Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2011, Article ID 158219, 7 pages
doi:10.1155/2011/158219

Research Article

L^p Approximation by Multivariate Baskakov-Durrmeyer Operator

Feilong Cao and Yongfeng An

Department of Mathematics, China Jiliang University, Hangzhou 310018, Zhejiang Province, China

Correspondence should be addressed to Feilong Cao, feilongcao@gmail.com

Received 14 November 2010; Accepted 17 January 2011

Academic Editor: Jewgeni Dshalalow

The main aim of this paper is to introduce and study multivariate Baskakov-Durrmeyer operator, which is nontensor product generalization of the one variable. As a main result, the strong direct inequality of L^p approximation by the operator is established by using a decomposition technique.

1. Introduction

Let $P_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}$, $x \in [0,\infty)$, $n \in \mathbb{N}$. The Baskakov operator defined by

$$B_{n,1}(f,x) = \sum_{k=0}^{\infty} P_{n,k}(x) f\left(\frac{k}{n}\right) \tag{1.1}$$

was introduced by Baskakov [1] and can be used to approximate a function f defined on $[0,\infty)$. It is the prototype of the Baskakov-Kantorovich operator (see [2]) and the Baskakov-Durrmeyer operator defined by (see [3, 4])

$$M_{n,1}(f,x) = \sum_{k=0}^{\infty} P_{n,k}(x)(n-1)\int_0^{\infty} P_{n,k}(t) f(t)\,dt, \quad x \in [0,\infty), \tag{1.2}$$

where $f \in L^p[0,\infty)(1 \le p < \infty)$.

By now, the approximation behavior of the Baskakov-Durrmeyer operator is well understood. It is characterized by the second-order Ditzian-Totik modulus (see [3])

$$\omega_\varphi^2(f,t)_p = \sup_{0<h\le t} \left\| f(\cdot + 2h\varphi(\cdot)) - 2f(\cdot + h\varphi(\cdot)) + f(\cdot) \right\|_p, \quad \varphi(x) = \sqrt{x(1+x)}. \tag{1.3}$$

More precisely, for any function defined on $L^p[0,\infty)(1 \le p < \infty)$, there is a constant such that

$$\|M_{n,1}(f) - f\|_p \le \text{const.}\left(\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right)_p + \frac{1}{n}\|f\|_p\right), \tag{1.4}$$

$$\omega_\varphi^2(f,t)_p = O\left(t^{2\alpha}\right) \Longleftrightarrow \|M_{n,1}(f) - f\|_p = O(n^{-\alpha}), \tag{1.5}$$

where $0 < \alpha < 1$.

Let $T \subset \mathbb{R}^d$ $(d \in \mathbb{N})$, which is defined by

$$T := T_d := \{\mathbf{x} := (x_1, x_2, \ldots, x_d) : 0 \le x_i < \infty, 1 \le i \le d\}. \tag{1.6}$$

Here and in the following, we will use the standard notations

$$\begin{gathered}\mathbf{x} := (x_1, x_2, \ldots, x_d), \qquad \mathbf{k} := (k_1, k_2, \ldots, k_d) \in \mathbb{N}_0^d, \\ \mathbf{x}^{\mathbf{k}} := x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}, \qquad \mathbf{k}! = k_1! k_2! \cdots k_d!, \qquad |\mathbf{x}| := \sum_{i=1}^{d} x_i, \qquad |\mathbf{k}| := \sum_{i=1}^{d} k_i, \\ \binom{n}{\mathbf{k}} := \frac{n!}{\mathbf{k}!(n - |\mathbf{k}|)!}, \qquad \sum_{\mathbf{k}=0}^{\infty} := \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_d=0}^{\infty}.\end{gathered} \tag{1.7}$$

