3.10), we have

$$v\left(F'_{\overline{x}}(\overline{x})(\overline{x}-\overline{x})+t'c\right)+u\left(g(\overline{x})+g'(\overline{x})(\overline{x}-\overline{x})+t'k\right)=t'v(c)+u\left(g(\overline{x})\right)+t'u(k)>0. \tag{3.21}$$

Letting $t' \to 0$, we obtain $u(g(\overline{x})) \geq 0$. Noting that $g(\overline{x}) \in -C_Z$ and $u \in C_Z^*$, we have $u(g(\overline{x})) \leq 0$. Thus,

$$u\left(g(\overline{x})\right)=0. \tag{3.22}$$

From (3.19) and (3.22), we have

$$v\left(F'_{\overline{x}}(\overline{x})(x-\overline{x})\right)+u\left(g'(\overline{x})(x-\overline{x})\right)\geq 0, \quad \forall x \in S_1. \tag{3.23}$$

Thus, we have

$$\begin{gathered}\left(v \circ F'_{\overline{x}}(\overline{x})+u \circ g'(\overline{x})\right)(x-\overline{x})\geq 0, \quad \forall x \in S_1, \\ u \circ g(\overline{x})=0.\end{gathered} \tag{3.24}$$

This completes the proof. □

Remark 3.3. The condition that there exists $x_1 \in S_1$ such that $g(\overline{x})+g'(\overline{x})(x_1-\overline{x}) \in -\operatorname{int} C_Z$ is a generalized Slater constraint condition. In fact, from the proof of Theorem 3.2, we obtain that there exists $0<\lambda<1$ such that $\overline{x}+\lambda(x_1-\overline{x}) \in S_1$ and $g(\overline{x}+\lambda(x_1-\overline{x})) \in -\operatorname{int} C_Z$.

On the other hand, if $g(\cdot)$ is Gâteaux differentiable at $\overline{x} \in S_1$, and $g(\cdot)$ is C_Z-convex on S_1, and $g(x_1) \in -\operatorname{int} C_Z$, then $g(\overline{x})+g'(\overline{x})(x_1-\overline{x}) \in -\operatorname{int} C_Z$. In fact, by Lemma 3.1, there exists $k \in C_Z$ such that

$$g(\overline{x})+g'(\overline{x})(x_1-\overline{x})=g(\overline{x})+g(x_1)-g(\overline{x})-k \in -\operatorname{int} C_Z. \tag{3.25}$$

Theorem 3.4. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, and $(Z, \|\cdot\|_Z)$ be real normed spaces, let $C_Y \subset Y$ be a closed convex pointed cone with $\operatorname{int} C_Y \neq \emptyset$, and let $C_Z \subset Z$ be a closed convex pointed cone with $\operatorname{int} C_Z \neq \emptyset$, and let $F(\overline{x}, \overline{x}) = 0$. Assume that $F_{\overline{x}}(\cdot)$ and $g(\cdot)$ are Gâteaux differentiable at $\overline{x} \in S$, $F_{\overline{x}}(\cdot)$ is C_Y-convex on S_1, and $g(\cdot)$ is C_Z-convex on S_1. If there exist $v \in C_Y^* \setminus \{0_{Y^*}\}$, $u \in C_Z^*$ such that*

$$\left(v \circ F'_{\overline{x}}(\overline{x}) + u \circ g'(\overline{x})\right)(x - \overline{x}) \geq 0, \quad \forall x \in S_1, \tag{3.26}$$

$$u \circ g(\overline{x}) = 0, \tag{3.27}$$

then $\overline{x} \in S$ is a weakly efficient solution to the VEPC.

Proof. Since the mappings $F_{\overline{x}}(\cdot)$ and $g(\cdot)$ are Gâteaux differentiable at $\overline{x} \in S$, $F_{\overline{x}}(\cdot)$ is C_Y-convex on S_1, and $g(\cdot)$ is C_Z-convex on S_1, from Lemma 3.1, we have

$$\begin{gathered} F'_{\overline{x}}(\overline{x})(x - \overline{x}) \in F_{\overline{x}}(x) - F_{\overline{x}}(\overline{x}) - C_Y = F_{\overline{x}}(x) - C_Y, \quad \forall x \in S_1, \\ g'(\overline{x})(x - \overline{x}) \in g(x) - g(\overline{x}) - C_Z, \quad \forall x \in S_1. \end{gathered} \tag{3.28}$$

