

Research Article

Some Properties of Orthogonal Polynomials for Laguerre-Type Weights

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Let $\mathbb{R}^+ = [0, \infty)$, let $R : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous, nonnegative, and increasing function, and let $p_{n,\rho}(x)$ be the orthonormal polynomials with the weight $w_\rho(x) = x^\rho e^{-R(x)}$, $\rho > -1/2$. For the zeros $\{x_{k,n,\rho}\}_{k=1}^n$ of $p_{n,\rho}(x) = p_n(w_\rho^2; x)$, we estimate $p_{n,\rho}^{(j)}(x_{k,n,\rho})$, where j is a positive integer. Moreover, we investigate the various weighted L_p -norms ($0 < p \leq \infty$) of $p_{n,\rho}(x)$.

1. Introduction and Main Results

Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}^+ = [0, \infty)$. Let $R : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous, nonnegative, and increasing function. Consider the exponential weights $w_\rho(x) = x^\rho \exp(-R(x))$, $\rho > -1/2$, and then we construct the orthonormal polynomials $\{p_{n,\rho}(x)\}_{n=0}^\infty$ with the weight $w_\rho(x)$. In this paper, for the zeros $\{x_{kn}\}_{k=1}^n$ of $p_{n,\rho}(x) = p_n(w_\rho^2; x)$ we estimate $p_{n,\rho}^{(j)}(x_{kn})$, where j is a positive integer. Moreover, we investigate the various weighted L_p -norms ($0 < p \leq \infty$) of $p_{n,\rho}(x)$.

We say that $f : \mathbb{R} \rightarrow \mathbb{R}^+$ is quasi-increasing if there exists $C > 0$ such that $f(x) \leq Cf(y)$ for $0 < x < y$. The notation $f(x) \sim g(x)$ means that there are positive constants C_1, C_2 such that for the relevant range of x , $C_1 \leq f(x)/g(x) \leq C_2$. The similar notation is used for sequences and sequences of functions.

Throughout, C, C_1, C_2, \dots denote positive constants independent of n, x, t . The same symbol does not necessarily denote the same constant in different occurrences. We denote the class of polynomials with degree n by \mathcal{P}_n .

First, we introduce some classes of weights.

Levin and Lubinsky [1, 2] introduced the class of weights on \mathbb{R}^+ as follows. Let $I = [0, d)$, where $0 < d \leq \infty$.

Definition 1.1 (see [1, 2]). We assume that $R : I \rightarrow [0, \infty)$ has the following properties. Let $Q(t) = R(t^2)$,

- (a) $\sqrt{x}R(x)$ is continuous in I , with limit 0 at 0 and $R(0) = 0$,
- (b) $R''(x)$ exists in $(0, d)$, while $Q''(t)$ is positive in $(0, \sqrt{d})$,
- (c)

$$\lim_{x \rightarrow d^-} R(x) = \infty; \quad (1.1)$$

- (d) the function

$$T(x) := \frac{xR'(x)}{R(x)} \quad (1.2)$$

is quasi-increasing in $(0, d)$, with

$$T(x) \geq \Lambda > \frac{1}{2}, \quad x \in (0, d); \quad (1.3)$$

- (e) there exists $C_1 > 0$ such that

$$\frac{|R''(x)|}{R'(x)} \leq C_1 \frac{R'(x)}{R(x)}, \quad \text{a.e. } x \in (0, d). \quad (1.4)$$

Then, we write $w \in \mathcal{L}(C^2)$. If there also exists a compact subinterval $J^* \ni 0$ of $I^* = (-\sqrt{d}, \sqrt{d})$, and $C_2 > 0$ such that

$$\frac{Q''(t)}{|Q'(t)|} \geq C_2 \frac{|Q'(t)|}{Q(t)}, \quad \text{a.e. } t \in I^* \setminus J^*, \quad (1.5)$$

then we write $w \in \mathcal{L}(C^2+)$.

We consider the case $d = \infty$, that is, the space $\mathbb{R}^+ = [0, \infty)$, and we strengthen *Definition 1.1* slightly.

Definition 1.2. We assume that $R : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ has the following properties:

- (a) $R(x), R'(x)$ are continuous, positive in \mathbb{R}^+ , with $R(0) = 0, R'(0) = 0$,
- (b) $R''(x) > 0$ exists in $\mathbb{R}^+ \setminus \{0\}$,
- (c)

$$\lim_{x \rightarrow \infty} R(x) = \infty, \quad (1.6)$$

- (d) the function

$$T(x) := \frac{xR'(x)}{R(x)} \quad (1.7)$$

is quasi-increasing in $\mathbb{R}^+ \setminus \{0\}$, with

$$T(x) \geq \Lambda > \frac{1}{2}, \quad x \in \mathbb{R}^+ \setminus \{0\}, \quad (1.8)$$

(e) there exists $C_1 > 0$ such that

$$\frac{R''(x)}{R'(x)} \leq C_1 \frac{R'(x)}{R(x)}, \quad \text{a.e. } x \in \mathbb{R}^+ \setminus \{0\}. \quad (1.9)$$

There exists a compact subinterval $J \ni 0$ of \mathbb{R}^+ , and $C_2 > 0$ such that

$$\frac{R''(x)}{R'(x)} \geq C_2 \frac{R'(x)}{R(x)}, \quad \text{a.e. } x \in \mathbb{R}^+ \setminus J, \quad (1.10)$$

then we write $w \in \mathcal{L}_2$.

Let us consider the weight $w \in \mathcal{L}(C^2+)$ in Definition 1.2. Levin and Lubinsky [2, Theorem 1.3] have given the following theorem.

Theorem A (see [3, Theorem 1.3]). *Let $w \in \mathcal{L}(C^2+)$ and $\rho > -1/2$. There exists n_0 such that uniformly for $n \geq n_0$, $1 \leq j \leq n$,*

$$\left| p'_{n,\rho} w_\rho \right| (x_{jn}) \sim \varphi_n(x_{jn})^{-1} [x_{jn}(a_n - x_{jn})]^{-1/4}. \quad (1.11)$$

Now, we will estimate the higher-order derivatives of the orthonormal polynomials $p_n(w_\rho^2; x)$. However, we need to focus on a smaller class of weights.

Definition 1.3. Let $w = \exp(-R) \in \mathcal{L}_2$ and $\nu \geq 2$ be an integer. For the exponent R , we assume the following:

- (a) $R^{(j)}(x) > 0$, for $0 \leq j \leq \nu$ and $x > 0$, and $R^{(j)}(0) = 0$, $0 \leq j \leq \nu - 1$.
- (b) there exist positive constants $C_i > 0$ such that for $i = 1, 2, \dots, \nu - 1$

$$R^{(i+1)}(x) \leq C_i R^{(i)}(x) \frac{R'(x)}{R(x)}, \quad \text{a.e. } x \in \mathbb{R}^+ \setminus \{0\}, \quad (1.12)$$

- (c) there exist positive constants C , $0 < c_1 \leq 1$ and $0 \leq \delta < 1$ such that for $x \in (0, c_1)$

$$R^{(\nu)}(x) \leq C \left(\frac{1}{x} \right)^\delta, \quad (1.13)$$

(d) there exists $c_2 > 0$ such that we have one among the following:

- (d1) $T(x)/\sqrt{x}$ is quasi-increasing on (c_2, ∞) ,
- (d2) $R^{(v)}(x)$ is nondecreasing on (c_2, ∞) .

Then, we write $w(x) = e^{-R(x)} \in \tilde{\mathcal{L}}_v$.

Example 1.4 (see [1, 4]). Let $v \geq 2$ be a fixed integer. There are some typical examples satisfying all conditions of Definition 1.3 constructed as follows: let $\alpha > 1$, $l \geq 1$, where l is an integer. Then, we define

$$R_{l,\alpha}(x) = \exp_l(x^\alpha) - \exp_l(0), \quad (1.14)$$

where $\exp_l(x) = \exp(\exp(\exp \cdots \exp(x)) \cdots)$ is the l th iterated exponential.

- (1) When $\alpha > v$, we consider $w(x) = e^{-R_{l,\alpha}(x)}$, then $w \in \tilde{\mathcal{L}}_v$.
- (2) When $\alpha \leq v$, α is an integer, we define

$$R_{l,\alpha}(x) = \exp_l(|x|^\alpha) - \exp_l(0) - \sum_{j=1}^r \frac{R_{l,\alpha}^{(j)}(0)}{j!} x^j. \quad (1.15)$$

Then, $w(x) = e^{-R_{l,\alpha}(x)} \in \tilde{\mathcal{L}}_v$.