By means of the notations, for a function f defined on T the multivariate Baskakov operator is defined as (see [5])

$$B_{n,d}(f, \mathbf{x}) := \sum_{\mathbf{k}=0}^{\infty} f\left(\frac{\mathbf{k}}{n}\right) P_{n,\mathbf{k}}(\mathbf{x}), \tag{1.8}$$

where

$$P_{n,\mathbf{k}}(\mathbf{x}) = \binom{n + |\mathbf{k}| - 1}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1 + |\mathbf{x}|)^{-n-|\mathbf{k}|}. \tag{1.9}$$

Naturally, we can modify the multivariate Baskakov operator as multivariate Baskakov-Durrmeyer operator

$$M_{n,d} f := M_{n,d}(f, \mathbf{x}) := \sum_{\mathbf{k}=0}^{\infty} P_{n,\mathbf{k}}(\mathbf{x}) \phi_{n,\mathbf{k},d}(f), \quad f \in L^p(T), \tag{1.10}$$

where

$$\phi_{n,\mathbf{k},d}(f) := \frac{\int_T P_{n,\mathbf{k}}(\mathbf{u}) f(\mathbf{u}) d\mathbf{u}}{\int_T P_{n,\mathbf{k}}(\mathbf{u}) d\mathbf{u}} = (n-1)(n-2)\cdots(n-d) \int_T P_{n,\mathbf{k}}(\mathbf{u}) f(\mathbf{u}) d\mathbf{u}. \tag{1.11}$$

It is a multivariate generalization of the univariate Baskakov-Durrmeyer operators given in (1.2) and can be considered as a tool to approximate the function in $L^p(T)$.

2. Main Result

We will show a direct inequality of L^p approximation by the Baskakov-Durrmeyer operator given in (1.10). By means of K-functional and modulus of smoothness defined in [5], we will extend (1.4) to the case of higher dimension by using a decomposition technique.

Fox $\mathbf{x} \in T$, we define the weight functions

$$\varphi_i(\mathbf{x}) = \sqrt{x_i(1+|\mathbf{x}|)}, \quad 1 \le i \le d. \tag{2.1}$$

Let

$$D_i^r = \frac{\partial^r}{\partial x_i^r}, \quad r \in \mathbb{N}, \qquad D^{\mathbf{k}} = D_1^{k_1} D_2^{k_2} \cdots D_d^{k_d}, \quad \mathbf{k} \in \mathbb{N}_0^d \tag{2.2}$$

denote the differential operators. For $1 \le p < \infty$, we define the weighted Sobolev space as follows:

$$W_\varphi^{r,p}(T) = \left\{ f \in L^p(T) : D^{\mathbf{k}} f \in L_{\text{loc}}(\dot{T}),\ \varphi_i^r D_i^r f \in L^p(T) \right\}, \tag{2.3}$$

where $|\mathbf{k}| \le r$, $\mathbf{k} \in \mathbb{N}_0^d$, and $\dot{T}$ denotes the interior of T. The Peetre K-functional on $L^p(T)$ $(1 \le p < \infty)$, are defined by

$$K_\varphi^r(f, t^r)_p = \inf \left\{ \|f - g\|_p + t^r \sum_{i=1}^d \|\varphi_i^r D_i^r g\|_p \right\}, \quad t > 0, \tag{2.4}$$

where the infimum is taken over all $g \in W_\varphi^{r,p}(T)$.

For any vector $\mathbf{e}$ in $\mathbb{R}^d$, we write the rth forward difference of a function f in the direction of $\mathbf{e}$ as

$$\Delta_{h\mathbf{e}}^r f(\mathbf{x}) = \begin{cases} \displaystyle\sum_{i=0}^r \binom{r}{i} (-1)^i f(\mathbf{x} + ih\mathbf{e}), & \mathbf{x}, \mathbf{x} + rh\mathbf{e} \in T, \\ 0, & \text{otherwise}. \end{cases} \tag{2.5}$$

We then can define the modulus of smoothness of $f \in L^p(T) (1 \le p < \infty)$, as

$$\omega_\varphi^r(f, t)_p = \sup_{0<h\le t} \sum_{i=1}^d \|\Delta_h^r \varphi_i \mathbf{e}_i f\|_p, \tag{2.6}$$

where $\mathbf{e}_i$ denotes the unit vector in $\mathbb{R}^d$, that is, its ith component is 1 and the others are 0.

