

Research Article

Refinements of Results about Weighted Mixed Symmetric Means and Related Cauchy Means

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A recent refinement of the classical discrete Jensen inequality is given by Horváth and Pečarić. In this paper, the corresponding weighted mixed symmetric means and Cauchy-type means are defined. We investigate the exponential convexity of some functions, study mean value theorems, and prove the monotonicity of the introduced means.

1. Introduction and Preliminary Results

A new refinement of the discrete Jensen inequality is given in [1]. The following notations are also introduced in [1].

Let X be a set, $P(X)$ its power set, and $|X|$ denotes the number of elements in X . Let $u \geq 1$ and $v \geq 2$ be fixed integers. Define the functions

$$\begin{aligned} S_{v,w} &: \{1, \dots, u\}^v \longrightarrow \{1, \dots, u\}^{v-1}, \quad 1 \leq w \leq v, \\ S_v &: \{1, \dots, u\}^v \longrightarrow P(\{1, \dots, u\}^{v-1}), \\ T_v &: P(\{1, \dots, u\}^v) \longrightarrow P(\{1, \dots, u\}^{v-1}) \end{aligned} \tag{1.1}$$

by

$$S_{v,w}(i_1, \dots, i_v) := (i_1, \dots, i_{w-1}, i_{w+1}, \dots, i_v), \quad 1 \leq w \leq v,$$

$$S_v(i_1, \dots, i_v) = \bigcup_{w=1}^v \{S_{v,w}(i_1, \dots, i_v)\}, \quad (1.2)$$

$$T_v(I) = \begin{cases} \bigcup_{(i_1, \dots, i_v) \in I} S_v(i_1, \dots, i_v), & I \neq \phi, \\ \phi, & I = \phi. \end{cases}$$

Further, introduce the function

$$\alpha_{v,i} : \{1, \dots, u\}^v \longrightarrow \mathbb{N}, \quad 1 \leq i \leq u, \quad (1.3)$$

via

$$\alpha_{v,i}(i_1, \dots, i_v) := \text{Number of occurrences of } i \text{ in the sequence } (i_1, \dots, i_v). \quad (1.4)$$

For each $I \in P(\{1, \dots, u\}^v)$, let

$$\alpha_{I,i} := \sum_{(i_1, \dots, i_v) \in I} \alpha_{v,i}(i_1, \dots, i_v), \quad 1 \leq i \leq u. \quad (1.5)$$

It is easy to observe from the construction of the functions $S_v, S_{v,w}, T_v$ and $\alpha_{v,i}$ that they do not depend essentially on u , so we can write for short S_v for S_v^u , and so on.

(H₁) The following considerations concern a subset I_k of $\{1, \dots, n\}^k$ satisfying

$$\alpha_{I_k,i} \geq 1, \quad 1 \leq i \leq n, \quad (1.6)$$

where $n \geq 1$ and $k \geq 2$ are fixed integers.

Next, we proceed inductively to define the sets $I_l \subset \{1, \dots, n\}^l$ ($k-1 \geq l \geq 1$) by

$$I_{l-1} := T_l(I_l), \quad k \geq l \geq 2. \quad (1.7)$$

By (1.6), $I_1 = \{1, \dots, n\}$ and this implies that $\alpha_{I_1,i} = 1$ for $1 \leq i \leq n$. From (1.6), again, we have $\alpha_{I_l,i} \geq 1$ ($k-1 \geq l \geq 1, 1 \leq i \leq n$).

For every $k \geq l \geq 2$ and for any $(j_1, \dots, j_{l-1}) \in I_{l-1}$, let

$$H_{I_l}(j_1, \dots, j_{l-1}) := \{(i_1, \dots, i_l), m\} \in I_l \times \{1, \dots, l\} \mid S_{l,m}(i_1, \dots, i_l) = (j_1, \dots, j_{l-1})\}. \quad (1.8)$$

Using these sets we define the functions $t_{I_k,l} : I_l \rightarrow \mathbb{N}$ ($k \geq l \geq 1$) inductively by

$$\begin{aligned}
 t_{I_k,k}(i_1, \dots, i_k) &:= 1, \quad (i_1, \dots, i_k) \in I_k, \\
 t_{I_k,l-1}(j_1, \dots, j_{l-1}) &:= \sum_{((i_1, \dots, i_l), m) \in H_{I_l}(j_1, \dots, j_{l-1})} t_{I_k,l}(i_1, \dots, i_l).
 \end{aligned}
 \tag{1.9}$$

Let J be an interval in \mathbb{R} , let $\mathbf{x} := (x_1, \dots, x_n) \in J^n$, let $\mathbf{p} := (p_1, \dots, p_n)$ such that $p_i > 0$ ($1 \leq i \leq n$) and $\sum_{i=1}^n p_i = 1$, and let $f : J \rightarrow \mathbb{R}$ be a convex function. For any $k \geq l \geq 1$, set

$$A_{l,l} = A_{l,l}(I_k; \mathbf{x}; \mathbf{p}) := \sum_{(i_1, \dots, i_l) \in I_l} \left(\sum_{s=1}^l \frac{p_{i_s}}{\alpha_{I_l, i_s}} \right) f \left(\frac{\sum_{s=1}^l (p_{i_s} / \alpha_{I_l, i_s}) x_{i_s}}{\sum_{s=1}^l p_{i_s} / \alpha_{I_l, i_s}} \right),
 \tag{1.10}$$

and associate to each $k - 1 \geq l \geq 1$ the number

$$\begin{aligned}
 A_{k,l} &= A_{k,l}(I_k; \mathbf{x}; \mathbf{p}) \\
 &:= \frac{1}{(k-1)} \sum_{(i_1, \dots, i_l) \in I_l} t_{I_k,l}(i_1, \dots, i_l) \left(\sum_{s=1}^l \frac{p_{i_s}}{\alpha_{I_k, i_s}} \right) f \left(\frac{\sum_{s=1}^l (p_{i_s} / \alpha_{I_k, i_s}) x_{i_s}}{\sum_{s=1}^l p_{i_s} / \alpha_{I_k, i_s}} \right).
 \end{aligned}
 \tag{1.11}$$

We need the following hypotheses.

- (H₂) Let $\mathbf{x} := (x_1, \dots, x_n)$ and $\mathbf{p} := (p_1, \dots, p_n)$ be positive n -tuples such that $\sum_{i=1}^n p_i = 1$.
- (H₃) Let $J \subset \mathbb{R}$ be an interval, let $\mathbf{x} := (x_1, \dots, x_n) \in J^n$, let $\mathbf{p} := (p_1, \dots, p_n)$ be a positive n -tuples such that $\sum_{i=1}^n p_i = 1$, and let $h, g : J \rightarrow \mathbb{R}$ be continuous and strictly monotone functions.
- (H₄) Let $J \subset \mathbb{R}$ be an interval, let $\mathbf{x} := (x_1, \dots, x_n) \in J^n$, and let $\mathbf{p} := (p_1, \dots, p_2)$ be positive n -tuples such that $\sum_{p_i} p_i = 1$. Further, let $f : J \rightarrow \mathbb{R}$ be a convex function.