In the rest of this paper, we consider the classes \mathcal{L}_2 and $\tilde{\mathcal{L}}_v$; let $w \in \mathcal{L}_2$ or $w \in \tilde{\mathcal{L}}_v$ ($v \geq 2$). For $\rho > -1/2$, we set $w_\rho(x) := x^\rho w(x)$. Then, we can construct the orthonormal polynomials $p_{n,\rho}(x) = p_n(w_\rho^2; x)$ of degree n with respect to $w_\rho^2(x)$. That is,

$$\int_0^\infty p_{n,\rho}(u) p_{m,\rho}(u) w_\rho^2(u) du = \delta_{nm} \quad (\text{Kronecker's delta}) \quad n, m = 0, 1, 2, \dots \quad (1.16)$$

Let us denote the zeros of $p_{n,\rho}(x)$ by

$$0 < x_{n,n,\rho} < \cdots < x_{2,n,\rho} < x_{1,n,\rho} < \infty. \quad (1.17)$$

The Mhaskar-Rahmanov-Saff numbers a_v are defined as follows:

$$v = \frac{1}{\pi} \int_0^1 a_v t R'(a_v t) \{t(1-t)\}^{-1/2} dt, \quad v > 0. \quad (1.18)$$

In this paper, we will consider the orthonormal polynomials $p_{n,\rho}(x)$ with respect to the weight class $\tilde{\mathcal{L}}_v$. Our main themes in this paper are to estimate the higher-order derivatives of $p_{n,\rho}(x)$ at the zeros of $p_{n,\rho}(x)$ and to investigate the various weighted L_p -norms ($0 < p \leq \infty$) of $p_{n,\rho}(x)$. More precisely, we will estimate the higher-order derivatives of $p_{n,\rho}(x)$ at all zeros of $p_{n,\rho}(x)$ for two cases of an odd order and of an even order. In addition, we will give

asymptotic relation of the odd order derivatives of $p_{n,\rho}(x)$ at the zeros of $p_{n,\rho}(x)$ in a certain finite interval. These estimations will play an important role in investigating convergence or divergence of higher-order Hermite-Fejér interpolation polynomials (see [3, 5–17]).

Then, our main purpose is to obtain estimations with respect to $p_{n,\rho}^{(j)}(x_{k,n,\rho})$, $k = 1, 2, \dots, n$, $j = 1, 2, \dots, \nu$ as follows.

Theorem 1.5. *Let $w(x) = \exp(-R(x)) \in \mathcal{L}(C^{2+})$ and $\rho > -1/2$. For each $k = 1, 2, \dots, n$ and $j = 0, 1, \dots, \nu - 1$ one has*

$$\left| p_{n,\rho}^{(j)}(x_{k,n,\rho}) \right| \leq C \left(\frac{n}{\sqrt{a_{2n} - x_{k,n,\rho}}} \right)^{j-1} x_{k,n,\rho}^{-(j-1)/2} \left| p'_{n,\rho}(x_{k,n,\rho}) \right|. \tag{1.19}$$

Theorem 1.6. *Let $w(x) = \exp(-R(x)) \in \tilde{\mathcal{L}}_\nu$ and $\rho > -1/2$. Assume that $1 + 2\rho - \delta/2 \geq 0$ for $\rho < -1/4$, and if $T(x)$ is bounded, then assume that*

$$a_n \leq Cn^{2/(1+\nu-\delta)}, \tag{1.20}$$

where $0 \leq \delta < 1$ is defined in (1.13). For each $k = 1, 2, \dots, n$ and $j = 0, 1, \dots, \nu - 1$, one has

$$\left| p_{n,\rho}^{(j)}(x_{k,n,\rho}) \right| \leq C \left(\frac{n}{\sqrt{a_{2n} - \sqrt{x_{k,n,\rho}}} + \frac{T(a_n)}{\sqrt{a_n}}} \right)^{j-1} x_{k,n,\rho}^{-(j-1)/2} \left| p'_{n,\rho}(x_{k,n,\rho}) \right|, \tag{1.21}$$

and in particular if j is even, then

$$\begin{aligned} \left| p_{n,\rho}^{(j)}(x_{k,n,\rho}) \right| &\leq C \left(\frac{T(a_n)}{\sqrt{a_n x_{k,n,\rho}}} + R'(x_{k,n,\rho}) + \frac{1}{x_{k,n,\rho}} \right) \\ &\times \left(\frac{n}{\sqrt{a_{2n} - \sqrt{x_{k,n,\rho}}} + \frac{T(a_n)}{\sqrt{a_n}}} \right)^{j-2} x_{k,n,\rho}^{-(j-2)/2} \left| p'_{n,\rho}(x_{k,n,\rho}) \right|. \end{aligned} \tag{1.22}$$

Theorem 1.7. *Let $w(x) = \exp(-R(x)) \in \tilde{\mathcal{L}}_\nu$ and $\rho > -1/2$. Let $(1/\varepsilon)(a_n/n^2) \leq x_{k,n,\rho} \leq \varepsilon a_n$, $0 < \varepsilon < 1/4$, and $\nu = 2, 3, \dots$, $s = 0, 1, \dots, (\nu - 1)/2$. Then, under the same conditions as the assumptions of Theorem 1.6, there exist $\beta(n, k)$, $0 < D_1 \leq \beta(n, k) \leq D_2$ for absolute constants D_1, D_2 such that the following equality holds:*

$$p_{n,\rho}^{(2s+1)}(x_{k,n,\rho}) = (-1)^s \beta_{2n}^s(2n, k) \left(\frac{n}{\sqrt{a_n}} \right)^{2s} (1 + \rho_s(\varepsilon, x_{k,n,\rho}, n)) p'_n(x_{k,n,\rho}) x_{k,n,\rho}^{-s} \tag{1.23}$$

and $|\rho_s(\varepsilon, x_{k,n,\rho}, n)| \rightarrow 0$ as $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$.

Define

$$\Phi_n(x) = \max \left\{ \eta_n, 1 - \left(\frac{x}{a_n} \right)^{1/2} \right\}, \quad \eta_n = \{nT(a_n)\}^{-2/3},$$

$$z^+ = \begin{cases} z, & z > 0, \\ 0, & z \leq 0. \end{cases} \quad (1.24)$$

Let us define

$$\Theta_n(x) = \frac{x/a_n}{1 + x/a_n}. \quad (1.25)$$

We consider the class of weights, $\mathcal{F}(C^2)$, which is defined in Definition 2.1 below. Levin and Lubinsky have obtained the following theorem.

Theorem B (see [18, Theorem 13.6]). *Assume that $W \in \mathcal{F}(C^2)$. Let $0 < p < \infty$. Then uniformly for $n \geq 1$,*

$$\|P_n W\|_{L_p(I)} \sim \begin{cases} a_n^{1/p-1/2}, & p < 4, \\ a_n^{-1/4} (\log(n+1))^{1/4}, & p = 4, \\ a_n^{1/p-1/2} (nT(a_n))^{(2/3)(1/4-1/p)}, & p > 4. \end{cases} \quad (1.26)$$

We remark that Levin and Lubinsky have shown Theorem B for more wider class $\mathcal{F}(\text{lip}(1/2)) \supseteq \mathcal{F}(C^2)$. In the following, we investigate the various weighted L_p -norms ($0 < p \leq \infty$) of $p_{n,\rho}(x)$.

Theorem 1.8. *Let $w \in \mathcal{L}_2$. Let $0 < p < \infty$ and $\rho > -1/2$. Then one has for $n \geq 1$,*

$$\left\| \Theta_n^{1/4}(x) p_{n,\rho}(x) w(x) \left(x + \frac{a_n}{n^2} \right)^\rho \right\|_{L_p(\mathbb{R}^+)} \sim a_n^{1/p-1/2} \begin{cases} 1, & p < 4, \\ (\log(1 + nT(a_n)))^{1/4}, & p = 4, \\ (nT(a_n))^{(2/3)(1/4-1/p)}, & p > 4. \end{cases} \quad (1.27)$$

Theorem 1.9. *Let $w \in \mathcal{L}_2$. Let $0 < p < \infty$ and $\rho > -1/2$. Then one has for $n \geq 1$,*

$$\left\| p_{n,\rho}(x) w(x) \left(x + \frac{a_n}{n^2} \right)^\rho \right\|_{L_p(\mathbb{R}^+)} \sim a_n^{1/p-1/2} \begin{cases} 1, & p < 4, \\ (\log n)^{1/4}, & p = 4, \\ n^{2(1/4-1/p)}, & p > 4. \end{cases} \quad (1.28)$$

Theorem 1.10. Let $w \in \mathcal{L}_2$, $0 \leq s \leq r$, and $n \geq 1$. Suppose that $\rho > -1/2$. For $0 \leq s \leq r$ and $n \geq 1$, one has

$$\left\| \Theta_n^{r/4}(x) \Phi_n^{(r/4-1/p)^+}(x) \left| p_{n,\rho}(x) w(x) \left(x + \frac{a_n}{n} \right)^\rho \right|^s \right\|_{L_p(\mathbb{R}^+)} \sim \begin{cases} a_n^{1/p-r/2} \log n, & \text{if } s = r, pr \geq 4, \\ a_n^{1/p-s/2}, & \text{otherwise,} \end{cases} \tag{1.29}$$

and for $p = \infty$

$$\left\| \Theta_n^{r/4}(x) \Phi_n^{(r/4-1/p)^+}(x) \left| p_{n,\rho}(x) w(x) \left(x + \frac{a_n}{n} \right)^\rho \right|^s \right\|_{L_p(\mathbb{R}^+)} \sim a_n^{-s/4}. \tag{1.30}$$

This paper is organized as follows. In Section 2, we will introduce the weight class $\tilde{\mathcal{F}}_v$ as an analogy of the class $\tilde{\mathcal{L}}_v$, and the known results of orthonormal polynomials with respect to $\tilde{\mathcal{F}}_v$, in order to prove the main results. In Section 3, we will prove Theorems 1.5, 1.6, and 1.7. Finally, we will prove the results for the various weighted L_p -norms ($0 < p \leq \infty$) of $p_{n,\rho}(x)$, that is, Theorems 1.8, 1.9, and 1.10, in Section 4.