In [5], the following result has been proved.

Lemma 2.1. *There exists a positive constant, dependent only on p and r, such that for any $f \in L^p(T)$, $1 \le p < \infty$*

$$\frac{1}{\text{const.}}\omega_\varphi^r(f,t)_p \le K_\varphi^r(f,t^r)_p \le \text{const.}\,\omega_\varphi^r(f,t)_p. \tag{2.7}$$

Now we state the main result of this paper.

Theorem 2.2. *If $f \in L^p(T)$, $1 \le p < \infty$, then there is a positive constant independent of n and f such that*

$$\|M_{n,d}f - f\|_p \le \text{const.}\left(\omega_\varphi^2\left(f,\frac{1}{\sqrt{n}}\right)_p + \frac{1}{n}\|f\|_p\right). \tag{2.8}$$

Proof. Our proof is based on an induction argument for the dimension d. We will also use a decomposition method of the operator $M_{n,d}f$. We report the detailed proof only for two dimensions. The higher dimensional cases are similar.

Our proof depends on Lemma 2.1 and the following estimates:

$$\|M_{n,2}f - f\|_p \le \text{const.}\begin{cases} \|f\|_p, & f \in L^p(T), \\ \dfrac{1}{n}\left(\displaystyle\sum_{i=1}^{2}\|\varphi_i^2 D_i^2 f\|_p + \|f\|_p\right), & f \in W_\varphi^{2,p}(T). \end{cases} \tag{2.9}$$

The first estimate is evident as the $M_{n,d}f$ are positive and linear contractions on $L^p(T)(1 \le p < \infty)$. We can demonstrate the second estimate by reducing it to the one dimensional inequality

$$\|M_{n,1}f - f\|_p \le \frac{\text{const.}}{n}\left(\left\|\varphi^2 f''\right\|_p + \|f\|_p\right), \tag{2.10}$$

which has been proved in [3]

Now we give the following decomposition formula:

$$\begin{aligned} M_{n,2}(f,\mathbf{x}) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} P_{n,k_1}(x_1)P_{n+k_1,k_2}\left(\frac{x_2}{1+x_1}\right)(n-1)(n-2) \\ &\quad\times \iint_0^{\infty} P_{n,k_1}(u_1)P_{n+k_1,k_2}\left(\frac{u_2}{1+u_1}\right)f(u_1,u_2)du_1\,du_2 \\ &= \sum_{k_1=0}^{\infty} P_{n,k_1}(x_1)(n-2)\int_0^{\infty} P_{n-1,k_1}(u_1)\sum_{k_2=0}^{\infty}P_{n+k_1,k_2}\left(\frac{x_2}{1+x_1}\right) \\ &\quad\times (n+k_1-1)\int_0^{\infty} P_{n+k_1,k_2}(t)f(u_1,(1+u_1)t)dt\,du_1 \\ &= \sum_{k_1=0}^{\infty} P_{n,k_1}(x_1)(n-2)\int_0^{\infty} P_{n-1,k_1}(u_1)M_{n+k_1,1}(g_{u_1},z)du_1, \end{aligned} \tag{2.11}$$

where

$$g_{u_1}(t) = f(u_1, (1+u_1)t), \quad 0 \le t < \infty, \quad z = \frac{x_2}{1+x_1}, \tag{2.12}$$

which can be checked directly and will play an important role in the following proof.