From $v \in C_Y^*$, $u \in C_Z^*$, and (3.26), we get

$$v(F_{\overline{x}}(x)) + u\left(g(x) - g(\overline{x})\right) \geq \left(v \circ F'_{\overline{x}}(\overline{x}) + u \circ g'(\overline{x})\right)(x - \overline{x}) \geq 0, \quad \forall x \in S_1. \tag{3.29}$$

By (3.27), we have

$$v(F_{\overline{x}}(x)) + u\left(g(x)\right) \geq 0, \quad \forall x \in S_1. \tag{3.30}$$

We will show that $\overline{x} \in S$ is a weakly efficient solution to the VEPC. If not, then there exists $y_0 \in S$ such that

$$F(\overline{x}, y_0) \in -\operatorname{int} C_Y. \tag{3.31}$$

From $v \in C_Y^* \setminus \{0_{Y^*}\}$ and the above statement, we have

$$v\left(F(\overline{x}, y_0)\right) < 0. \tag{3.32}$$

Noticing $y_0 \in S$, we have $g(y_0) \in -C_Z$; so $u(g(y_0)) \leq 0$ because of $u \in C_Z^*$. Hence,

$$v\left(F_{\overline{x}}(y_0)\right) + u\left(g(y_0)\right) < 0. \tag{3.33}$$

This contradicts (3.30). Hence, $\overline{x} \in S$ is a weakly efficient solution to the VEPC. □

From Theorems 3.2 and 3.4, and Remark 2.7, we get the following corollary.

Corollary 3.5. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, and $(Z, \|\cdot\|_Z)$ be real normed spaces, let $C_Y \subset Y$ be a closed convex pointed cone with $\operatorname{int} C_Y \neq \emptyset$, and let $C_Z \subset Z$ be a closed convex pointed cone with $\operatorname{int} C_Z \neq \emptyset$, and let $F(\overline{x}, \overline{x}) = 0$. Assume that $F_{\overline{x}}(\cdot)$ and $g(\cdot)$ are Fréchet differentiable at $\overline{x} \in S$, $F_{\overline{x}}(\cdot)$ is C_Y-convex on S_1, and $g(\cdot)$ is C_Z-convex on S_1. Furthermore, assume that there exists $x_1 \in S_1$ such that $g(\overline{x}) + g'(\overline{x})(x_1 - \overline{x}) \in -\operatorname{int} C_Z$. Then $\overline{x} \in S$ is a weakly efficient solution to the VEPC if and only if there exist $v \in C_Y^* \setminus \{0_{Y^*}\}$, $u \in C_Z^*$, such that*

$$\begin{gathered}\left(v \circ F'_{\overline{x}}(\overline{x}) + u \circ g'(\overline{x})\right)(x - \overline{x}) \geq 0, \quad \forall x \in S_1, \\ u \circ g(\overline{x}) = 0.\end{gathered} \tag{3.34}$$

The concept of weakly efficient solution to the vector equilibrium problem requires the condition that the ordering cone has an nonempty interior. If the ordering cone has an empty interior, we cannot discuss the property of weakly efficient solutions to the vector equilibrium problem. However, if the ordering cone has a base, we can give necessary conditions and sufficient conditions for Henig efficient solutions and globally efficient solutions to the vector equilibrium problems with constraints.

Theorem 3.6. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, and $(Z, \|\cdot\|_Z)$ be real normed spaces, let C_Y be a closed convex pointed cone in Y with a base, let C_Z be a closed convex pointed cone in Z with $\operatorname{int} C_Z \neq \emptyset$, and let $F(\overline{x}, \overline{x}) = 0$. Assume that $F_{\overline{x}}(\cdot)$ and $g(\cdot)$ are Fréchet differentiable at $\overline{x} \in S$. Furthermore, Assume that there exists $x_1 \in S_1$ such that $g(\overline{x}) + g'(\overline{x})(x_1 - \overline{x}) \in -\operatorname{int} C_Z$. If $\overline{x} \in S$ is a Henig efficient solution to the VEPC, then there exist $v \in C^{\Delta}(B)$, $u \in C_Z^*$ such that*

$$\begin{gathered}\left(v \circ F'_{\overline{x}}(\overline{x}) + u \circ g'(\overline{x})\right)(x - \overline{x}) \geq 0, \quad \forall x \in S_1, \\ u \circ g(\overline{x}) = 0.\end{gathered} \tag{3.35}$$