2. Preliminaries

Levin and Lubinsky introduced the classes $\mathcal{L}(C^2)$ and $\mathcal{L}(C^{2+})$ as an analogy of the classes $\mathcal{F}(C^2)$ and $\mathcal{F}(C^{2+})$ which they already defined on $I^* = (-\sqrt{d}, \sqrt{d})$. They defined the following.

Definition 2.1 (see [18]). We assume that $Q : I^* \rightarrow [0, \infty)$ has the following properties:

- (a) $Q(t)$ is continuous in I^* , with $Q(0) = 0$,
- (b) $Q''(t)$ exists and is positive in $I^* \setminus \{0\}$,
- (c)

$$\lim_{t \rightarrow \sqrt{d}^-} Q(t) = \infty, \tag{2.1}$$

- (d) the function

$$T^*(t) := \frac{tQ'(t)}{Q(t)} \tag{2.2}$$

is quasi-increasing in $(0, \sqrt{d})$, with

$$T^*(t) \geq \Lambda^* > 1, \quad t \in I^* \setminus \{0\}, \tag{2.3}$$

(e) there exists $C_1 > 0$ such that

$$\frac{Q''(t)}{|Q'(t)|} \leq C_1 \frac{|Q'(t)|}{Q(t)}, \quad \text{a.e. } t \in I^* \setminus \{0\}. \quad (2.4)$$

Then, we write $W \in \mathcal{F}(C^2)$. If there also exists a compact subinterval $J^* \ni 0$ of I^* , and $C_2 > 0$ such that

$$\frac{Q''(t)}{|Q'(t)|} \geq C_2 \frac{|Q'(t)|}{Q(t)}, \quad \text{a.e. } t \in I^* \setminus J^*, \quad (2.5)$$

then, we write $W \in \mathcal{F}(C^{2+})$.

Then we see that $w \in \mathcal{L}(C^2) \Leftrightarrow \mathcal{F}(C^2)$ and $w \in \mathcal{L}(C^{2+}) \Leftrightarrow \mathcal{F}(C^{2+})$ from [1, Lemma 2.2]. In addition, we easily have the following.

Lemma 2.2. *Let $Q(t) = R(t^2)$, $x = t^2$. Then one has*

$$w \in \mathcal{L}_2 \implies W \in \mathcal{F}(C^{2+}), \quad (2.6)$$

where $W(t) = w(x)$, $x = t^2$.

On \mathbb{R} , we can consider the corresponding class to $\tilde{\mathcal{L}}_\nu$ as follows.

Definition 2.3 (cf. [19]). Let $W = \exp(-Q) \in \mathcal{F}(C^{2+})$ and $\nu \geq 2$ be an integer. Let Q be a continuous and even function on \mathbb{R} . For the exponent Q , we assume the following:

- (a) $Q^{(j)}(t) > 0$, for $0 \leq j \leq \nu$ and $t \in \mathbb{R}^+ \setminus \{0\}$,
- (b) there exist positive constants $C_i > 0$ such that for $i = 1, 2, \dots, \nu - 1$

$$Q^{(i+1)}(t) \leq C_i Q^{(i)}(t) \frac{Q'(t)}{Q(t)}, \quad \text{a.e. } x \in \mathbb{R}^+ \setminus \{0\}, \quad (2.7)$$

- (c) there exist positive constants $C, c_1 > 0$ and $0 \leq \delta^* < 1$ such that for $t \in (0, c_1)$

$$Q^{(\nu)}(t) \leq C \left(\frac{1}{t}\right)^{\delta^*}, \quad (2.8)$$

- (d) there exists $c_2 > 0$ such that one has one among the following:

- (d1) $T^*(t)/t$ is quasi-increasing on (c_2, ∞) ,
- (d2) $Q^{(\nu)}(t)$ is nondecreasing on (c_2, ∞) .

Then, we write $W(t) = e^{-Q(t)} \in \tilde{\mathcal{F}}_\nu$.

Let $W \in \tilde{\mathcal{F}}_\nu$ and $\nu \geq 2$. For $\rho^* > -1/2$, we set

$$W_{\rho^*}(t) := |t|^{\rho^*} W(t). \quad (2.9)$$

Then, we can construct the orthonormal polynomials $P_{n,\rho^*}(t) = P_n(W_{\rho^*}^2; t)$ of degree n with respect to $W_{\rho^*}(t)$. That is,

$$\int_{-\infty}^{\infty} P_{n,\rho^*}(v) P_{m,\rho^*}(v) W_{\rho^*}^2(v) dt = \delta_{nm}, \quad n, m = 0, 1, 2, \dots \quad (2.10)$$

Let us denote the zeros of $P_{n,\rho^*}(t)$ by

$$-\infty < t_{nn} < \dots < t_{2n} < t_{1n} < \infty. \quad (2.11)$$

Jung and Sakai [5, Theorems 3.3 and 3.6] estimate $P_{n,\rho^*}^{(j)}(t_{k,n})$, $k = 1, 2, \dots, n$, $j = 1, 2, \dots, \nu$, and we will obtain analogous estimations with respect to $p_{n,\rho}^{(j)}(x_{k,n})$, $k = 1, 2, \dots, n$, $j = 1, 2, \dots, \nu$ in Theorems 1.6 and 1.7.

There are many properties of $P_{n,\rho^*}(t) = P_n(W_{\rho^*}; t)$ with respect to $W_{\rho^*}(t)$, $W \in \tilde{\mathcal{F}}_\nu$, $\nu = 2, 3, \dots$ of Definition 2.3 in [4–6, 19–21]. They were obtained by transformations from the results in [1, 2]. In this paper, we consider $w = \exp(-R) \in \tilde{\mathcal{L}}_\nu$ and $p_{n,\rho}(x) = p_n(w_\rho; x)$. In [5] we got the estimations of $P_{n,\rho^*}^{(j)}(t_{kn})$, $k = 1, 2, \dots, n$, $j = 1, 2, \dots, \nu - 1$ with the weight $W_{\rho^*}(t) \in \tilde{\mathcal{F}}_\nu$. By a transformation of the results with respect to $P_{n,\rho^*}(t)$, we estimate $p_{n,\rho}^{(j)}(x_{kn})$, $k = 1, 2, \dots, n$, $j = 1, 2, \dots, \nu - 1$. In order to it we will give the transformation theorems in this section. In the following, we will give some applications of them.

Theorem 2.4 (see [21, Theorem 2.1]). *Let $W(t) = w(x)$ with $x = t^2$. Then, the orthonormal polynomials $P_{n,\rho^*}(t)$ on \mathbb{R} can be entirely reduced to the orthonormal polynomials $p_{n,\rho}(x)$ in \mathbb{R}^+ as follows: for $n = 0, 1, 2, \dots$,*

$$P_{2n,2\rho+(1/2)}(t) = p_{n,\rho}(x), \quad P_{2n+1,2\rho-(1/2)}(t) = tp_{n,\rho}(x). \quad (2.12)$$

In this paper, we will use the fact that $w_\rho(x) = x^\rho \exp(-R(x))$ is transformed into $W_{2\rho+1/2}(t) = |t|^{2\rho+1/2} \exp(-Q(t))$ as meaning that

$$\begin{aligned} \int_0^\infty p_{n,\rho}(x) p_{m,\rho}(x) w_\rho^2(x) dx &= 2 \int_0^\infty p_{n,\rho}(t^2) p_{m,\rho}(t^2) t^{4\rho+1} W^2(t) dt \\ &= \int_{-\infty}^\infty P_{2n,2\rho+1/2}(t) P_{2m,2\rho+1/2}(t) W_{2\rho+1/2}^2(t) dt. \end{aligned} \quad (2.13)$$

Theorem 2.5. *Let $Q(t) = R(x)$, $x = t^2$. Then one has*

$$w(x) = \exp(-R(x)) \in \tilde{\mathcal{L}}_\nu \implies W(t) = \exp(-Q(t)) \in \tilde{\mathcal{F}}_\nu. \quad (2.14)$$

In particular, one has

$$Q^{(\nu)}(t) \leq C \left(\frac{1}{t} \right)^\delta, \quad (2.15)$$

where $0 \leq \delta < 1$ is defined in (1.13).