From the decomposition formula, it follows that

$$\begin{aligned} M_{n,2}(f,\mathbf{x}) - f(\mathbf{x}) &= \sum_{k_1=0}^{\infty} P_{n,k_1}(x_1)(n-2) \\ &\quad \times \left\{ \int_0^\infty P_{n-1,k_1}(u_1)\left(M_{n+k_1,1}(g_{u_1}, z) - g_{u_1}(z)\right) du_1 \right\} + M^*_{n,1}(h(\cdot), x_1) - h(x_1) \\ &:= J + L, \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} h(u_1) &:= h(u_1, \mathbf{x}) := f\left(u_1, (1+u_1)\frac{x_2}{1+x_1}\right), \quad 0 \le u_1 < \infty, \\ M^*_{n,1}(g, y) &= \sum_{l=0}^{\infty} P_{n,l}(y)(n-2)\int_0^\infty P_{n-1,l}(t) g(t)\,dt. \end{aligned} \tag{2.14}$$

Then by the Jensen's inequality, we have

$$\begin{aligned} \|J\|_p^p &\le \int_T \sum_{k_1=0}^{\infty} P_{n,k_1}(x_1) \left| (n-2) \int_0^\infty P_{n-1,k_1}(u_1)(M_{n+k_1,1}(g_{u_1}, z) - g_{u_1}(z)) du_1 \right|^p dx \\ &\le \int_T \sum_{k_1=0}^{\infty} P_{n,k_1}(x_1)(n-2) \int_0^\infty P_{n-1,k_1}(u_1) \left| (M_{n+k_1,1}(g_{u_1}, z) - g_{u_1}(z)) \right|^p du_1 dx \\ &= \int_0^\infty \sum_{k_1=0}^{\infty} P_{n,k_1}(x_1)(1+x_1) dx_1 (n-2) \iint_0^\infty P_{n-1,k_1}(u_1) \\ &\quad \times \left| (M_{n+k_1,1}(g_{u_1}, z) - g_{u_1}(z)) \right|^p dz du_1 \\ &\le \sum_{k_1=0}^{\infty} \frac{n+k_1-1}{n-1} \int_0^\infty P_{n-1,k_1}(u_1) \int_0^\infty \left| (M_{n+k_1,1}(g_{u_1}, z) - g_{u_1}(z)) \right|^p dz du_1 \\ &\le \text{const.} \sum_{k_1=0}^{\infty} \frac{n+k_1-1}{n-1} \int_0^\infty P_{n-1,k_1}(u_1) \left(\frac{1}{n+k_1} \right)^p \left(\left\| \varphi^2 g''_{u_1} \right\|_p^p + \left\| g_{u_1} \right\|_p^p \right) du_1. \end{aligned} \tag{2.15}$$

However, by definition, one also has

$$\varphi^2(t) g''_{u_1}(t) = t(1+t)(1+u_1)^2 D_2^2 f(u_1, (1+u_1)t) = \left(\varphi_2^2 D_2^2 f\right)(u_1, (1+u_1)t). \tag{2.16}$$

Therefore,

$$\begin{aligned}
\|J\|_p^p &\le \text{const.}\sum_{k_1=0}^{\infty}\frac{n+k_1-1}{(n-1)(n+k_1)^p}\iint_0^{\infty}P_{n-1,k_1}(u_1)\\
&\quad\times\left(\left|\left(\varphi_2^2D_2^2f\right)(u_1,(1+u_1)t)\right|^p+\left|f(u_1,(1+u_1)t)\right|^p\right)dt\,du_1\\
&= \text{const.}\sum_{k_1=0}^{\infty}\frac{n+k_1-1}{(n-1)(n+k_1)^p}\int_0^{\infty}\frac{1}{1+u_1}P_{n-1,k_1}(u_1)\\
&\quad\times\int_0^{\infty}\left(\left|(\varphi_2^2(u_1,u_2)D_2^2f(u_1,u_2)\right|^p+\left|f(u_1,u_2)\right|^p\right)du_1\,du_2\\
&\le\frac{\text{const.}}{n^p}\sum_{k_1=0}^{\infty}\int_0^{\infty}P_{n,k_1}(u_1)\int_0^{\infty}\left(\left|\left(\varphi_2^2(u_1,u_2)D_2^2f(u_1,u_2)\right)\right|^p+\left|f(u_1,u_2)\right|^p\right)du_1\,du_2\\
&=\frac{\text{const.}}{n^p}\left(\left\|\varphi_2^2D_2^2f\right\|_p^p+\|f\|_p^p\right).
\end{aligned}\tag{2.17}$$