Assume (H₁) and (H₂). The power means of order $r \in \mathbb{R}$ corresponding to $\mathbf{i}^l := (i_1, \dots, i_n) \in I_l$ ($l = 1, \dots, k$) are given as

$$M_r(I_k, \mathbf{i}^l) := \begin{cases} \left(\frac{\sum_{s=1}^l (p_{i_s} / \alpha_{I_k, i_s}) x_{i_s}^r}{\sum_{s=1}^l p_{i_s} / \alpha_{I_k, i_s}} \right)^{1/r}, & r \neq 0, \\ \left(\prod_{s=1}^l x_{i_s}^{p_{i_s} / \alpha_{I_k, i_s}} \right)^{1 / \sum_{s=1}^l (p_{i_s} / \alpha_{I_k, i_s})}, & r = 0. \end{cases}
 \tag{1.12}$$

We also use the means

$$M_r := \begin{cases} \left(\sum_{i=1}^n p_i x_i^r \right)^{1/r}, & r \neq 0, \\ \prod_{i=1}^n x_i^{p_i}, & r = 0. \end{cases}
 \tag{1.13}$$

For $\gamma, \eta \in \mathbb{R}$, we introduce the mixed symmetric means with positive weights as follows:

$$M_{\eta, \gamma}^1(I_k, k) := \begin{cases} \left[\sum_{\mathbf{i}^k=(i_1, \dots, i_k) \in I_k} \left(\sum_{s=1}^k \frac{p_{i_s}}{\alpha_{I_k, i_s}} \right) (M_Y(I_k, \mathbf{i}^k))^\eta \right]^{1/\eta}, & \eta \neq 0, \\ \prod_{\mathbf{i}^k=(i_1, \dots, i_k) \in I_k} (M_Y(I_k, \mathbf{i}^k))^{\left(\sum_{s=1}^k p_{i_s} / \alpha_{I_k, i_s} \right)}, & \eta = 0, \end{cases} \quad (1.14)$$

and, for $k-1 \geq l \geq 1$,

$$M_{\eta, \gamma}^1(I_k, l) := \begin{cases} \left[\frac{1}{(k-1) \dots l} \sum_{\mathbf{i}^l=(i_1, \dots, i_l) \in I_l} t_{I_k, l}(\mathbf{i}^l) \left(\sum_{s=1}^l \frac{p_{i_s}}{\alpha_{I_k, i_s}} \right) (M_Y(I_k, \mathbf{i}^l))^\eta \right]^{1/\eta}, & \eta \neq 0, \\ \left[\prod_{\mathbf{i}^l=(i_1, \dots, i_l) \in I_l} (M_Y(I_k, \mathbf{i}^l))^{t_{I_k, l}(\mathbf{i}^l) \left(\sum_{s=1}^l p_{i_s} / \alpha_{I_k, i_s} \right)} \right]^{1/(k-1) \dots l}, & \eta = 0. \end{cases} \quad (1.15)$$

We deduce the monotonicity of these means from the following refinement of the discrete Jensen inequality in [1].

Theorem 1.1. *Assume (H_1) and (H_4) . Then,*

$$f\left(\sum_{i=1}^n p_i x_i\right) \leq A_{k, k} \leq A_{k, k-1} \leq \dots \leq A_{k, 2} \leq A_{k, 1} = \sum_{i=1}^n p_i f(x_i), \quad (1.16)$$

where the numbers $A_{k, l}$ ($k \geq l \geq 1$) are defined in (1.10) and (1.11). If f is a concave function, then the inequalities in (1.16) are reversed.

Under the conditions of the previous theorem,

$$\begin{aligned} \Upsilon^1(\mathbf{x}, \mathbf{p}, f) &:= A_{k, m} - A_{k, l} \geq 0, \quad k \geq l > m \geq 1, \\ \Upsilon^2(\mathbf{x}, \mathbf{p}, f) &:= A_{k, l} - f\left(\sum_{i=1}^n p_i x_i\right) \geq 0, \quad k \geq l \geq 1. \end{aligned} \quad (1.17)$$

Corollary 1.2. *Assume (H_1) and (H_2) . Let $\eta, \gamma \in \mathbb{R}$ such that $\eta \leq \gamma$, then*

$$M_Y = M_{\gamma, \eta}^1(I_k, 1) \geq \dots \geq M_{\gamma, \eta}^1(I_k, k) \geq M_\eta, \quad (1.18)$$

$$M_\eta = M_{\eta, \gamma}^1(I_k, 1) \leq \dots \leq M_{\eta, \gamma}^1(I_k, k) \leq M_Y. \quad (1.19)$$

Proof. Assume $\eta, \gamma \neq 0$. To obtain (1.18), we can apply Theorem 1.1 to the function $f(x) = x^{\eta/\gamma}$ ($x > 0$) and the n -tuples $(x_1^\eta, \dots, x_n^\eta)$ to get the analogue of (1.16) and to raise the power $1/\gamma$. Equation (1.19) can be proved in a similar way by using $f(x) = x^{\eta/\gamma}$ ($x > 0$) and $(x_1^\gamma, \dots, x_n^\gamma)$ and raising the power $1/\eta$.

When $\eta = 0$ or $\gamma = 0$, we get the required results by taking limit. □

Assume (H_1) and (H_3) . Then, we define the quasiarithmetic means with respect to (1.10) and (1.11) as follows:

$$M_{h,g}^1(I_k, k) := h^{-1} \left(\sum_{(i_1, \dots, i_k) \in I_k} \left(\sum_{s=1}^k \frac{p_{i_s}}{\alpha_{I_k, i_s}} \right) h \circ g^{-1} \left(\frac{\sum_{s=1}^k (p_{i_s} / \alpha_{I_k, i_s}) g(x_{i_s})}{\sum_{s=1}^k p_{i_s} / \alpha_{I_k, i_s}} \right) \right), \tag{1.20}$$

and, for $k - 1 \geq l \geq 1$,

$$M_{h,g}^1(I_k, l) = h^{-1} \left(\frac{1}{(k-1) \dots l} \sum_{i^l = (i_1, \dots, i_l) \in I_l} t_{I_k, l}(i^l) \left(\sum_{s=1}^l \frac{p_{i_s}}{\alpha_{I_k, i_s}} \right) h \circ g^{-1} \left(\frac{\sum_{s=1}^l (p_{i_s} / \alpha_{I_k, i_s}) g(x_{i_s})}{\sum_{s=1}^l p_{i_s} / \alpha_{I_k, i_s}} \right) \right). \tag{1.21}$$

The monotonicity of these generalized means is obtained in the next corollary.

Corollary 1.3. *Assume (H_1) and (H_3) . For a continuous and strictly monotone function $q : J \rightarrow \mathbb{R}$, one defines*

$$M_q := q^{-1} \left(\sum_{i=1}^n p_i q(x_i) \right). \tag{1.22}$$

Then,

$$M_h = M_{h,g}^1(I_k, 1) \geq \dots \geq M_{h,g}^1(I_k, k) \geq M_g, \tag{1.23}$$

if either $h \circ g^{-1}$ is convex and h is increasing or $h \circ g^{-1}$ is concave and h is decreasing,

$$M_g = M_{g,h}^1(I_k, 1) \leq \dots \leq M_{g,h}^1(I_k, k) \leq M_h, \tag{1.24}$$

if either $g \circ h^{-1}$ is convex and g is decreasing or $g \circ h^{-1}$ is concave and g is increasing.