Proof. Let $w \in \tilde{\mathcal{L}}_2$. Then, from Lemma 2.2, one has $W \in \mathcal{F}(C^2+)$. Let $[x]$ denote the maximum integer as $[x] \leq x$ (Gaussian symbol). For $1 \leq j \leq \nu$, one has

$$Q^{(j)}(t) = \sum_{i=0}^{[j/2]} c_{ij} R^{(j-i)}(x) t^{j-2i}, \quad c_{ij} > 0 \quad (i = 0, 1, \dots, [j/2]), \quad x = t^2. \quad (2.16)$$

Therefore, we easily see that (a) of Definition 2.3 holds. Let $x = t^2$. Since $R^{(\ell)}(x)$ is increasing for $x > 0$ and $\ell = 0, 1, \dots, \nu - 1$, there exists ξ with $0 < \xi < x$ such that for $k = 0, 1, \dots, \nu - 2$,

$$\frac{R^{(k)}(x)}{x} = R^{(k+1)}(\xi) \leq C R^{(k+1)}(x). \quad (2.17)$$

Then, since for $0 \leq k < j \leq \nu - 1$,

$$R^{(k)}(x) \leq C x^{j-k} R^{(j)}(x), \quad (2.18)$$

one has by (b) of Definition 1.3 that

$$\begin{aligned} Q^{(j)}(t) &= \sum_{i=0}^{[j/2]} c_{ij} R^{(j-i)}(x) t^{j-2i} \leq C R^{(j)}(x) t^j \leq C R^{(j-1)}(x) t^{j-1} \left(\frac{t R'(x)}{R(x)} \right) \\ &\leq C Q^{(j-1)}(t) \frac{Q'(t)}{Q(t)}, \quad 1 \leq j \leq \nu - 1. \end{aligned} \quad (2.19)$$

Similarly, one has by (2.16), (d) of Definition 1.2, and (b) of Definition 1.3 that

$$\begin{aligned} Q^{(\nu)}(t) &= c_{0,\nu} R^{(\nu)}(x) t^\nu + \sum_{i=1}^{[\nu/2]} c_{i,\nu} R^{(\nu-i)}(x) t^{\nu-2i} \\ &\leq c_{0,\nu} R^{(\nu)}(x) t^\nu + C R^{(\nu-1)}(x) t^{\nu-2} \\ &\leq C R^{(\nu-1)}(x) t^{\nu-1} \left(\frac{t R'(x)}{R(x)} \right) \\ &\leq C Q^{(\nu-1)}(t) \frac{Q'(t)}{Q(t)}. \end{aligned} \quad (2.20)$$

Consequently, one has (b) in Definition 2.3. We know that

$$\sum_{i=1}^{[v/2]} c_{i,v} R^{(v-i)}(x) t^{v-2i} \leq C, \quad t \in (0, c_1), \quad (2.21)$$

and since $t^{v-\delta} \leq C$ on $t \in (0, c_1)$, one has from (1.13) that

$$R^{(v)}(x) t^v \leq C \left(\frac{1}{t^2} \right)^\delta t^v \leq C \left(\frac{1}{t^2} \right)^\delta t^\delta \leq C \left(\frac{1}{t} \right)^\delta. \quad (2.22)$$

Therefore, one has by (2.16)

$$Q^{(v)}(t) \leq C \left(\frac{1}{t} \right)^\delta, \quad (2.23)$$

where $0 \leq \delta < 1$ is defined in (1.13). The inequalities (d1) and (d2) of Definition 2.3 follow easily from (d1) and (d2) of Definition 1.3. Therefore, one has (2.14). \square

3. Proofs of Theorems 1.5, 1.6, and 1.7

For convenience, in the rest of this paper, we put as follows:

$$\rho > -\frac{1}{2}, \quad \rho^* := 2\rho + \frac{1}{2}, \quad p_n(x) := p_{n,\rho}(x), \quad P_n(t) := P_{n,\rho^*}(t), \quad (3.1)$$

and $x_{kn} = x_{k,n,\rho}$, $t_{kn} = t_{k,n,\rho^*}$. Then, we know that $\rho^* > -1/2$ and

$$p_n(x) = P_{2n}(t), \quad x = t^2, \quad x_{kn} = t_{k,2n}^2, \quad t_{k,2n} > 0, \quad k = 1, 2, \dots, n. \quad (3.2)$$

In the following, we introduce some useful notations.

- (a) The Mhaskar-Rahmanov-Saff numbers a_v and a_u^* are defined as the positive roots of the following equations:

$$\begin{aligned} v &= \frac{1}{\pi} \int_0^1 a_v t R'(a_v t) \{t(1-t)\}^{-1/2} dt, \quad v > 0, \\ u &= \frac{2}{\pi} \int_0^1 a_u^* t Q'(a_u^* t) (1-t^2)^{-1/2} dt, \quad u > 0. \end{aligned} \quad (3.3)$$

- (b) Let

$$\eta_n = \{nT(a_n)\}^{-2/3}, \quad \eta_n^* = \{nT^*(a_n^*)\}^{-2/3}. \quad (3.4)$$

Then, one has the following.

Lemma 3.1 (see [1, (2.5), (2.7), (2.9)]).

$$a_n = a_{2n}^*, \quad \eta_n = 4^{2/3} \eta_{2n}^*, \quad T(a_n) = \frac{1}{2} T^*(a_{2n}^*). \quad (3.5)$$

To prove Theorem 1.6, we need some lemmas as follows.

Lemma 3.2 (see [21, Theorem 2.2, Lemma 3.7]). *For the minimum positive zero $t_{[n/2],n}$ ($[n/2]$ is the largest integer $n/2$), one has*

$$t_{[n/2],n} \sim a_n^* n^{-1}, \quad (3.6)$$

and for the maximum zero x_{1n} , one has for large enough n ,

$$1 - \frac{t_{1n}}{a_n^*} \sim \eta_n^*, \quad \eta_n^* = (nT^*(a_n^*))^{-2/3}. \quad (3.7)$$

Moreover, for some constant $0 < \varepsilon \leq 2$, one has

$$T^*(a_n^*) \leq Cn^{2-\varepsilon}. \quad (3.8)$$

Lemma 3.3 (see [6, Theorem 2.5]). *Let $W \in \mathcal{F}(C^{2+})$ and $r = 1, 2, \dots$. Then, uniformly for $1 \leq k \leq n$,*

$$\left| \frac{P_{n,\rho}^{(r)}(t_{k,\rho,n})}{P_{n,\rho}'(t_{k,\rho,n})} \right| \leq C \left(\frac{n}{\sqrt{a_{2n}^*{}^2 - t_{k,\rho,n}^2}} \right)^{r-1}. \quad (3.9)$$

Lemma 3.4 (see [5, Theorem 3.6 and Lemma 3.7 (3.20)]). *Let $\rho^* > -1/2$ and $W(x) = \exp(-Q(x)) \in \tilde{\mathcal{F}}_\nu$, $\nu \geq 2$. Assume that $1 + 2\rho^* - \delta^* \geq 0$ for $\rho^* < 0$ and if $T^*(t)$ is bounded, then assume*

$$a_n^* \leq Cn^{1/(1+\nu-\delta^*)}, \quad (3.10)$$

where $0 \leq \delta^* < 1$ is defined in (2.8). If $t_{kn} \neq 0$, then one has for $j = 1, 2, \dots, \nu$

$$\left| P_n^{(j)}(t_{kn}) \right| \leq C \left(\frac{n}{a_{2n}^* - |t_{kn}|} + \frac{T^*(a_n^*)}{a_n^*} \right)^{j-1} |P_n'(t_{kn})|, \quad (3.11)$$

and in particular, if j is even, then

$$\left| P_n^{(j)}(t_{kn}) \right| \leq C \left(\frac{T^*(a_n^*)}{a_n^*} + |Q'(t_{kn})| + \frac{1}{|t_{kn}|} \right) \left(\frac{n}{a_{2n}^* - |t_{kn}|} + \frac{T^*(a_n^*)}{a_n^*} \right)^{j-2} |P_n'(t_{kn})|. \quad (3.12)$$

Remark 3.5. Let $W(t) \in \mathcal{F}(C^2+)$. Then, from [19, Theorem 1.6] we know that when $T^*(t)$ is unbounded, for any $\eta > 0$, there exists $C(\eta) > 0$ such that for $t \geq 1$,

$$a_t^* \leq C(\eta)t^\eta. \tag{3.13}$$

In addition, since $T(x) = T^*(t)/2$, we know that

- (i) $T(x)$ is bounded $\Leftrightarrow T^*(t)$ is bounded,
- (ii) $T(x)$ is unbounded $\Rightarrow a_n \leq Cn^\eta$ for any $\eta > 0$,
- (iii) $T(a_n) \leq Cn^{2-\varepsilon}$ for some constant $0 < \varepsilon \leq 2$.