To estimate the second term L, we use a similar method as to estimate (2.10) (see [3]) and can get

$$\|L\|_p\le\frac{\text{const.}}{n}\left(\left\|\varphi^2h''\right\|_p+\|h\|_p\right).\tag{2.18}$$

Denoting $\varphi_{12}(\mathbf{x})=\varphi_{21}(\mathbf{x}):=\sqrt{x_1x_2}$, $D_{12}^2:=\partial^2/(\partial x_1\partial x_2)$, and $D_{21}^2:=\partial^2/(\partial x_2\partial x_1)$, we have

$$\begin{aligned}
&\left|\varphi^2(s)h''(s)\right|\\
&=\left|s(1+s)\left(D_1^2f+\frac{x_2}{1+x_1}D_{12}^2f+\frac{x_2}{1+x_1}D_{21}^2f+\frac{x_2^2}{(1+x_1)^2}D_{22}^2f\right)\times\left(s,(1+s)\frac{x_2}{1+x_1}\right)\right|\\
&=\left|\left(\frac{1+x_1}{1+x_1+x_2}\varphi_1^2D_1^2f+\varphi_{12}^2D_{12}^2f+\varphi_{21}^2D_{21}^2f+\frac{s}{1+s}\frac{x_2}{1+x_1+x_2}\varphi_2^2D_2^2f\right)\left(s,(1+s)\frac{x_2}{1+x_1}\right)\right|.
\end{aligned}\tag{2.19}$$

Recalling that $\varphi_{12}(\mathbf{x})$ is no bigger than $\varphi_1(\mathbf{x})$ or $\varphi_2(\mathbf{x})$, and the fact

$$\left|D_{12}^2f(\mathbf{x})\right|\le\sup\left(\left|D_1^2f(\mathbf{x})\right|,\left|D_2^2f(\mathbf{x})\right|\right)\tag{2.20}$$

proved in [6] (see [6, Lemma 2.1]), we obtain

$$\left\|\varphi^2h''\right\|_p\le\text{const.}\sum_{i=1}^{2}\left\|\varphi_i^2D_i^2f\right\|_p,\tag{2.21}$$

and hence

$$\|L\|_p \le \frac{\text{const.}}{n}\left(\sum_{i=1}^{2}\left\|\varphi_i^2 D_i^2 f\right\|_p + \|f\|_p\right). \tag{2.22}$$

The second inequality of (2.9) has thus been established, and the proof of Theorem 2.2 is finished. □

Acknowledgment

The research was supported by the National Natural Science Foundation of China (no. 90818020).

References

[1] V. A. Baskakov, "An instance of a sequence of linear positive operators in the space of continuous functions," *Doklady Akademii Nauk SSSR*, vol. 113, pp. 249–251, 1957.

[2] Z. Ditzian and V. Totik, *Moduli of Smoothness*, vol. 9 of *Springer Series in Computational Mathematics*, Springer, New York, NY, USA, 1987.

[3] M. Heilmann, "Direct and converse results for operators of Baskakov-Durrmeyer type," *Approximation Theory and its Applications*, vol. 5, no. 1, pp. 105–127, 1989.

[4] A. Sahai and G. Prasad, "On simultaneous approximation by modified Lupas operators," *Journal of Approximation Theory*, vol. 45, no. 2, pp. 122–128, 1985.

[5] F. Cao, C. Ding, and Z. Xu, "On multivariate Baskakov operator," *Journal of Mathematical Analysis and Applications*, vol. 307, no. 1, pp. 274–291, 2005.

[6] W. Chen and Z. Ditzian, "Mixed and directional derivatives," *Proceedings of the American Mathematical Society*, vol. 108, no. 1, pp. 177–185, 1990.