Proof. First, we can apply Theorem 1.1 to the function $h \circ g^{-1}$ and the n -tuples $(g(x_1), \dots, g(x_n))$, then we can apply h^{-1} to the inequality coming from (1.16). This gives (1.23). A similar argument gives (1.24): $g \circ h^{-1}, (h(x_1), \dots, h(x_n))$ and g^{-1} can be used. □

Throughout Examples 1.4-1.5, 1.9-1.12, which are based on examples in [1], the conditions (H_2) , in the mixed symmetric means, and (H_3) , in the quasiarithmetic means, will be assumed.

Example 1.4. Suppose

$$I_2 := \left\{ (i_1, i_2) \in \{1, \dots, n\}^2 \mid i_1 | i_2 \right\}, \quad (1.25)$$

where $i_1 | i_2$ means that i_1 divides i_2 . Since $i | i$ ($i = 1, \dots, n$), therefore (1.6) holds. We note that

$$\alpha_{I_2, i} = \left[\frac{n}{i} \right] + d(i), \quad i = 1, \dots, n, \quad (1.26)$$

where $[n/i]$ is the largest positive integer not greater than n/i , and $d(i)$ means the number of positive divisors of i . Then, (1.14) gives for $\eta, \gamma \in \mathbb{R}$

$$M_{\eta, \gamma}^1(I_2, 2) = \begin{cases} \left[\sum_{i^2=(i_1, i_2) \in I_2} \left(\sum_{s=1}^2 \frac{p_{i_s}}{[n/i_s] + d(i_s)} \right) (M_\gamma(I_2, \mathbf{i}^k))^n \right]^{1/n}, & \eta \neq 0, \\ \prod_{i^2=(i_1, i_2) \in I_2} (M_\gamma(I_2, \mathbf{i}^2))^{\sum_{s=1}^2 p_{i_s} / ([n/i_s] + d(i_s))} & \eta = 0, \end{cases} \quad (1.27)$$

while (1.20) gives

$$M_{h, g}^1(I_2, 2) = h^{-1} \left(\sum_{(i_1, i_2) \in I_2} \left(\sum_{s=1}^2 \frac{p_{i_s}}{[n/i_s] + d(i_s)} \right) h \circ g^{-1} \left(\frac{\sum_{s=1}^2 (p_{i_s} / ([n/i_s] + d(i_s))) g(x_{i_s})}{\sum_{s=1}^2 (p_{i_s} / ([n/i_s] + d(i_s)))} \right) \right). \quad (1.28)$$

Assume (H_4) holds, and consider the set I_2 in Example 1.4. Then, Theorem 1.1 implies that

$$f \left(\sum_{r=1}^n p_r x_r \right) \leq \sum_{(i_1, i_2) \in I_2} \left(\sum_{s=1}^2 \frac{p_{i_s}}{[n/i_s] + d(i_s)} \right) f \left(\frac{\sum_{s=1}^2 (p_{i_s} / ([n/i_s] + d(i_s))) x_{i_s}}{\sum_{s=1}^2 (p_{i_s} / ([n/i_s] + d(i_s)))} \right) \leq \sum_{r=1}^n p_r f(x_r), \quad (1.29)$$

and thus

$$\begin{aligned} Y^3(\mathbf{x}, \mathbf{p}, f) &:= \sum_{(i_1, i_2) \in I_2} \left(\sum_{s=1}^2 \frac{p_{i_s}}{[n/i_s] + d(i_s)} \right) f \left(\frac{\sum_{s=1}^2 (p_{i_s} / ([n/i_s] + d(i_s))) x_{i_s}}{\sum_{s=1}^2 (p_{i_s} / ([n/i_s] + d(i_s)))} \right) - f \left(\sum_{r=1}^n p_r x_r \right) \geq 0, \\ Y^4(\mathbf{x}, \mathbf{p}, f) &:= \sum_{r=1}^n p_r f(x_r) - \sum_{(i_1, i_2) \in I_2} \left(\sum_{s=1}^2 \frac{p_{i_s}}{[n/i_s] + d(i_s)} \right) f \left(\frac{\sum_{s=1}^2 (p_{i_s} / ([n/i_s] + d(i_s))) x_{i_s}}{\sum_{s=1}^2 (p_{i_s} / ([n/i_s] + d(i_s)))} \right) \geq 0. \end{aligned} \quad (1.30)$$

Example 1.5. Let $c_i \geq 1$ be an integer ($i = 1, \dots, n$), let $k := \sum_{i=1}^n c_i$, and also let $I_k = P_k^{c_1, \dots, c_n}$ consist of all sequences (i_1, \dots, i_k) in which the number of occurrences of $i \in \{1, \dots, n\}$ is c_i ($i = 1, \dots, n$). Obviously, (1.6) holds, and, by simple calculations, we have

$$I_{k-1} = \bigcup_{i=1}^n P_{k-1}^{c_1, \dots, c_{i-1}, c_i-1, c_{i+1}, \dots, c_n}, \quad \alpha_{I_k, i} = \frac{k!}{c_1! \dots c_n!} c_i, \quad i = 1, \dots, n. \tag{1.31}$$

Moreover, $t_{I_k, k-1}(i_1, \dots, i_{k-1}) = k$ for

$$(i_1, \dots, i_{k-1}) \in P_{k-1}^{c_1, \dots, c_{i-1}, c_i-1, c_{i+1}, \dots, c_n}, \quad i = 1, \dots, n. \tag{1.32}$$

Under the above settings, (1.15) can be written as

$$M_{\eta, \gamma}^1(I_k, k-1) = \begin{cases} \left[\frac{1}{k-1} \sum_{i=1}^n (c_i - p_i) \left(\frac{\sum_{r=1}^n p_r x_r^\gamma - (p_i/c_i) x_i^\gamma}{1 - (p_i/c_i)} \right)^{\eta/\gamma} \right]^{1/\eta}, & \eta \neq 0, \gamma \neq 0, \\ \left(\prod_{i=1}^n \left(\frac{\sum_{r=1}^n p_r x_r^\gamma - (p_i/c_i) x_i^\gamma}{1 - (p_i/c_i)} \right)^{(c_i - p_i)/\gamma} \right)^{1/(k-1)}, & \gamma \neq 0, \eta = 0, \\ \left(\prod_{i=1}^n \left(x_i^{-p_i} \left(\prod_{r=1}^n x_r^{p_r} \right)^{c_i} \right) \right)^{1/(k-1)}, & \gamma = 0, \eta = 0, \end{cases} \tag{1.33}$$

while (1.21) becomes

$$M_{h, g}^1(I_k, k-1) = h^{-1} \left(\frac{1}{k-1} \sum_{i=1}^n (c_i - p_i) h \circ g^{-1} \left(\frac{\sum_{r=1}^n p_r g(x_r) - (p_i/c_i) g(x_i)}{1 - (p_i/c_i)} \right) \right). \tag{1.34}$$

Assume (H_4) holds, and consider the set I_k in Example 1.5. Then, Theorem 1.1 yields that

$$A_{k, k-1} = \frac{1}{k-1} \sum_{i=1}^n (c_i - p_i) f \left(\frac{\sum_{r=1}^n p_r x_r - (p_i/c_i) x_i}{1 - (p_i/c_i)} \right), \tag{1.35}$$

$$f \left(\sum_{r=1}^n p_r x_r \right) \leq A_{k, k-1} \leq \sum_{r=1}^n p_r f(x_r).$$

This shows that

$$\Upsilon^5(\mathbf{x}, \mathbf{p}, f) := A_{k, k-1} - f \left(\sum_{r=1}^n p_r x_r \right) \geq 0, \tag{1.36}$$

$$\Upsilon^6(\mathbf{x}, \mathbf{p}, f) := \sum_{r=1}^n p_r f(x_r) - A_{k, k-1} \geq 0.$$

The following result is also given in [1].