Lemma 3.6. For $j = 1, 2, 3, \dots$, one has

$$p_n^{(j)}(x) = \sum_{i=1}^j (-1)^{j-i} c_{j,i} P_{2n}^{(i)}(t) t^{-2j+i}, \tag{3.14}$$

where $c_{j,i} > 0$ satisfy that for $k = 1, 2, \dots$,

$$c_{k+1,1} = \frac{2k-1}{2} c_{k,1}, \quad c_{k+1,k+1} = \frac{1}{2^{k+1}}, \quad c_{1,1} = \frac{1}{2}, \tag{3.15}$$

and for $2 \leq i \leq k$

$$c_{k+1,i} = \frac{c_{k,i-1} + (2k-i)c_{k,i}}{2}. \tag{3.16}$$

Proof. It is easily proved, using the mathematical induction on j . □

Proof of Theorem 1.5. By Lemmas 3.3, 3.6 and (3.2), one has

$$\begin{aligned} |p_n^{(j)}(x_{kn})| &\leq C \sum_{i=1}^j |P_{2n}^{(i)}(t_{k,2n})| |t_{k,2n}^{-2j+i}| \\ &\leq C \sum_{i=1}^j \left(\frac{2n}{\sqrt{a_{4n}^*{}^2 - t_{k,2n}^2}} \right)^{i-1} |P'_{2n}(t_{k,2n})| |t_{k,2n}^{-2j+i}|. \end{aligned} \tag{3.17}$$

Since by Lemma 3.2

$$\begin{aligned}
 \sum_{i=1}^j \left(\frac{2n}{\sqrt{a_{4n}^{*2} - t_{k,2n}^2}} \right)^{i-1} |t_{k,2n}^{i-1}| &\leq C \left\{ 1 + \left(\frac{2n}{\sqrt{a_{4n}^{*2} - t_{k,2n}^2}} \right)^{j-1} |t_{k,2n}^{j-1}| \right\} \\
 &\leq C \left\{ |t_{k,2n}^{-j+1}| + \left(\frac{2n}{\sqrt{a_{4n}^{*2} - t_{k,2n}^2}} \right)^{j-1} \right\} |t_{k,2n}^{j-1}| \\
 &\leq C \left(\frac{n}{\sqrt{a_{4n}^{*2} - t_{k,2n}^2}} \right)^{j-1} |t_{k,2n}^{j-1}|,
 \end{aligned} \tag{3.18}$$

one has from Lemma 3.1 that

$$\begin{aligned}
 |p_n^{(j)}(x_{kn})| &\leq C \left(\frac{n}{\sqrt{a_{4n}^{*2} - t_{k,2n}^2}} \right)^{j-1} |P'_{2n}(t_{k,2n})| |t_{k,2n}^{-j}| \\
 &\leq C \left(\frac{n}{\sqrt{a_{2n} - x_{k,n}}} \right)^{j-1} x_{k,n}^{-(j-1)/2} |p'_n(x_{k,n})|.
 \end{aligned} \tag{3.19}$$

□

Proof of Theorem 1.6. Since $w(x) \in \tilde{\mathcal{L}}_\nu$, we know that $W(t) \in \tilde{\mathcal{F}}_\nu$ and we know that $\delta^* = \delta$ by Theorem 2.5 and from (3.1), (3.2), and Lemma 3.1 that

- (i) $\rho > -1/2 \Rightarrow \rho^* > -1/2$,
- (ii) $1 + 2\rho - \delta/2 \geq 0$ for $\rho < -1/4 \Rightarrow 1 + 2\rho^* - \delta^* \geq 0$ for $\rho^* < 0$,
- (iii) $a_n \leq Cn^{2/(1+\nu-\delta)} \Rightarrow a_n^* \leq Cn^{1/(1+\nu-\delta^*)}$.

Then, using Remark 3.5, we can apply Lemma 3.4 to $p_n(x) = P_{2n,\rho^*}(t)$, $x = t^2$. In a similar way to the proof of Theorem 1.5, one has from Lemma 3.4 and Lemma 3.1

$$\begin{aligned}
 |p_n^{(j)}(x_{kn})| &\leq C \sum_{i=1}^j |P_{2n}^{(i)}(t_{k,2n})| |t_{k,2n}^{-2j+i}| \\
 &\leq C \sum_{i=1}^j \left(\frac{n}{a_{4n}^* - |t_{k,2n}|} + \frac{T^*(a_{2n}^*)}{a_{2n}^*} \right)^{i-1} |P'_{2n}(t_{k,2n})| |t_{k,2n}^{-2j+i}| \\
 &\leq C \left(\frac{n}{a_{4n}^* - |t_{k,2n}|} + \frac{T^*(a_{2n}^*)}{a_{2n}^*} \right)^{j-1} |P'_{2n}(t_{k,2n})| |t_{k,2n}^{-j}| \\
 &\leq C \left(\frac{n}{\sqrt{a_{2n} - x_{k,n}}} + \frac{T(a_n)}{\sqrt{a_n}} \right)^{j-1} x_{k,n}^{-(j-1)/2} |p'_n(x_{k,n})|.
 \end{aligned} \tag{3.20}$$

Let j be even. Then, one has from Lemma 3.4 that

$$\begin{aligned} \sum_{i:\text{even}} \left| P_{2n}^{(i)}(t_{k,2n}) \right| |t_{k,2n}^{-2j+i}| &\leq C \left(\frac{T^*(a_{2n}^*)}{a_{2n}^*} + |Q'(t_{k,2n})| + \frac{1}{|t_{k,2n}|} \right) |P'_{2n}(t_{k,2n})| |t_{k,2n}^{-2j+2}| \\ &\times \sum_{i:\text{even}} \left(\frac{n}{a_{4n}^* - |t_{k,2n}|} + \frac{T^*(a_{2n}^*)}{a_{2n}^*} \right)^{i-2} |t_{k,2n}^{i-2}|. \end{aligned} \quad (3.21)$$

Since by Lemma 3.2 and

$$\sum_{i:\text{even}} \left(\frac{n}{a_{4n}^* - |t_{k,2n}|} + \frac{T^*(a_{2n}^*)}{a_{2n}^*} \right)^{i-2} |t_{k,2n}^{i-2}| \leq C \left(\frac{n}{a_{4n}^* - |t_{k,2n}|} + \frac{T^*(a_{2n}^*)}{a_{2n}^*} \right)^{j-2} |t_{k,2n}^{j-2}|, \quad (3.22)$$

one has

$$\begin{aligned} \sum_{i:\text{even}} \left| P_{2n}^{(i)}(t_{k,2n}) \right| |t_{k,2n}^{-2j+i}| &\leq C \left(\frac{T^*(a_{2n}^*)}{a_{2n}^*} + |Q'(t_{k,2n})| + \frac{1}{|t_{k,2n}|} \right) |P'_{2n}(t_{k,2n})| |t_{k,2n}^{-j}| \\ &\times \left(\frac{n}{a_{4n}^* - |t_{k,2n}|} + \frac{T^*(a_{2n}^*)}{a_{2n}^*} \right)^{j-2}, \\ \sum_{i:\text{odd}}^{j-1} \left| P_{2n}^{(i)}(t_{k,2n}) \right| |t_{k,2n}^{-2j+i}| &\leq C \left(\frac{n}{a_{4n}^* - |t_{k,2n}|} + \frac{T^*(a_{2n}^*)}{a_{2n}^*} \right)^{j-2} |P'_{2n}(t_{k,2n})| |t_{k,2n}^{-j-1}| \\ &\leq C \left(\frac{T^*(a_{2n}^*)}{a_{2n}^*} + |Q'(t_{k,2n})| + \frac{1}{|t_{k,2n}|} \right) \\ &\times \left(\frac{n}{a_{4n}^* - |t_{k,2n}|} + \frac{T^*(a_{2n}^*)}{a_{2n}^*} \right)^{j-2} |P'_{2n}(t_{k,2n})| |t_{k,2n}^{-j}|. \end{aligned} \quad (3.23)$$

Therefore, when j is even, one has by Lemma 3.1 that

$$\begin{aligned} |p_n^{(j)}(x_{kn})| &\leq C \left(\frac{T^*(a_{2n}^*)}{a_{2n}^* |t_{k,2n}|} + \frac{|Q'(t_{k,2n})|}{|t_{k,2n}|} + \frac{1}{t_{k,2n}^2} \right) \\ &\times \left(\frac{n}{a_{4n}^* - |t_{k,2n}|} + \frac{T^*(a_{2n}^*)}{a_{2n}^*} \right)^{j-2} |P'_{2n}(t_{k,2n})| |t_{k,2n}^{-j+1}| \\ &\leq C \left(\frac{T(a_n)}{\sqrt{a_n} x_{k,n}} + R'(x_{k,n}) + \frac{1}{x_{k,n}} \right) \\ &\times \left(\frac{n}{\sqrt{a_{2n}} - \sqrt{x_{k,n}}} + \frac{T(a_n)}{\sqrt{a_n}} \right)^{j-2} x_{k,n}^{-(j-2)/2} |p'_n(x_{k,n})|. \end{aligned} \quad (3.24)$$

□

Next, we will prove Theorem 1.7. To prove it, we need two lemmas as follows.

Lemma 3.7 ([5, Theorem 3.3]). *Let $W(x) = \exp(-Q(x)) \in \tilde{\mathcal{F}}_\nu$, $\nu \geq 2$. Let $(1/\varepsilon)(a_n^*/n) \leq |t_{kn}| \leq \varepsilon a_n^*$, $0 < \varepsilon < 1/2$, and $s = 1, 2, \dots, (\nu - 1)/2$. Then, under the same conditions as the assumptions of Lemma 3.4, there exist $\beta(n, k)$, $0 < D_1 \leq \beta(n, k) \leq D_2$ for absolute constants D_1, D_2 such that the following equality holds:*

$$P_n^{(2s+1)}(t_{kn}) = (-1)^s \beta_n^s(n, k) \left(\frac{n}{a_n^*}\right)^{2s} (1 + \tilde{\rho}_{2s+1}(\varepsilon, t_{kn}, n)) P_n'(t_{kn}), \quad (3.25)$$

and $|\tilde{\rho}_{2s+1}(\varepsilon, t_{kn}, n)| \rightarrow 0$ as $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$.