Proof. Assume that $\overline{x} \in S$ is a Henig efficient solution to the VEPC. By the definition, there exists some $0 < \varepsilon < \delta$ such that

$$F(\overline{x}, S) \cap (-\operatorname{int} C_\varepsilon(B)) = \emptyset. \tag{3.36}$$

Define the set

$$\begin{aligned}M = \{(y, z) \in Y \times Z : \text{there exists } x \in S_1 \text{ such that } y - F'_{\overline{x}}(\overline{x})(x - \overline{x}) \in \operatorname{int} C_\varepsilon(B), \\ z - \left(g(\overline{x}) + g'(\overline{x})(x - \overline{x})\right) \in \operatorname{int} C_Z\}.\end{aligned} \tag{3.37}$$

It is clear that M is a nonempty open convex set and $(0,0) \notin M$. By the separation theorem of convex sets, there exists $(0,0) \neq (v, u) \in (Y \times Z)^* = Y^* \times Z^*$ such that

$$v(y) + u(z) > 0, \quad \forall (y, z) \in M. \tag{3.38}$$

Let $(y, z) \in M$. Then there exists $x \in S_1$ such that

$$y - F'_{\overline{x}}(\overline{x})(x - \overline{x}) \in \operatorname{int} C_\varepsilon(B), \qquad z - \left(g(\overline{x}) + g'(\overline{x})(x - \overline{x})\right) \in \operatorname{int} C_Z. \tag{3.39}$$

Hence, for every $c \in \operatorname{int} C_\varepsilon(B)$, $k \in \operatorname{int} C_Z$, $t' > 0$, $t'' > 0$, we have $(y + t'c, z) \in M$ and $(y, z + t''k) \in M$; this implies that $v \in (C_\varepsilon(B))^*$, $u \in C_Z^*$. In a way similar to the proof of Theorem 3.2, we have $v \neq 0_{Y^*}$. By Lemma 2.9, we can see that $v \in C^\Delta(B)$. It is clear that $(F'_{\overline{x}}(\overline{x})(x - \overline{x}) + c, g(\overline{x}) + g'(\overline{x})(x - \overline{x}) + k) \in M$ for all $x \in S_1$, $c \in \operatorname{int} C_\varepsilon(B)$, $k \in \operatorname{int} C_Z$. By (3.38), we get

$$v\big(F'_{\overline{x}}(\overline{x})(x - \overline{x})\big) + u\big(g(\overline{x}) + g'(\overline{x})(x - \overline{x})\big) \geq 0, \quad \forall x \in S_1. \tag{3.40}$$

In a way similar to the proof of Theorem 3.2, we have

$$u\big(g(\overline{x})\big) = 0. \tag{3.41}$$

From (3.40) and (3.41), we have

$$\big(v \circ F'_{\overline{x}}(\overline{x}) + u \circ g'(\overline{x})\big)(x - \overline{x}) \geq 0, \quad \forall x \in S_1. \tag{3.42}$$

This completes the proof. □

Theorem 3.7. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, and $(Z, \|\cdot\|_Z)$ be real normed spaces, let C_Y be a closed convex pointed cone in Y with a base, let C_Z be a closed convex pointed cone in Z with $\operatorname{int} C_Z \neq \emptyset$, and let $F(\overline{x}, \overline{x}) = 0$. Assume that $F_{\overline{x}}(\cdot)$ and $g(\cdot)$ are Gâteaux differentiable at $\overline{x} \in S$, $F_{\overline{x}}(\cdot)$ is C_Y-convex on S_1, and $g(\cdot)$ is C_Z-convex on S_1. If there exist $v \in C^\Delta(B)$, $u \in C_Z^*$ such that*

$$\big(v \circ F'_{\overline{x}}(\overline{x}) + u \circ g'(\overline{x})\big)(x - \overline{x}) \geq 0, \quad \forall x \in S_1, \tag{3.43}$$

$$u \circ g(\overline{x}) = 0, \tag{3.44}$$

then $\overline{x} \in S$ is a Henig efficient solution to the VEPC.