Theorem 1.6. Assume (H_1) and (H_4) , and suppose $|H_{I_l}(j_1, \dots, j_{l-1})| = \beta_{l-1}$ for any $(j_1, \dots, j_{l-1}) \in I_{l-1}$ ($k \geq l \geq 2$). Then,

$$A_{k,l} = A_{l,l} = \frac{n}{l|I_l|} \sum_{(i_1, \dots, i_l) \in I_l} \left(\sum_{s=1}^l p_{i_s} \right) f \left(\frac{\sum_{s=1}^l p_{i_s} x_{i_s}}{\sum_{s=1}^l p_{i_s}} \right), \quad k \geq l \geq 1, \quad (1.37)$$

and thus

$$f \left(\sum_{r=1}^n p_r x_r \right) \leq A_{k,k} \leq A_{k-1,k-1} \leq \dots \leq A_{2,2} \leq A_{1,1} = \sum_{r=1}^n p_r f(x_r). \quad (1.38)$$

If f is a concave function then the inequalities (1.38) are reversed.

Under the conditions of the previous theorem, we have, from (1.38), that

$$\begin{aligned} \Upsilon^7(\mathbf{x}, \mathbf{p}, f) &:= A_{m,m} - A_{l,l} \geq 0, \quad k \geq l > m \geq 1, \\ \Upsilon^8(\mathbf{x}, \mathbf{p}, f) &:= A_{l,l} - f \left(\sum_{r=1}^n p_r x_r \right) \geq 0, \quad k \geq l \geq 1. \end{aligned} \quad (1.39)$$

Assume (H_1) and (H_2) , and suppose $|H_{I_l}(j_1, \dots, j_{l-1})| = \beta_{l-1}$ for any $(j_1, \dots, j_{l-1}) \in I_{l-1}$ ($k \geq l \geq 2$). In this case, the power means of order $r \in \mathbb{R}$ corresponding to $\mathbf{i}^l := (i_1, \dots, i_l) \in I_l$ ($l = 1, \dots, k$) has the form

$$M_r(I_l, \mathbf{i}^l) = M_r(I_k, \mathbf{i}^l) = \begin{cases} \left(\frac{\sum_{s=1}^l p_{i_s} x_{i_s}^r}{\sum_{s=1}^l p_{i_s}} \right)^{1/r}, & r \neq 0, \\ \left(\prod_{s=1}^l x_{i_s}^{p_{i_s}} \right)^{1/\sum_{s=1}^l p_{i_s}}, & r = 0. \end{cases} \quad (1.40)$$

Now, for $\gamma, \eta \in \mathbb{R}$ and $k \geq l \geq 1$, we introduce the mixed symmetric means with positive weights related to (1.37) as follows:

$$M_{\eta, \gamma}^2(I_l) := \begin{cases} \left[\frac{n}{l|I_l|} \sum_{\mathbf{i}^l = (i_1, \dots, i_l) \in I_l} \left(\sum_{s=1}^l p_{i_s} \right) (M_\gamma(I_l, \mathbf{i}^l))^\eta \right]^{1/\eta}, & \eta \neq 0, \\ \left[\prod_{\mathbf{i}^l = (i_1, \dots, i_l) \in I_l} (M_\gamma(I_l, \mathbf{i}^l))^{\sum_{s=1}^l p_{i_s}} \right]^{n/l|I_l|}, & \eta = 0. \end{cases} \quad (1.41)$$

Corollary 1.7. Assume (H_1) and (H_2) , and suppose $|H_{I_l}(j_1, \dots, j_{l-1})| = \beta_{l-1}$ for any $(j_1, \dots, j_{l-1}) \in I_{l-1}$ ($k \geq l \geq 2$). Let $\eta, \gamma \in \mathbb{R}$ such that $\eta \leq \gamma$. Then,

$$\begin{aligned} M_\gamma &= M_{\gamma, \eta}^2(I_1) \geq \dots \geq M_{\gamma, \eta}^2(I_k) \geq M_\eta, \\ M_\eta &= M_{\eta, \gamma}^2(I_1) \leq \dots \leq M_{\eta, \gamma}^2(I_k) \leq M_\gamma. \end{aligned} \quad (1.42)$$

Proof. The proof comes from Corollary 1.2. \square

Assume (H_1) and (H_3) , and suppose $|H_{I_l}(j_1, \dots, j_{l-1})| = \beta_{l-1}$ for any $(j_1, \dots, j_{l-1}) \in I_{l-1}$ ($k \geq l \geq 1$). We define for $k \geq l \geq 1$ the quasiarithmetic means with respect to (1.37) as follows:

$$M_{h,g}^2(I_l) := h^{-1} \left(\frac{n}{l|I_l|} \sum_{(i_1, \dots, i_l) \in I_l} \left(\sum_{s=1}^l p_{i_s} \right) h \circ g^{-1} \left(\frac{\sum_{s=1}^l p_{i_s} g(x_{i_s})}{\sum_{s=1}^l p_{i_s}} \right) \right). \quad (1.43)$$

Corollary 1.8. Assume (H_1) and (H_3) , and suppose $|H_{I_l}(j_1, \dots, j_{l-1})| = \beta_{l-1}$ for any $(j_1, \dots, j_{l-1}) \in I_{l-1}$ ($k \geq l \geq 2$). Then,

$$M_h = M_{h,g}^2(I_1) \geq \dots \geq M_{h,g}^2(I_k) \geq M_g, \quad (1.44)$$

where either $h \circ g^{-1}$ is convex and h is increasing or $h \circ g^{-1}$ is concave and h is decreasing,

$$M_g = M_{g,h}^2(I_1) \leq \dots \leq M_{g,h}^2(I_k) \leq M_h, \quad (1.45)$$

where either $g \circ h^{-1}$ is convex and g is decreasing or $g \circ h^{-1}$ is concave and g is increasing.

Proof. The proof is a consequence of Corollary 1.3. \square

Example 1.9. If we set

$$I_k := \left\{ (i_1, \dots, i_k) \in \{1, \dots, n\}^k \mid i_1 < \dots < i_k \right\}, \quad 1 \leq k \leq n, \quad (1.46)$$

then $\alpha_{I_n, i} = 1$ ($i = 1, \dots, n$), that is, (1.6) is satisfied for $k = n$. It comes easily that $T_k(I_k) = I_{k-1}$ ($k = 2, \dots, n$), $|I_k| = \binom{n}{k}$ ($k = 1, \dots, n$), and for each $k = 2, \dots, n$

$$|H_{I_k}(j_1, \dots, j_{k-1})| = n - (k - 1), \quad (j_1, \dots, j_{k-1}) \in I_{k-1}. \quad (1.47)$$

In this case, (1.41) becomes for $n \geq k \geq 1$

$$M_{\eta, \gamma}^2(I_k) = \begin{cases} \left[\frac{1}{\binom{n-1}{k-1}} \sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\sum_{s=1}^k p_{i_s} \right) (M_\gamma(I_k, \mathbf{i}^k))^\eta \right]^{1/\eta}, & \eta \neq 0, \\ \left[\prod_{1 \leq i_1 < \dots < i_k \leq n} (M_\gamma(I_k, \mathbf{i}^k))^{\left(\sum_{s=1}^k p_{i_s} \right)} \right]^{1/\binom{n-1}{k-1}}, & \eta = 0, \end{cases} \quad (1.48)$$

and (1.43) has the form

$$M_{h, g}^2(I_k) = h^{-1} \left(\frac{1}{\binom{n-1}{k-1}} \sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\sum_{s=1}^k p_{i_s} \right) h \circ g^{-1} \left(\frac{\sum_{s=1}^k p_{i_s} g(x_{i_s})}{\sum_{s=1}^k p_{i_s}} \right) \right). \quad (1.49)$$

Equation (1.48) is a weighted mixed symmetric mean and (1.49) is a generalized mean, as given in [2]. Therefore, Corollaries 1.7 and 1.8 are more general than the Corollaries 1.2 and 1.3 given in [2].