From Lemma 3.3, we easily have the following.

Lemma 3.8. *Let $W \in \mathcal{F}(C^2+)$ and $j = 1, 2, \dots$. Then, uniformly for $|t_{kn}| \leq a_n^*/2$,*

$$|P_n^{(j)}(t_{kn})| \leq C \left(\frac{n}{a_n^*}\right)^{j-1} |P_n'(t_{kn})|. \quad (3.26)$$

Proof of Theorem 1.7. By Lemmas 3.4, 3.6 and Theorem 2.4, one has

$$\begin{aligned} p_n^{(2s+1)}(x_{kn}) &= \sum_{i=1}^{2s+1} (-1)^{2s+1-i} c_{2s+1,i} P_{2n}^{(i)}(t_{k,2n}) t_{k,2n}^{-2(2s+1)+i} \\ &= \sum_{p=0}^s c_{2s+1,2p+1} P_{2n}^{(2p+1)}(t_{k,2n}) t_{k,2n}^{-4s+2p-1} - \sum_{p=1}^s c_{2s+1,2p} P_{2n}^{(2p)}(t_{k,2n}) t_{k,2n}^{-4s+2p-2} \\ &=: \sum_{\text{odd}} - \sum_{\text{even}}. \end{aligned} \quad (3.27)$$

Since we know that

$$\frac{1}{\varepsilon} \frac{a_n}{n^2} \leq x_{k,n} \leq \varepsilon a_n \implies \frac{2}{\sqrt{\varepsilon}} \frac{a_{2n}^*}{2n} \leq |t_{k,2n}| \leq \sqrt{\varepsilon} a_{2n}^*, \quad 0 < \sqrt{\varepsilon} < \frac{1}{2}, \quad (3.28)$$

by the same reason as the proof of Theorem 1.6, we can apply Lemma 3.7 to $P_{2n}^{(2p+1)}(t_{k,2n})$.

Then, using Lemmas 3.7 and 3.6, one has

$$\begin{aligned}
 \sum_{\text{odd}} &= \sum_{p=0}^s c_{2s+1,2p+1} (-1)^p \beta_{2n}^p (2n, k) \left(\frac{2n}{a_{2n}^*} \right)^{2p} (1 + \tilde{\rho}_{2p+1}) P'_{2n}(t_{k,2n}) t_{k,2n}^{-4s+2p-1} \\
 &= (-1)^s \beta_{2n}^s (2n, k) \left(\frac{n}{\sqrt{a_n}} \right)^{2s} \left(\frac{1}{2} + \frac{\tilde{\rho}_{2s+1}}{2} \right) P'_{2n}(t_{k,2n}) t_{k,2n}^{-2s-1} \\
 &\quad + \sum_{p=0}^{s-1} c_{2s+1,2p+1} (-1)^p \beta_{2n}^p (2n, k) \left(\frac{2n}{a_{2n}^*} \right)^{2p} (1 + \tilde{\rho}_{2p+1}) P'_{2n}(t_{k,2n}) t_{k,2n}^{-4s+2p-1} \\
 &= (-1)^s \beta_{2n}^s (2n, k) \left(\frac{n}{\sqrt{a_n}} \right)^{2s} \left(\frac{1}{2} + \frac{\tilde{\rho}_{2s+1}}{2} \right) P'_{2n}(t_{k,2n}) t_{k,2n}^{-2s-1} \\
 &\quad + (-1)^s \beta_{2n}^s (2n, k) \left(\frac{n}{\sqrt{a_n}} \right)^{2s} P'_{2n}(t_{k,2n}) t_{k,2n}^{-2s-1} \\
 &\quad \times \sum_{p=0}^{s-1} c_{2s+1,2p+1} (-1)^{p-s} 2^{2s} \beta_{2n}^{p-s} (2n, k) \left(\frac{2n}{a_{2n}^*} \right)^{2(p-s)} (1 + \tilde{\rho}_{2p+1}) t_{k,2n}^{2(p-s)}.
 \end{aligned} \tag{3.29}$$

Here, $\tilde{\rho}_{2p+1} := \tilde{\rho}_{2p+1}(\varepsilon, t_{k,2n}, 2n)$, $p = 0, 1, \dots, s$. Since from (3.28) we see that for $0 \leq p \leq s - 1$,

$$\left(\frac{a_{2n}^*}{2n} \frac{1}{t_{k,2n}} \right)^{2(s-p)} \leq \left(\frac{\sqrt{\varepsilon}}{2} \right)^{2(s-p)} \leq \left(\frac{1}{4} \right)^{s-p} \varepsilon, \tag{3.30}$$

one has that

$$\left| \sum_{p=0}^{s-1} c_{2s+1,2p+1} (-1)^{p-s} 2^{2s} \beta_{2n}^{p-s} (2n, k) \left(\frac{a_{2n}^*}{2n} \frac{1}{t_{k,2n}} \right)^{2(s-p)} (1 + \tilde{\rho}_{2p+1}) \right| \rightarrow 0, \tag{3.31}$$

as $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$. If we let

$$\xi'_{n,1}(s; x_{kn}) := \tilde{\rho}_{2s+1} + \sum_{p=0}^{s-1} c_{2s+1,2p+1} (-1)^{p-s} 2^{2s+1} \beta_{2n}^{p-s} (2n, k) \left(\frac{2n}{a_{2n}^*} \right)^{2(p-s)} (1 + \tilde{\rho}_{2p+1}) t_{k,2n}^{2(p-s)}, \tag{3.32}$$

then one has

$$\begin{aligned}
 \sum_{\text{odd}} &= (-1)^s \beta_{2n}^s (2n, k) \left(\frac{n}{\sqrt{a_n}} \right)^{2s} \left(\frac{1}{2} + \frac{\xi'_{n,1}(s; x_{kn})}{2} \right) P'_{2n}(t_{k,2n}) t_{k,2n}^{-2s-1} \\
 &= (-1)^s \beta_{2n}^s (2n, k) \left(\frac{n}{\sqrt{a_n}} \right)^{2s} \left(1 + \xi'_{n,1}(s; x_{kn}) \right) p'_n(x_{k,n}) x_{k,n}^{-s}
 \end{aligned} \tag{3.33}$$

and $|\xi'_{n,1}(s; x_{kn})| \rightarrow 0$ as $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$. On the other hand, we obtain

$$\begin{aligned}
 \sum_{\text{even}} &= \sum_{p=1}^s c_{2s+1,2p} \frac{P_{2n}^{(2p)}(t_{k,2n})}{P'_{2n}(t_{k,2n})} P'_{2n}(t_{k,2n}) t_{k,2n}^{-4s+2p-2} \\
 &= \sum_{p=1}^s 2c_{2s+1,2p} \frac{P_{2n}^{(2p)}(t_{k,2n})}{P'_{2n}(t_{k,2n})} p'_n(x_{k,n}) t_{k,2n}^{-4s+2p-1} \\
 &= (-1)^s \beta_{2n}^s(2n, k) \left(\frac{n}{\sqrt{a_n}} \right)^{2s} p'_n(x_{k,n}) x_{k,n}^{-s} \\
 &\quad \times \sum_{p=1}^s 2c_{2s+1,2p} (-1)^s \beta_{2n}^{-s}(2n, k) \left(\frac{n}{a_{2n}^*} \right)^{-2s} \frac{P_{2n}^{(2p)}(t_{k,2n})}{P'_{2n}(t_{k,2n})} t_{k,2n}^{-2s+2p-1} \\
 &:= \xi'_{n,2}(s; x_{k,n}) (-1)^s \beta_{2n}^s(2n, k) \left(\frac{n}{\sqrt{a_n}} \right)^{2s} p'_n(x_{k,n}) x_{k,n}^{-s}.
 \end{aligned} \tag{3.34}$$

Here, one has from Lemma 3.8 and (3.28) that

$$\begin{aligned}
 \left| \xi'_{n,2}(s; x_{k,n}) \right| &= \left| \sum_{p=1}^s 2c_{2s+1,2p} (-1)^s \beta_{2n}^{-s}(2n, k) \left(\frac{n}{a_{2n}^*} \right)^{-2s} \frac{P_{2n}^{(2p)}(t_{k,2n})}{P'_{2n}(t_{k,2n})} t_{k,2n}^{-2s+2p-1} \right| \\
 &\leq C \sum_{p=1}^s \left(\frac{a_{2n}^*}{n} \frac{1}{t_{k,2n}} \right)^{2s-2p+1} \leq C\sqrt{\varepsilon}.
 \end{aligned} \tag{3.35}$$

Finally, if we let $\rho_s(\varepsilon, x_{kn}, n) := \xi'_{n,1}(s; x_{k,n}) - \xi'_{n,2}(s; x_{k,n})$, then the result is proved. \square