Proof. From $v \in C^\Delta(B) \subset C_Y^*$, $u \in C_Z^*$, (3.43), and (3.44), in a way similar to the proof of Theorem 3.4, we have

$$v(F_{\overline{x}}(x)) + u\big(g(x)\big) \geq 0, \quad \forall x \in S_1. \tag{3.45}$$

We will show that $\overline{x} \in S$ is a Henig efficient solution to the VEPC; that is, there exists some $0 < \varepsilon < \delta$ such that

$$F(\overline{x}, S) \cap (-\operatorname{int} C_\varepsilon(B)) = \emptyset. \tag{3.46}$$

Suppose to the contrary that for any $0 < \varepsilon < \delta$,

$$F_{\overline{x}}(S) \cap (-\operatorname{int} C_\varepsilon(B)) \neq \emptyset. \tag{3.47}$$

Then by $v \in C^{\Delta}(B)$, and by Lemma 2.9, there exists some $0 < \varepsilon_0 < \delta$ such that $v \in (C_{\varepsilon_0}(B))^* \setminus \{0_{Y^*}\}$. For this $0 < \varepsilon_0 < \delta$, we have $F_{\overline{x}}(S) \cap (-\operatorname{int} C_{\varepsilon_0}(B)) \neq \emptyset$. Thus, there exists $y_0 \in S$ such that

$$F_{\overline{x}}(y_0) \in -\operatorname{int} C_{\varepsilon_0}(B). \tag{3.48}$$

By $v \in (C_{\varepsilon_0}(B))^* \setminus \{0_{Y^*}\}$, we have

$$v(F_{\overline{x}}(y_0)) < 0. \tag{3.49}$$

Notice $y_0 \in S$, we have $g(y_0) \in -C_Z$, and thus we obtain $u(g(y_0)) \leq 0$ because of $u \in C_Z^*$. Hence,

$$v(F_{\overline{x}}(y_0)) + u(g(y_0)) < 0. \tag{3.50}$$

This contradicts (3.45). Hence, $\overline{x} \in S$ is a Henig efficient solution to the VEPC. □

Corollary 3.8. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, and $(Z, \|\cdot\|_Z)$ be real normed spaces, and $C_Y \subset Y$ be a closed convex pointed cone with a base, let $C_Z \subset Z$ be a closed convex pointed cone with $\operatorname{int} C_Z \neq \emptyset$, and let $F(\overline{x}, \overline{x}) = 0$. Assume that $F_{\overline{x}}(\cdot)$ and $g(\cdot)$ are Fréchet differentiable at $\overline{x} \in S$, $F_{\overline{x}}(\cdot)$ is C_Y-convex on S_1, and $g(\cdot)$ is C_Z-convex on S_1. Furthermore, assume that there exists $x_1 \in S_1$ such that $g(\overline{x}) + g'(\overline{x})(x_1 - \overline{x}) \in -\operatorname{int} C_Z$. Then $\overline{x} \in S$ is a Henig efficient solution to the VEPC if and only if there exist $v \in C^{\Delta}(B)$, $u \in C_Z^*$, such that*

$$\begin{gathered}
(v \circ F'_{\overline{x}}(\overline{x}) + u \circ g'(\overline{x}))(x - \overline{x}) \geq 0, \quad \forall x \in S_1, \\
u \circ g(\overline{x}) = 0.
\end{gathered} \tag{3.51}$$

Remark 3.9. If C_Y has a bounded closed base B, in view of Lemma 2.9, we have $C^{\Delta}(B) = \operatorname{int} C_Y^*$; besides, $\overline{x} \in S$ is a superefficient solution to the VEPC if and only if $\overline{x}$ is a Henig efficient solution to the VEPC (see [22]). Hence, by Corollary 3.8, we have the following corollary.

Corollary 3.10. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, and $(Z, \|\cdot\|_Z)$ be real normed spaces, let $C_Y \subset Y$ be a closed convex pointed cone with a bounded closed base, let $C_Z \subset Z$ be a closed convex pointed cone with $\operatorname{int} C_Z \neq \emptyset$, and let $F(\overline{x}, \overline{x}) = 0$. Assume that $F_{\overline{x}}(\cdot)$ and $g(\cdot)$ are Fréchet differentiable at $\overline{x} \in S$, $F_{\overline{x}}(\cdot)$ is C_Y-convex on S_1, and $g(\cdot)$ is C_Z-convex on S_1. Furthermore, assume that there exists $x_1 \in S_1$ such that $g(\overline{x}) + g'(\overline{x})(x_1 - \overline{x}) \in -\operatorname{int} C_Z$. Then $\overline{x} \in S$ is a superefficient efficient solution to the VEPC if and only if there exists $v \in \operatorname{int} C_Y^*$ (with respect to the norm topology of Y^*), $u \in C_Z^*$ such that*

$$\begin{gathered}
(v \circ F'_{\overline{x}}(\overline{x}) + u \circ g'(\overline{x}))(x - \overline{x}) \geq 0, \quad \forall x \in S_1, \\
u \circ g(\overline{x}) = 0.
\end{gathered} \tag{3.52}$$