Assume (H_4) holds, and consider the set I_k in Example 1.9. Then, Theorem 1.6 shows that

$$A_{k,k} = \frac{1}{\binom{n-1}{k-1}} \sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\sum_{s=1}^k p_{i_s} \right) f \left(\frac{\sum_{s=1}^k p_{i_s} x_{i_s}}{\sum_{s=1}^k p_{i_s}} \right), \quad k = 1, \dots, n, \quad (1.50)$$

$$f \left(\sum_{r=1}^n p_r x_r \right) = A_{n,n} \leq A_{n-1, n-1} \leq \dots \leq A_{1,1} = \sum_{r=1}^n p_r f(x_r).$$

Thus, we have

$$\Upsilon^9(\mathbf{x}, \mathbf{p}, f) := A_{m,m} - A_{l,l} \geq 0, \quad n \geq l > m \geq 1. \quad (1.51)$$

Example 1.10. If we set

$$I_k := \left\{ (i_1, \dots, i_k) \in \{1, \dots, n\}^k \mid i_1 \leq \dots \leq i_k \right\}, \quad k \geq 1, \quad (1.52)$$

then $\alpha_{I_k, i} \geq 1$ ($i = 1, \dots, n$) and thus (1.6) is satisfied. It is easy to see that $T_k(I_k) = I_{k-1}$ ($k = 2, \dots$), $|I_k| = \binom{n+k-1}{k}$ ($k = 1, \dots$), and for each $l = 2, \dots, k$

$$|H_{I_l}(j_1, \dots, j_{l-1})| = n, \quad (j_1, \dots, j_{l-1}) \in I_{l-1}. \quad (1.53)$$

Under these settings (1.41) becomes

$$M_{\eta,\gamma}^2(I_k) = \begin{cases} \left[\frac{1}{\binom{n+k-1}{k-1}} \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} \left(\sum_{s=1}^k p_{i_s} \right) (M_\gamma(I_k, \mathbf{i}^k))^\eta \right]^{1/\eta}, & \eta \neq 0, \\ \left[\prod_{1 \leq i_1 \leq \dots \leq i_k \leq n} (M_\gamma(I_k, \mathbf{i}^k))^{\left(\sum_{s=1}^k p_{i_s} \right)} \right]^{1/\binom{n+k-1}{k-1}}, & \eta = 0, \end{cases} \tag{1.54}$$

and (1.43) has the form

$$M_{h,g}^2(I_k) = h^{-1} \left(\frac{1}{\binom{n+k-1}{k-1}} \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} \left(\sum_{s=1}^k p_{i_s} \right) h \circ g^{-1} \left(\frac{\sum_{s=1}^k p_{i_s} g(x_{i_s})}{\sum_{s=1}^k p_{i_s}} \right) \right). \tag{1.55}$$

Equation (1.54) represents weighted mixed symmetric means, and (1.55) defines generalized means, as given in [2]. Therefore, Corollaries 1.7 and 1.8 are more general than the Corollaries 1.9 and 1.10 given in [2].

Assume (H₄) holds, and consider the set I_k in Example 1.10. Then, it follows from Theorem 1.6 that

$$f \left(\sum_{r=1}^n p_r x_r \right) \leq \dots \leq A_{k,k} \leq \dots \leq A_{1,1} = \sum_{r=1}^n p_r f(x_r), \tag{1.56}$$

where

$$A_{k,k} = \frac{1}{\binom{n+k-1}{k-1}} \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} \left(\sum_{s=1}^k p_{i_s} \right) f \left(\frac{\sum_{s=1}^k p_{i_s} x_{i_s}}{\sum_{s=1}^k p_{i_s}} \right), \quad k \geq 1. \tag{1.57}$$

This yields that

$$\begin{aligned} \Upsilon^{10}(\mathbf{x}, \mathbf{p}, f) &= A_{k,k} - A_{l,l} \geq 0, \quad l > k \geq 1, \\ \Upsilon^{11}(\mathbf{x}, \mathbf{p}, f) &:= A_{k,k} - f \left(\sum_{r=1}^n p_r x_r \right) \geq 0, \quad k \geq 1. \end{aligned} \tag{1.58}$$

Example 1.11. We set

$$I_k := \{1, \dots, n\}^k, \quad k \geq 1. \tag{1.59}$$

Then, $\alpha_{I_k,i} \geq 1$ ($i = 1, \dots, n$), hence (1.6) holds. It is not hard to see that $T_k(I_k) = I_{k-1}$ ($k = 2, \dots$), $|I_k| = n^k$ ($k = 1, \dots$), and for each $l = 2, \dots, k$,

$$|H_l(j_1, \dots, j_{l-1})| = n^l, \quad (j_1, \dots, j_{l-1}) \in I_{l-1}. \tag{1.60}$$

Therefore, under these settings, for $k \geq 1$, (1.41) leads to

$$M_{\eta, \gamma}^2(I_k) = \begin{cases} \left[\frac{1}{kn^{k-1}} \sum_{\mathbf{i}^k=(i_1, \dots, i_k) \in I_k} \left(\sum_{s=1}^k p_{i_s} \right) (M_{\gamma}(I_k, \mathbf{i}^k))^{\eta} \right]^{1/\eta}, & \eta \neq 0, \\ \left[\prod_{\mathbf{i}^k=(i_1, \dots, i_k) \in I_k} (M_{\gamma}(I_k, \mathbf{i}^k))^{\left(\sum_{s=1}^k p_{i_s} \right)} \right]^{1/kn^{k-1}}, & \eta = 0, \end{cases} \quad (1.61)$$

and (1.43) gives

$$M_{h, g}^2(I_k) = h^{-1} \left(\frac{1}{kn^{k-1}} \sum_{\mathbf{i}^k=(i_1, \dots, i_k) \in I_k} \left(\sum_{s=1}^k p_{i_s} \right) h \circ g^{-1} \left(\frac{\sum_{s=1}^k p_{i_s} g(x_{i_s})}{\sum_{s=1}^k p_{i_s}} \right) \right), \quad (1.62)$$

respectively.