4. Proofs of Theorems 1.8, 1.9, and 1.10

Lemma 4.1. Let $W(t) \in \mathcal{F}(C^2)$, and let $0 < p < \infty$ and $\rho^* > -1/2$. Then, one has for $n \geq 1$ that

$$\begin{aligned}
 \left\| P_{n,\rho^*}(t) W(t) \left(|t| + \frac{a_n^*}{n} \right)^{\rho^*} \right\|_{L_p(\mathbb{R})} &\sim \left\| P_{n,\rho^*}(t) W(t) \left(|t| + \frac{a_n^*}{n} \right)^{\rho^*} \right\|_{L_p(a_n^*/2 \leq |t| \leq 2a_n^*)} \\
 &\sim a_n^{*1/p-1/2} \begin{cases} 1, & p < 4, \\ \{\log(1 + n\Gamma^*(a_n^*))\}^{1/4}, & p = 4, \\ \{n\Gamma^*(a_n^*)\}^{2/3(1/4-1/p)}, & p > 4. \end{cases} \tag{4.1}
 \end{aligned}$$

Proof. In [21, theorem 2.6] we showed that

$$\left\| P_{n,\rho^*}(t)W(t)\left(|t| + \frac{a_n^*}{n}\right)^{\rho^*} \right\|_{L_p(\mathbb{R})} \sim a_n^{*1/p-1/2} \begin{cases} 1, & p < 4, \\ \{\log(1 + nT^*(a_n^*))\}^{1/4}, & p = 4, \\ \{nT^*(a_n^*)\}^{2/3(1/4-1/p)}, & p > 4. \end{cases} \quad (4.2)$$

But, seeing our proof of [21, Theorem 2.6] carefully, we can easily prove the first equivalence. \square

Lemma 4.2 (see [21, Theorem 2.4]). *Let $W(t) \in \mathcal{F}(C^2)$, $0 < p \leq \infty$ and $L \geq 0$. Let $\beta \in \mathbb{R}$. Then, given $r > 1$, there exists a positive constant C_2 such that one has for any polynomial $P \in \mathcal{D}_n$ that*

$$\|(PW_\beta)(t)\|_{L_p(a_n^* \leq |t|)} \leq \exp(-C_2 n^\alpha) \|(PW_\beta)(t)\|_{L_p(L(a_n^*/n) \leq |t| \leq a_n^*(1-L/n))}. \quad (4.3)$$

Proof of Theorem 1.8. From Theorem 2.4 and Lemmas 4.2 and 4.1, one has

$$\begin{aligned} & \left\| \Theta_n^{1/4}(x)p_{n,\rho}(x)w(x)\left(x + \frac{a_n}{n^2}\right)^\rho \right\|_{L_p(\mathbb{R}^+)} \\ & \leq C \left\| \left(\frac{t^2}{a_{2n}^{*2}}\right)^{1/4} P_{2n,\rho^*}(t)W(t)\left(t^2 + \frac{a_{2n}^{*2}}{n^2}\right)^\rho |t|^{1/p} \right\|_{L_p(\mathbb{R})} \\ & \leq C \left(\frac{1}{a_{2n}^*}\right)^{1/2} \left\| P_{2n,\rho^*}(t)W(t)\left(|t| + \frac{a_{2n}^*}{n}\right)^{\rho^*} |t|^{1/p} \right\|_{L_p(|t| \leq 2a_{2n}^*)} \\ & \leq C a_{2n}^{*1/p-1/2} \left\| P_{2n,\rho^*}(t)W(t)\left(|t| + \frac{a_{2n}^*}{n}\right)^{\rho^*} \right\|_{L_p(\mathbb{R})} \\ & \sim a_{2n}^{*2/p-1} \begin{cases} 1, & p < 4, \\ \{\log(1 + nT^*(a_{2n}^*))\}^{1/4}, & p = 4, \\ \{nT^*(a_{2n}^*)\}^{2/3(1/4-1/p)}, & p > 4. \end{cases} \end{aligned} \quad (4.4)$$

On the other hand, one has by Theorem 2.4 and Lemma 4.1 that

$$\begin{aligned}
& \left\| \Theta_n^{1/4}(x) p_{n,\rho}(x) w(x) \left(x + \frac{a_n}{n^2} \right)^\rho \right\|_{L_p(\mathbb{R}^+)} \\
& \geq \left\| \Theta_n^{1/4}(t^2) P_{2n,\rho^*}(t) W(t) \left(t^2 + \frac{a_{2n}^{*2}}{n^2} \right)^\rho |t|^{1/p} \right\|_{L_p(a_{2n}^*/2 \leq |t| \leq 2a_{2n}^*)} \\
& \sim a_{2n}^{*1/p-1/2} \left\| P_{2n,\rho^*}(t) W(t) \left(|t| + \frac{a_{2n}^*}{n} \right)^\rho \right\|_{L_p(a_{2n}^*/2 \leq |t| \leq 2a_{2n}^*)} \\
& \sim a_{2n}^{*2/p-1} \begin{cases} 1, & p < 4, \\ \{\log(1 + nT^*(a_{2n}^*))\}^{1/4}, & p = 4, \\ \{nT^*(a_{2n}^*)\}^{2/3(1/4-1/p)}, & p > 4. \end{cases} \tag{4.5}
\end{aligned}$$

Consequently, using Lemma 3.1, one has the result. \square

Lemma 4.3. Let $\rho > -1/2$, and let $w(x) \in \mathcal{L}(C^2+)$. Then, uniformly for $n \geq 1$ and $1 \leq j \leq n$, one has the following:

(a)

$$\sup_{x \in I} |p_{n,\rho}(x) w(x)| \left(x + \frac{a_n}{n^2} \right)^\rho \left| (x + a_n n^{-2})(a_n - x) \right|^{1/4} \sim 1, \tag{4.6}$$

(b) for $j \leq n - 1$ and $x \in [x_{j+1,n}, x_{jn}]$,

$$|p_{n,\rho}(x) w(x)| \left(x + \frac{a_n}{n^2} \right)^\rho \sim \min\{|x - x_{jn}|, |x - x_{j+1,n}|\} \varphi_n(x_{jn})^{-1} [x_{jn}(a_n - x_{jn})]^{-1/4}, \tag{4.7}$$

(c) for $1 \leq j \leq n - 1$,

$$x_{jn} - x_{j+1,n} \sim \varphi_n(x_{jn}), \tag{4.8}$$

where

$$\varphi_u(x) = \begin{cases} \frac{\sqrt{x + a_u u^{-2}}(a_{2u} - x)}{u\sqrt{a_u - x + a_u \eta_u}}, & 0 \leq x \leq a_u, \\ \varphi_u(a_u), & a_u < x. \end{cases} \tag{4.9}$$

Proof. (a) It is from [1, Theorem 1.2]. (b) It is from [2, Theorem 1.3]. (c) It is from [2, Theorem 1.4]. \square

Proof of Theorem 1.9. By Theorem 1.8, one has for $0 < p \leq 4$,

$$\begin{aligned} \left\| p_{n,\rho}(x)\omega(x)\left(x + \frac{a_n}{n^2}\right)^\rho \right\|_{L_p(\mathbb{R}^+)} &\geq C \left\| \Theta_n^{1/4}(x)p_{n,\rho}(x)\omega(x)\left(x + \frac{a_n}{n^2}\right)^\rho \right\|_{L_p(\mathbb{R}^+)} \\ &\sim a_n^{1/p-1/2} \begin{cases} 1, & p < 4, \\ \{\log(n\Gamma(a_n))\}^{1/4}, & p = 4. \end{cases} \end{aligned} \tag{4.10}$$

For $p > 4$, we know by (4.7) and (4.8) that

$$\begin{aligned} &\int_{x_m}^{a_n/3} \left| p_{n,\rho}(x)\omega(x)\left(x + \frac{a_n}{n^2}\right)^\rho \right|^p dx \\ &\geq C \sum_{x_m \leq x_{jn} \leq a_n/3} \int_{x_{j+1,n}}^{x_{jn}} |x - x_{j+1,n}|^p dx \varphi_n^{-p}(x_{jn}) [x_{jn}(a_n - x_{jn})]^{-p/4} \\ &\sim a_n^{-p/4} \sum_{x_m \leq x_{jn} \leq a_n/3} \varphi_n(x_{jn}) x_{jn}^{-p/4} \\ &\sim a_n^{-p/4} \int_{a_n n^{-2}}^{a_n/3} t^{-p/4} dt \\ &\sim a_n^{-p/2+1} n^{2(p/4-1)}. \end{aligned} \tag{4.11}$$

Then, for $p > 4$

$$\left\| p_{n,\rho}(x)\omega(x)\left(x + \frac{a_n}{n^2}\right)^\rho \right\|_{L_p(\mathbb{R}^+)} \geq C a_n^{-1/2+1/p} n^{2(1/4-1/p)}. \tag{4.12}$$

Therefore, one has

$$\left\| p_{n,\rho}(x)\omega(x)\left(x + \frac{a_n}{n^2}\right)^\rho \right\|_{L_p(\mathbb{R}^+)} \geq C a_n^{1/p-1/2} \begin{cases} 1, & p < 4, \\ \{\log(n\Gamma(a_n))\}^{1/4}, & p = 4, \\ n^{2(1/4-1/p)}, & p > 4. \end{cases} \tag{4.13}$$