Theorem 3.11. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, and $(Z, \|\cdot\|_Z)$ be real normed spaces, let C_Y be a closed convex pointed cone in Y with a base, let C_Z be a closed convex pointed cone in Z with $\operatorname{int} C_Z \neq \emptyset$, and let $F(\overline{x}, \overline{x}) = 0$. Assume that $F_{\overline{x}}(\cdot)$ and $g(\cdot)$ are Fréchet differentiable at $\overline{x} \in S$, and there exists $x_1 \in S_1$ such that $g(\overline{x}) + g'(\overline{x})(x_1 - \overline{x}) \in -\operatorname{int} C_Z$. If $\overline{x} \in S$ is a globally efficient solution to the VEPC, then there exist $v \in C_Y^{\#}$, $u \in C_Z^*$ such that*

$$\begin{gathered}\left(v \circ F'_{\overline{x}}(\overline{x}) + u \circ g'(\overline{x})\right)(x - \overline{x}) \geq 0, \quad \forall x \in S_1,\\ u \circ g(\overline{x}) = 0.\end{gathered} \tag{3.53}$$

Proof. Assume that $\overline{x} \in S$ is a globally efficient solution to the VEPC. By the definition, there exists a pointed convex cone H such that $C_Y \setminus \{0_Y\} \subset \operatorname{int} H$ and

$$F(\overline{x}, S) \cap (-H \setminus \{0_Y\}) = \emptyset. \tag{3.54}$$

Define the set

$$\begin{aligned}M = \{ (y,z) \in Y \times Z : &\text{ there exists } x \in S_1 \text{ such that } y - F'_{\overline{x}}(\overline{x})(x - \overline{x}) \in \operatorname{int} H,\\ &z - \left(g(\overline{x}) + g'(\overline{x})(x - \overline{x})\right) \in \operatorname{int} C_Z \}.\end{aligned} \tag{3.55}$$

Since H is a convex cone, we know that M is a nonempty open convex set and $(0,0) \notin M$. By the separation theorem of convex sets, there exists $(0,0) \neq (v,u) \in (Y \times Z)^* = Y^* \times Z^*$ such that

$$v(y) + u(z) > 0, \quad \forall (y,z) \in M. \tag{3.56}$$

Let $(y,z) \in M$. Then there exists $x \in S_1$ such that

$$y - F'_{\overline{x}}(\overline{x})(x - \overline{x}) \in \operatorname{int} H, \qquad z - \left(g(\overline{x}) + g'(\overline{x})(x - \overline{x})\right) \in \operatorname{int} C_Z. \tag{3.57}$$

Hence, for every $c \in \operatorname{int} H$, $k \in \operatorname{int} C_Z$, $t' > 0$, $t'' > 0$, we have $(y + t'c, z) \in M$ and $(y, z + t''k) \in M$. This implies that $v \in H^* \setminus \{0_{Y^*}\}$, $u \in C_Z^*$, and therefore $v \in C_Y^{\#}$ because of $C \setminus \{0\} \subset \operatorname{int} H$. It is clear that

$$\left(F'_{\overline{x}}(\overline{x})(x - \overline{x}) + c, g(\overline{x}) + g'(\overline{x})(x - \overline{x}) + k\right) \in M \tag{3.58}$$

for all $x \in S_1$, $c \in \operatorname{int} H$, $k \in \operatorname{int} C_Z$. By (3.56), we get

$$v\left(F'_{\overline{x}}(\overline{x})(x - \overline{x}) + c\right) + u\left(g(\overline{x}) + g'(\overline{x})(x - \overline{x}) + k\right) > 0 \tag{3.59}$$

for all $x \in S_1$, $c \in \operatorname{int} H$, $k \in \operatorname{int} C_Z$.

Letting $c \to 0,\ k \to 0$, we get

$$v\big(F'_{\overline{x}}(\overline{x})(x-\overline{x})\big) + u\big(g(\overline{x}) + g'(\overline{x})(x-\overline{x})\big) \geq 0, \quad \forall x \in S_1. \tag{3.60}$$

In a way similar to the proof of Theorem 3.2, we have

$$u\big(g(\overline{x})\big) = 0. \tag{3.61}$$

From (3.60) and (3.61), we have

$$\big(v \circ F'_{\overline{x}}(\overline{x}) + u \circ g'(\overline{x})\big)(x-\overline{x}) \geq 0, \quad \forall x \in S_1. \tag{3.62}$$