Assume (H_4) holds, and consider the set I_k in Example 1.11. Then, Theorem 1.6 implies that

$$f \left(\sum_{r=1}^n p_r x_r \right) \leq \dots \leq A_{k,k} \leq \dots \leq A_{1,1} = \sum_{r=1}^n p_r f(x_r), \quad (1.63)$$

where

$$A_{k,k} = \frac{1}{kn^{k-1}} \sum_{(i_1, \dots, i_k) \in I_k} \left(\sum_{s=1}^k p_{i_s} \right) f \left(\frac{\sum_{s=1}^k p_{i_s} x_{i_s}}{\sum_{s=1}^k p_{i_s}} \right), \quad k \geq 1. \quad (1.64)$$

Therefore, we have

$$\begin{aligned} \Upsilon^{12}(\mathbf{x}, \mathbf{p}, f) &= A_{k,k} - A_{l,l} \geq 0, \quad l > k \geq 1, \\ \Upsilon^{13}(\mathbf{x}, \mathbf{p}, f) &:= A_{k,k} - f \left(\sum_{r=1}^n p_r x_r \right) \geq 0, \quad k \geq 1. \end{aligned} \quad (1.65)$$

Example 1.12. Let $1 \leq k \leq n$ and let I_k consist of all sequences (i_1, \dots, i_k) of k distinct numbers from $\{1, \dots, n\}$. Then, $\alpha_{I_n, i} \geq 1$ ($i = 1, \dots, n$), hence (1.6) holds. It is immediate that $T_k(I_k) = I_{k-1}$ ($k = 2, \dots$), $|I_k| = n(n-1) \cdots (n-k+1)$ ($k = 1, \dots, n$), and for every $k = 2, \dots, n$,

$$|H_{I_k}(j_1, \dots, j_{k-1})| = (n-k+1)k, \quad (j_1, \dots, j_{k-1}) \in I_{k-1}. \quad (1.66)$$

Therefore under these settings, for $k = 1, \dots, n$, (1.41) gives

$$M_{\eta, \gamma}^2(I_k) = \begin{cases} \left[\frac{n}{kn(n-1) \cdots (n-k+1)} \sum_{\mathbf{i}^k=(i_1, \dots, i_k) \in I_k} \left(\sum_{s=1}^k p_{i_s} \right) (M_{\gamma}(I_k, \mathbf{i}^k))^{\eta} \right]^{1/\eta}, & \eta \neq 0, \\ \left[\prod_{\mathbf{i}^k=(i_1, \dots, i_k) \in I_k} (M_{\gamma}(I_k, \mathbf{i}^k))^{\left(\sum_{s=1}^k p_{i_s}\right)} \right]^{n/kn(n-1) \cdots (n-k+1)}, & \eta = 0, \end{cases} \tag{1.67}$$

and (1.43) has the form

$$M_{h, g}^2(I_k) = h^{-1} \left(\frac{n}{kn(n-1) \cdots (n-k+1)} \sum_{\mathbf{i}^k=(i_1, \dots, i_k) \in I_k} \left(\sum_{s=1}^k p_{i_s} \right) h \circ g^{-1} \left(\frac{\sum_{s=1}^k p_{i_s} g(x_{i_s})}{\sum_{s=1}^k p_{i_s}} \right) \right), \tag{1.68}$$

respectively.

Assume (H_4) holds, and consider the set I_k in Example 1.12. Then, Theorem 1.6 yields that

$$f \left(\sum_{r=1}^n p_r x_r \right) \leq A_{n,n} \leq \dots \leq A_{k,k} \leq \dots \leq A_{1,1} = \sum_{r=1}^n p_r f(x_r), \tag{1.69}$$

where for $k = 1, \dots, n$,

$$A_{k,k} = \frac{n}{kn(n-1) \cdots (n-k+1)} \sum_{(i_1, \dots, i_k) \in I_k} \left(\sum_{s=1}^k p_{i_s} \right) f \left(\frac{\sum_{s=1}^k p_{i_s} x_{i_s}}{\sum_{s=1}^k p_{i_s}} \right). \tag{1.70}$$

Therefore, we have

$$\begin{aligned} \Upsilon^{14}(\mathbf{x}, \mathbf{p}, f) &= A_{m,m} - A_{l,l} \geq 0, \quad n \geq l > m \geq 1, \\ \Upsilon^{15}(\mathbf{x}, \mathbf{p}, f) &= A_{l,l} - f \left(\sum_{r=1}^n p_r x_r \right) \geq 0, \quad n \geq l \geq 1. \end{aligned} \tag{1.71}$$

2. Main Results

We have seen that

$$\Upsilon^i(\mathbf{x}, \mathbf{p}, f) \geq 0, \quad i = 1, \dots, 15. \tag{2.1}$$

From now on, (H_1) and (H_4) are assumed if we consider $\Upsilon^i(\mathbf{x}, \mathbf{p}, f)$ ($i = 1, 2$), further, the hypothesis $|H_{I_l}(j_1, \dots, j_{l-1})| = \beta_{l-1}$ for any $(j_1, \dots, j_{l-1}) \in I_{l-1}$ ($k \geq l \geq 2$) is also assumed if we

consider $Y^i(\mathbf{x}, \mathbf{p}, f)$ ($i = 7, 8$). The numbers $Y^i(\mathbf{x}, \mathbf{p}, f)$ ($i = 3, \dots, 6, 9, \dots, 15$) are generated by concrete examples, and (H_4) is assumed.

We need the following subclass of convex functions (see [3]).

Definition 2.1. A function $f : (a, b) \rightarrow \mathbb{R}$ is exponentially convex if it is continuous and

$$\sum_{i,j=1}^n \xi_i \xi_j f(x_i + x_j) \geq 0, \quad (2.2)$$

for all $n \in \mathbb{N}$ and all choices $\xi_i \in \mathbb{R}$ and $x_i + x_j \in (a, b)$ ($1 \leq i, j \leq n$).

We quote here useful propositions from [3].

Proposition 2.2. Let $f : (a, b) \rightarrow \mathbb{R}$ be a function. Then, the following statements are equivalent

- (i) f is exponentially convex.
- (ii) f is continuous and

$$\sum_{i,j=1}^n \xi_i \xi_j f\left(\frac{x_i + x_j}{2}\right) \geq 0, \quad (2.3)$$

for every $\xi_i \in \mathbb{R}$ and every $x_i \in (a, b)$ ($1 \leq i \leq n$).

Proposition 2.3. If $f : (a, b) \rightarrow (0, \infty)$ is an exponentially convex function, then f is log-convex which means that for every $x, y \in (a, b)$ and all $\lambda \in [0, 1]$

$$f(\lambda x + (1 - \lambda)y) \leq f(x)^\lambda f(y)^{1-\lambda}. \quad (2.4)$$

First, we introduce a special class of functions.

(H_5) Let $\{f_s : (c, d) \subset \mathbb{R} \rightarrow \mathbb{R} \mid s \in (a, b) \subset \mathbb{R}\}$ be a set of twice differentiable convex functions such that the function $s \rightarrow f_s''(x)s \in (a, b)$ is exponentially convex for every fixed $x \in (c, d)$.

As examples, consider two classes of functions $\varphi_s : (0, \infty) \rightarrow \mathbb{R}$ defined by

$$\varphi_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0, \\ -\log x, & s = 0, \\ x \log x & s = 1, \end{cases} \quad (2.5)$$

and $\phi_s : \mathbb{R} \rightarrow [0, \infty)$ defined by

$$\phi_s(x) = \begin{cases} \frac{1}{s^2} e^{sx}, & s \neq 0, \\ \frac{1}{2} x^2, & s = 0. \end{cases} \quad (2.6)$$

It is easy to see that the sets of functions $\{\varphi_s \mid s \in \mathbb{R}\}$ and $\{\phi_s \mid s \in \mathbb{R}\}$ satisfy (H_5) .