On the other hand, one has from Theorem 1.8 that

$$\begin{aligned} \left\| p_{n,\rho}(x)w(x)\left(x + \frac{a_n}{n^2}\right)^\rho \right\|_{L_p(x \geq a_n/3)} &\sim \left\| \Theta_n^{1/4}(x)p_{n,\rho}(x)w(x)\left(x + \frac{a_n}{n^2}\right)^\rho \right\|_{L_p(x \geq a_n/3)} \\ &\leq \left\| \Theta_n^{1/4}(x)p_{n,\rho}(x)w(x)\left(x + \frac{a_n}{n^2}\right)^\rho \right\|_{L_p(\mathbb{R}^+)} \\ &\sim a_n^{1/p-1/2} \begin{cases} 1, & p < 4, \\ \{\log(nT(a_n))\}^{1/4}, & p = 4, \\ \{nT(a_n)\}^{2/3(1/4-1/p)}, & p > 4, \end{cases} \end{aligned} \quad (4.14)$$

and by (4.6) that

$$\begin{aligned} \left\| p_{n,\rho}(x)w(x)\left(x + \frac{a_n}{n^2}\right)^\rho \right\|_{L_p(x \leq a_n/3)} &\leq Ca_n^{-1/4} \left(\int_0^{a_n/3} \left(x + \frac{a_n}{n^2}\right)^{-p/4} dx \right)^{1/p} \\ &\sim a_n^{1/p-1/2} \begin{cases} 1, & p < 4, \\ \{\log a_n\}^{1/4}, & p = 4, \\ n^{2(1/4-1/p)}, & p > 4. \end{cases} \end{aligned} \quad (4.15)$$

Therefore, one has

$$\left\| p_{n,\rho}(x)w(x)\left(x + \frac{a_n}{n^2}\right)^\rho \right\|_{L_p(\mathbb{R}^+)} \sim a_n^{1/p-1/2} \begin{cases} 1, & p < 4, \\ \{\log(nT(a_n))\}^{1/4}, & p = 4, \\ n^{2(1/4-1/p)}, & p > 4. \end{cases} \quad (4.16)$$

From Remark 3.5(iii), we see that $nT(a_n) < Cn^3$. So, consequently, one has the result. \square

Let

$$\Phi_n^*(t) = \max \left\{ \eta_n^* \left(1 - \frac{|t|}{a_n^*} \right), \eta_n^* \right\}, \quad \eta_n^* = \{nT^*(a_n^*)\}^{-2/3}. \quad (4.17)$$

Then, we obtain by Lemma 3.1 that

$$\Phi_n(x) \sim \Phi_{2n}^*(t), \quad x = t^2. \quad (4.18)$$

Lemma 4.4 (see [21, Theorem 2.7]). *Let $W_{\rho^*} \in \mathcal{F}(C^2)$ and $\rho^* > -1/2$. For $0 \leq s \leq r$ and $n \geq 1$, one has*

$$\begin{aligned} & \left\| \left| \Phi_n^*(t)^{(r/4-1/p)^+} \right| P_{n,\rho^*}(t) W(t) \left(|t| + \frac{a_n^*}{n} \right)^{\rho^*} \right|^s \Big\|_{L_p(\mathbb{R})} \\ & \sim \left\| \left| \Phi_n^*(t)^{(r/4-1/p)^+} \right| P_{n,\rho^*}(t) W(t) \left(|t| + \frac{a_n^*}{n} \right)^{\rho^*} \right|^s \Big\|_{L_p(a_n^*/2 \leq |t| \leq 2a_n^*)} \\ & \sim \begin{cases} a_n^{*1/p-r/2} \log n, & s = r, 4 \leq pr < \infty, \\ a_n^{*1/p-s/2}, & \text{otherwise.} \end{cases} \end{aligned} \tag{4.19}$$

Proof of Theorem 1.10. By Theorem 2.4, we can transform $p_{n,\rho}(x)$ on \mathbb{R}^+ to $P_{2n,\rho^*}(t)$ on \mathbb{R} .

$$\begin{aligned} & \left\| \Theta_n^{r/4}(x) \Phi_n^{*(r/4-1/p)^+}(x) \left| p_n \omega(x) \left(x + \frac{a_n}{n^2} \right)^\rho \right|^s \right\|_{L_p(\mathbb{R}^+)} \\ & = \left\| \Theta_n^{r/4}(t^2) \Phi_{2n}^{*(r/4-1/p)^+}(t) \left| P_{2n} W(t) \left(t^2 + \frac{a_{2n}^*}{n^2} \right)^\rho \right|^s (2|t|)^{1/p} \right\|_{L_p(\mathbb{R})}. \end{aligned} \tag{4.20}$$

Using Lemma 4.4 and noting (3.1), one has

$$\begin{aligned} & \left\| \Theta_n^{r/4}(t^2) \Phi_{2n}^{*(r/4-1/p)^+}(t) \left| P_{2n} W(t) \left(t^2 + \frac{a_{2n}^*}{n^2} \right)^\rho \right|^s (2|t|)^{1/p} \right\|_{L_p(\mathbb{R})} \\ & \geq \left\| \Theta_n^{r/4}(t^2) \Phi_{2n}^{*(r/4-1/p)^+}(t) \left| P_{2n} W(t) \left(t^2 + \frac{a_{2n}^*}{n^2} \right)^\rho \right|^s (2|t|)^{1/p} \right\|_{L_p(a_{2n}^*/2 \leq |t| \leq 2a_{2n}^*)} \\ & \geq C \left\| \Phi_{2n}^{*(r/4-1/p)^+}(t) \left| P_{2n} W(t) \left(|t| + \frac{a_{2n}^*}{n^2} \right)^{\rho^*} \right|^s |t|^{1/p-s/2} \right\|_{L_p(a_{2n}^*/2 \leq |t| \leq 2a_{2n}^*)} \\ & \sim a_{2n}^{*1/p-s/2} \left\| \Phi_{2n}^{*(r/4-1/p)^+}(t) \left| P_{2n} W(t) \left(|t| + \frac{a_{2n}^*}{n^2} \right)^{\rho^*} \right|^s \right\|_{L_p(a_{2n}^*/2 \leq |t| \leq 2a_{2n}^*)} \\ & \sim \begin{cases} a_{2n}^{*2/p-r} \log n, & \text{if } s = r, 4 \leq pr < \infty, \\ a_{2n}^{*2/p-s}, & \text{otherwise.} \end{cases} \end{aligned} \tag{4.21}$$

On the other hand, by Lemma 4.2, we see

$$\left\| \Theta_n^{r/4}(x) \Phi_{2n}^{*(r/4-1/p)^+}(t) \left| P_{2n}(t) W(t) \left(t^2 + \frac{a_{2n}^*}{n^2} \right)^\rho \right|^s (2t)^{1/p} \right\|_{L_p(a_{2n}^* \leq |t|)} \leq C n_{2n}^{(r/4-1/p)^+} e^{-cn}, \tag{4.22}$$

where $c > 0$ is a constant. Therefore, using Lemma 4.4 and noting (3.1) and the definition of Θ_n , one has

$$\begin{aligned}
& \left\| \Theta_n^{r/4} \left(t^2 \right) \Phi_{2n}^{*(r/4-1/p)^+} (t) \left| P_{2n} W(t) \left(t^2 + \frac{a_{2n}^{*2}}{n^2} \right)^\rho \right|^s (2|t|)^{1/p} \right\|_{L_p(\mathbb{R})} \\
& \leq C a_{2n}^{*-r/2} \left\| \Phi_{2n}^{*(r/4-1/p)^+} (t) \left| P_{2n} W(t) \left(|t| + \frac{a_{2n}^*}{n} \right)^{\rho^*} \right|^s |t|^{1/p+r/2-s/2} \right\|_{L_p(|t| \leq 2a_{2n}^*)} \\
& \leq C a_{2n}^{*1/p-s/2} \left\| \Phi_{2n}^{*(r/4-1/p)^+} (t) \left| P_{2n} W(t) \left(|t| + \frac{a_{2n}^*}{n} \right)^{\rho^*} \right|^s \right\|_{L_p(|t| \leq 2a_{2n}^*)} \quad (4.23) \\
& \leq C a_{2n}^{*1/p-s/2} \left\| \Phi_{2n}^{*(r/4-1/p)^+} (t) \left| P_{2n} W(t) \left(|t| + \frac{a_{2n}^*}{n} \right)^{\rho^*} \right|^s \right\|_{L_p(\mathbb{R})} \\
& \sim \begin{cases} a_{2n}^{*2/p-r} \log n, & \text{if } s = r, 4 \leq pr < \infty, \\ a_{2n}^{*2/p-s}, & \text{otherwise.} \end{cases}
\end{aligned}$$

Therefore, one has

$$\begin{aligned}
& \left\| \Theta_n^{r/4} (x) \Phi_n^{(r/4-1/p)^+} (x) \left| p_n \omega(x) \left(x + \frac{a_n}{n^2} \right)^\rho \right|^s \right\|_{L_p(\mathbb{R}^+)} \\
& \sim \begin{cases} a_{2n}^{*2/p-r} \log n, & \text{if } s = r, 4 \leq pr < \infty, \\ a_{2n}^{*2/p-s}, & \text{otherwise,} \end{cases} \quad (4.24) \\
& \sim \begin{cases} a_n^{1/p-r/2} \log n, & \text{if } s = r, 4 \leq pr < \infty, \\ a_n^{1/p-s/2}, & \text{otherwise.} \end{cases}
\end{aligned}$$

□

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