This completes the proof. □

Theorem 3.12. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, and $(Z, \|\cdot\|_Z)$ be real normed spaces, let C_Y be a closed convex pointed cone in Y with a base, let C_Z be a closed convex pointed cone in Z with $\operatorname{int} C_Z \neq \emptyset$, and let $F(\overline{x}, \overline{x}) = 0$. Assume that $F_{\overline{x}}(\cdot)$ and $g(\cdot)$ are Gâteaux differentiable at $\overline{x} \in S$, $F_{\overline{x}}(\cdot)$ is C_Y-convex on S_1, and $g(\cdot)$ is C_Z-convex on S_1. If there exist $v \in C_Y^{\#},\ u \in C_Z^*$ such that*

$$\big(v \circ F'_{\overline{x}}(\overline{x}) + u \circ g'(\overline{x})\big)(x-\overline{x}) \geq 0, \quad \forall x \in S_1, \tag{3.63}$$

$$u \circ g(\overline{x}) = 0, \tag{3.64}$$

then $\overline{x} \in S$ is a globally efficient solution to the VEPC.

Proof. From $v \in C_Y^{\#} \subset C_Y^*$, $u \in C_Z^*$, (3.63), and (3.64), in a way similar to the proof of Theorem 3.4, we have

$$v(F_{\overline{x}}(x)) + u\big(g(x)\big) \geq 0, \quad \forall x \in S_1. \tag{3.65}$$

We will show that $\overline{x} \in S$ is a globally efficient solution to the VEPC; that is, there exists a pointed convex cone $H \subset Y$ such that $C_Y \setminus \{0_Y\} \subset \operatorname{int} H$ and

$$F(\overline{x}, S) \cap (-\operatorname{int} H) = \emptyset. \tag{3.66}$$

Suppose to the contrary that for any pointed convex cone $H \subset Y$ with $C_Y \setminus \{0_Y\} \subset \operatorname{int} H$, we have that

$$F_{\overline{x}}(S) \cap (-\operatorname{int} H) \neq \emptyset. \tag{3.67}$$

By $v \in C_Y^{\#}$, we set

$$H_0 = \big\{y \in Y : v(y) > 0\big\} \cup \{0_Y\}. \tag{3.68}$$

We have $C_Y \setminus \{0_Y\} \subset \operatorname{int} H_0$, and H_0 is a pointed convex cone. By (3.67), there exists $y_0 \in S$ such that

$$F(\overline{x}, y_0) = F_{\overline{x}}(y_0) \in -H_0 \setminus \{0_Y\}. \tag{3.69}$$

By the definition of H_0, we have that $v(F_{\overline{x}}(y_0)) < 0$. Noticing $y_0 \in S$, we have $g(y_0) \in -C_Z$, and so $u(g(y_0)) \leq 0$ because of $u \in C_Z^*$. Hence

$$v(F_{\overline{x}}(y_0)) + u(g(y_0)) < 0. \tag{3.70}$$

This contradicts (3.65). Hence, $\overline{x} \in S$ is a globally efficient solution to the VEPC. □

Corollary 3.13. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, and $(Z, \|\cdot\|_Z)$ be real normed spaces, let $C_Y \subset Y$ be a closed convex pointed cone with a base, let $C_Z \subset Z$ be a closed convex pointed cone with $\operatorname{int} C_Z \neq \emptyset$, and let $F(\overline{x}, \overline{x}) = 0$. Assume that $F_{\overline{x}}(\cdot)$ and $g(\cdot)$ are Fréchet differentiable at $\overline{x} \in S$, $F_{\overline{x}}(\cdot)$ is C_Y-convex on S_1, and $g(\cdot)$ is C_Z-convex on S_1. Furthermore, assume that there exists $x_1 \in S_1$ such that $g(\overline{x}) + g'(\overline{x})(x_1 - \overline{x}) \in -\operatorname{int} C_Z$. Then $\overline{x} \in S$ is a globally efficient solution to the VEPC if and only if there exist $v \in C_Y^{\#}$, $u \in C_Z^*$ such that*

$$\begin{gathered} \left(v \circ F'_{\overline{x}}(\overline{x}) + u \circ g'(\overline{x})\right)(x - \overline{x}) \geq 0, \quad \forall x \in S_1, \\ u \circ g(\overline{x}) = 0. \end{gathered} \tag{3.71}$$

Acknowledgments

This research was partially supported by the National Natural Science Foundation of China and the Natural Science Foundation of Jiangxi Province (2008GZS0072), China.

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