Assume (H₅). If f is replaced by f_s in (2.1), we obtain

$$\widehat{Y}_s^i := Y^i(\mathbf{x}, \mathbf{p}, f_s) \geq 0, \quad s \in (a, b), \quad i = 1, \dots, 15. \quad (2.7)$$

Especially,

$$Y_s^i := Y^i(\mathbf{x}, \mathbf{p}, \varphi_s) \geq 0, \quad s \in \mathbb{R}, \quad i = 1, \dots, 15. \quad (2.8)$$

In this paper we prove the exponential convexity of the functions $s \rightarrow \widehat{Y}_s^i$ ($s \in (a, b)$), and we give mean value theorems for $Y^i(\mathbf{x}, \mathbf{p}, f)$ ($i = 1, \dots, 15$). We also define the respective means of Cauchy type and study their monotonicity. The results for Y^i ($i = 3, \dots, 6$) are special cases of the results for Y^i ($i = 1, 2$), and the results for Y^i ($i = 9, \dots, 15$) are special cases of results for Y^i ($i = 7, 8$). Especially, the results for Y^i ($i = 9, 10, 11$) are also given in [2].

Theorem 2.4. *Assume (H₅), and suppose that the functions $s \mapsto \widehat{Y}_s^i$ ($s \in (a, b)$) are continuous. The following statements hold for \widehat{Y}_s^i ($i = 1, \dots, 15$).*

- (a) *For every $q \in \mathbb{N}$ and $s_1, \dots, s_q \in (a, b)$, the matrix $[\widehat{Y}_{(s_l+s_m)/2}^i]_{l,m=1}^q$ is positive semidefinite. Particularly,*

$$\det \left[\widehat{Y}_{(s_l+s_m)/2}^i \right]_{l,m=1}^k \geq 0, \quad \text{for } k = 1, 2, \dots, q. \quad (2.9)$$

- (b) *The function $s \mapsto \widehat{Y}_s^i$ ($s \in (a, b)$) is exponentially convex.*

Proof. Fix $1 \leq i \leq 15$.

- (a) Let $u_l \in \mathbb{R}$ ($l = 1, \dots, q$), and define the functions $\mu_k : (0, \infty) \rightarrow \mathbb{R}$ by $\mu_k(x) := \sum_{l,m=1}^k u_l u_m f_{s_{lm}}(x)$ for $k = 1, \dots, q$, where $s_{lm} = (s_l + s_m)/2$ ($1 \leq l, m \leq q$). Then μ_k ($k = 1, \dots, q$) is a convex function since

$$\mu_k''(x) = \sum_{l,m=1}^k u_l u_m f''_{s_{lm}}(x) \geq 0, \quad x \in (c, d). \quad (2.10)$$

By taking $f = \mu_k$ in (2.1), we have

$$\sum_{l,m=1}^k u_l u_m \widehat{Y}_{s_{lm}}^i \geq 0, \quad k = 1, \dots, q. \quad (2.11)$$

This means that the matrix $[\widehat{Y}_{(s_l+s_m)/2}^i]_{l,m=1}^q$ is positive semidefinite, that is, (2.9) is valid.

- (b) It is assumed that the function $s \mapsto \widehat{Y}_s^i$ ($s \in (a, b)$) is continuous. By using Proposition 2.2 and (a), we get the exponential convexity of the function $s \mapsto \widehat{Y}_s^i$ ($s \in (a, b)$). \square

Since the functions $s \mapsto \Upsilon_s^i$ ($s \in \mathbb{R}$) are continuous ($i = 1, \dots, 15$), we have the following.

Corollary 2.5. *The function $s \mapsto \Upsilon_s^i$ ($s \in \mathbb{R}$, $i = 1, \dots, 15$) are exponentially convex. This remains valid if we replace φ_s by ϕ_s in (2.8).*

3. Cauchy Means

In this section, first, we are interested in mean value theorems.

Theorem 3.1. *Assume $f \in C^2[a, b]$ and $\Upsilon^i(\mathbf{x}, \mathbf{p}; x^2) \neq 0$ ($i = 1, \dots, 15$). Then, there exists $\xi_i \in [a, b]$ such that*

$$\Upsilon^i(\mathbf{x}, \mathbf{p}, f) = \frac{1}{2} f''(\xi_i) \Upsilon^i(\mathbf{x}, \mathbf{p}, x^2), \quad i = 1, \dots, 15. \quad (3.1)$$

Theorem 3.2. *Assume $f, g \in C^2[a, b]$. Then, there exists $\xi_i \in [a, b]$ such that*

$$\frac{\Upsilon^i(\mathbf{x}, \mathbf{p}, f)}{\Upsilon^i(\mathbf{x}, \mathbf{p}, g)} = \frac{f''(\xi_i)}{g''(\xi_i)}, \quad i = 1, \dots, 15, \quad (3.2)$$

provided that the denominators are nonzero.

The idea of the proofs of Theorems 3.1 and 3.2 is the same as the proofs of Theorems 2.3 and 2.4 in [2].

Corollary 3.3. *Let $f, g : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$, $f(x) = x^p$ and $g(x) = x^q$. Then, for distinct real numbers p and q , different from 0 and 1, there exists $\xi_i \in [a, b]$ such that*

$$\xi_i^{p-q} = \frac{q(q-1)}{p(p-1)} \frac{\Upsilon^i(\mathbf{x}, \mathbf{p}, f)}{\Upsilon^i(\mathbf{x}, \mathbf{p}, g)}, \quad i = 1, \dots, 15. \quad (3.3)$$

Proof. Theorem 3.2 can be applied. □

Remark 3.4. Suppose the conditions of Corollary 3.3 are satisfied.

- (a) Since the function $\xi \rightarrow \xi^{p-q}$ ($\xi \in (0, \infty)$), $p \neq q$ is invertible, then we get, from (3.3), that for $i = 1, \dots, 15$

$$a \leq \left(\frac{q(q-1)}{p(p-1)} \frac{\Upsilon^i(\mathbf{x}, \mathbf{p}, f)}{\Upsilon^i(\mathbf{x}, \mathbf{p}, g)} \right)^{1/(p-q)} \leq b. \quad (3.4)$$

- (b) By choosing $a := \min_{1 \leq i \leq n} x_i$ and $b := \max_{1 \leq i \leq n} x_i$, we can see that the expression between a and b in (3.4) defines a mean.

Corollary 3.5. Assume (H_1) and (H_2) , and suppose $x_i \in [a, b] \subset (0, \infty)$ ($1 \leq i \leq n$). In (3.6), it is also supposed that $|H_{I_l}(j_1, \dots, j_{l-1})| = \beta_{l-1}$ for any $(j_1, \dots, j_{l-1}) \in I_{l-1}$ ($k \geq l \geq 2$). Then, for distinct real numbers p, q , and r , all are different from 0 and 1, there exists $\xi_1, \xi_2 \in [a, b]$, such that

$$\xi_1^{p-q} = \frac{q(q-r)}{p(p-r)} \frac{\left(M_{p,r}^1(I_k, l-1)\right)^p - \left(M_{p,r}^1(I_k, l)\right)^p}{\left(M_{q,r}^1(I_k, l-1)\right)^q - \left(M_{q,r}^1(I_k, l)\right)^q}, \tag{3.5}$$

$$\xi_2^{p-q} = \frac{q(q-r)}{p(p-r)} \frac{\left(M_{p,r}^2(I_{l-1})\right)^p - \left(M_{p,r}^2(I_l)\right)^p}{\left(M_{q,r}^2(I_{l-1})\right)^q - \left(M_{q,r}^2(I_l)\right)^q}. \tag{3.6}$$

Proof. We can apply Theorem 3.2 to the functions $f, g : [a, b] \rightarrow \mathbb{R}$, $f(x) = x^{p/r}$, and $g(x) = x^{q/r}$, and the n -tuples (x_1^r, \dots, x_n^r) . □

Remark 3.6. Suppose the conditions of Corollary 3.5 are satisfied.

- (a) Since the function $\xi \rightarrow \xi^{p-q}$ ($\xi \in (0, \infty)$) is invertible, then we get, from (3.5) and (3.6) that

$$a \leq \left(\frac{q(q-r)}{p(p-r)} \frac{\left(M_{p,r}^1(I_k, l-1)\right)^p - \left(M_{p,r}^1(I_k, l)\right)^p}{\left(M_{q,r}^1(k, l-1)\right)^q - \left(M_{q,r}^1(k, l)\right)^q} \right)^{1/(p-q)} \leq b, \tag{3.7}$$

$$a \leq \left(\frac{q(q-r)}{p(p-r)} \frac{\left(M_{p,r}^2(I_{l-1})\right)^p - \left(M_{p,r}^2(I_l)\right)^p}{\left(M_{q,r}^2(I_{l-1})\right)^q - \left(M_{q,r}^2(I_l)\right)^q} \right)^{1/(p-q)} \leq b.$$

- (b) As in Remark 3.4 (b), the expressions in (3.7) define means.

By Remark 3.4 (b), we can define Cauchy means for $p, q \in \mathbb{R}$ as follows:

$$M_{p,q}^i := \left(\frac{\Upsilon^i(\mathbf{x}, \mathbf{p}, \varphi_p)}{\Upsilon^i(\mathbf{x}, \mathbf{p}, \varphi_q)} \right)^{1/(p-q)}, \quad p \neq q, \quad i = 1, \dots, 15. \tag{3.8}$$

Moreover, we have continuous extensions of these means in other cases. By taking the limit, we have

$$M_{q,q}^i := \exp\left(\frac{1-2q}{q(q-1)} - \frac{\Upsilon^i(\mathbf{x}, \mathbf{p}; \varphi_q \varphi_0)}{\Upsilon^i(\mathbf{x}, \mathbf{p}; \varphi_q)} \right), \quad q \neq 0, 1,$$

$$M_{1,1}^i := \exp\left(-1 - \frac{\Upsilon^i(\mathbf{x}, \mathbf{p}; \varphi_0 \varphi_1)}{2\Upsilon^i(\mathbf{x}, \mathbf{p}; \varphi_1)} \right), \tag{3.9}$$

$$M_{0,0}^i := \exp\left(1 - \frac{\Upsilon^i(\mathbf{x}, \mathbf{p}; \varphi_0^2)}{2\Upsilon^i(\mathbf{x}, \mathbf{p}; \varphi_0)} \right).$$

Now, we deduce the monotonicity of these means in the form of Dresher's inequality as follows.

Theorem 3.7. *Let $p, q, u, v \in \mathbb{R}$ such that $p \leq v$, $q \leq u$. Then*

$$M_{p,q}^i \leq M_{v,u}^i \quad i = 1, \dots, 15. \quad (3.10)$$

Proof. Fix $1 \leq i \leq 15$. Corollary 2.5 shows that the function $p \rightarrow \Upsilon_p^i$ ($p \in \mathbb{R}$) is exponentially convex, and hence, by Proposition 2.3, it is log-convex. Therefore, the function $p \rightarrow \log(\Upsilon_p^i)$ ($p \in \mathbb{R}$) is convex, which implies (see [4]) that

$$\frac{\log \Upsilon_p^i - \log \Upsilon_q^i}{p - q} \leq \frac{\log \Upsilon_v^i - \log \Upsilon_u^i}{v - u}. \quad (3.11)$$

This gives (3.10) if $p \neq q$ and $v \neq u$. The other cases come from this by taking limit. \square

By Remark 3.6 (b), we can define Cauchy means in the following form:

$$M_{p,q;r}^i := \left(\frac{\Upsilon^i(\mathbf{x}^r, \mathbf{p}; \varphi_{p/r})}{\Upsilon^i(\mathbf{x}^r, \mathbf{p}; \varphi_{q/r})} \right)^{1/(p-q)}, \quad p \neq q, \quad i = 1, 2, \quad (3.12)$$

where $\mathbf{x}^r = (x_1^r, \dots, x_n^r)$. By taking the limit, we have

$$\begin{aligned} M_{q,q;r}^i &:= \exp\left(\frac{(r-2q)}{q(q-r)} - \frac{\Upsilon^i(\mathbf{x}^r, \mathbf{p}; \varphi_{q/r}\varphi_0)}{r\Upsilon^i(\mathbf{x}^r, \mathbf{p}; \varphi_{q/r})}\right), \quad q(q-r) \neq 0, \quad r \neq 0, \\ M_{0,0;r}^i &= \exp\left(\frac{1}{r} - \frac{\Upsilon^i(\mathbf{x}^r, \mathbf{p}; \varphi_0^2)}{2r\Upsilon^i(\mathbf{x}^r, \mathbf{p}; \varphi_0)}\right), \quad r \neq 0, \\ M_{r,r;r}^i &= \exp\left(\frac{-1}{r} - \frac{\Upsilon^i(\mathbf{x}^r, \mathbf{p}; \varphi_0\varphi_1)}{2r\Upsilon^i(\mathbf{x}^r, \mathbf{p}; \varphi_1)}\right), \quad r \neq 0, \\ M_{q,q;0}^i &= \exp\left(\frac{-2}{q} + \frac{\Upsilon^i(\log \mathbf{x}, \mathbf{p}; x\phi_q)}{\Upsilon^i(\log \mathbf{x}, \mathbf{p}; \phi_q)}\right), \quad q \neq 0, \\ M_{0,0;0}^i &= \exp\left(\frac{\Upsilon^i(\log \mathbf{x}, \mathbf{p}; x\phi_0)}{3\Upsilon^i(\log \mathbf{x}, \mathbf{p}; \phi_0)}\right), \end{aligned} \quad (3.13)$$

where $\log \mathbf{x} = (\log x_1, \dots, \log x_n)$.

Now, we give the monotonicity of these new means.

Theorem 3.8. *Let $p, q, u, v \in \mathbb{R}$ such that $p \leq v$, $q \leq u$. Then,*

$$M_{p,q;r}^i \leq M_{v,u;r}^i, \quad i = 1, 2. \quad (3.14)$$

Proof. Suppose $i = 1, 2$ is fixed. The function $p \rightarrow Y_p^i$ ($p \in \mathbb{R}$) is exponentially convex, and hence, by Proposition 2.3, it is log-convex. Therefore, the function $p \rightarrow \log(Y_p^i)$ ($p \in \mathbb{R}$) is convex, which implies (as in the proof of Theorem 3.7) that

$$\left(\frac{Y_p^i}{Y_q^i}\right)^{1/(p-q)} \leq \left(\frac{Y_v^i}{Y_u^i}\right)^{1/(v-u)}. \quad (3.15)$$

If $r \neq 0$, set $x_i := x_i^r$, $p := p/r$, $q := q/r$, $u := u/r$, $v := v/r$ in (3.15) to obtain (3.14).

For $r = 0$, we get the required result by limit. \square

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