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Research Article

A Sharp Double Inequality for Sums of Powers

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It is established that the sequences $n \mapsto S(n) := \sum_{k=1}^{n}(k/n)^n$ and $n \mapsto n(e/(e-1)-S(n))$ are strictly increasing and converge to $e/(e-1)$ and $e(e+1)/2(e-1)^3$, respectively. It is shown that there holds the sharp double inequality $(1/(e-1))\cdot(1/n) \leqq e/(e-1)-S(n) < (e(e+1)/2(e-1)^3)\cdot(1/n)$, $(n \in \mathbb{N})$.

1. Introduction

The proof of the equality

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^n = \frac{e}{e-1}, \tag{1.1a}$$

published recently in the form [1]

$$\lim_{n\to\infty}\sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^n = \frac{1}{e-1}, \tag{1.1b}$$

was based on the equations $n^{1-k}\cdot n(n-1)\cdots(n-k+2) = (1-1/n)(1-2/n)\cdots(1-(k-2)/n) = 1+O(1/n)$ with the false hypothesis that big O is independent of k (see [1, pages 63-64] and [2, pages 54-55]). Deriving (1.1b) the author used the Euler-Maclaurin summation formula and a generating function for the Bernoulli numbers.

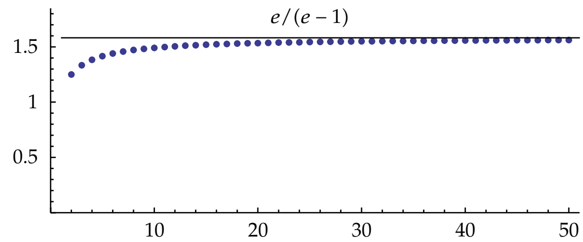

Figure 1: The graph of the sequence $n \mapsto S(n) \equiv \sum_{k=1}^{n} (k/n)^n$.

Subsequently, Spivey published the correction of his demonstration as the Letter to the Editor [2]. Additionally, Holland [3] published two different derivations of (1.1a) in the same issue as Spivey's correction appeared.

In this note, using only elementary techniques, we demonstrate that the sequence $S(n)$ is strictly increasing and that (1.1a) holds; in addition, we establish a sharp estimate of the rate of convergence.

2. Monotone Convergence

The formula (1.1a) is illustrated in Figure 1, where the sequence $n \mapsto S(n) := \sum_{k=1}^{n} (k/n)^n$ is depicted. Its monotonicity is seen very clearly.

To prove that the sequence $(S_n)_{n\in\mathbb{N}}$ is strictly increasing, we change the order of summation

$$S(n) \equiv \sum_{k=1}^{n} \left(\frac{k}{n}\right)^n \equiv \sum_{j=0}^{n} \left(\frac{n-j}{n}\right)^n \equiv 1 + \sum_{j=1}^{n} \left(1 + \frac{-j}{n}\right)^n. \tag{2.1}$$

Now, consider the function $t \mapsto E(x,t) := (1 + x/t)^t$ which is, for $x \neq 0$, strictly increasing on the open interval $(-\min\{0,x\}, \infty)$ and $\lim_{t\to\infty} E(x,t) = \sup_{t>|x|} E(x,t) = e^x$, for any $x \in \mathbb{R}$ [4, page 42]. Consequently, the sequence $(S(n))_{n\in\mathbb{N}}$ is strictly increasing. We use Tannery's theorem for series (see [5] or [6, item 49, page 136]) to determine its limit.

Lemma 2.1 (Tannery). *Let a double sequence $(j,n) \mapsto z_j(n)$ of complex numbers satisfy the following conditions:*

(1) *The finite limit $z_\infty(j) := \lim_{n\to\infty} z_n(j)$ exists for every fixed $j \in \mathbb{N}$.*

(2) *There exists a sequence of positive constants $M_1, M_2, M_3, \ldots$ such that $|z_n(j)| \leq M_j$ for every $(j,n) \in \mathbb{N} \times \mathbb{N}$ satisfying the estimate $j \leq n$, and the series $\sum_{j=1}^{\infty} M_j$ converges. (In [6, item 49, page 136], we have the stronger supposition that $|z_n(j)| \leq M_j$ for all $(j,n) \in \mathbb{N} \times \mathbb{N}$.)*

Then we have

$$\lim_{n\to\infty} \sum_{j=1}^{n} z_n(j) = \sum_{j=1}^{\infty} z_\infty(j). \tag{2.2}$$

Proof. Let all the conditions of the Lemma be satisfied and $\varepsilon \in \mathbb{R}^+$ be given. Then we estimate $|z_\infty(j)| \le M_j$ for $j \in \mathbb{N}$ and $\sum_{j=m_\varepsilon+1}^{\infty} M_j < \varepsilon/3$ for some $m_\varepsilon \in \mathbb{N}$. Moreover, for any $j \in \{1,\ldots,m_\varepsilon\}$, also $|z_\infty(j) - z_n(j)| < \varepsilon/(3m_\varepsilon)$ for $n \ge n_\varepsilon(j)$ at some $n_\varepsilon(j) \in \mathbb{N}$. Thus, for $n \ge n_\varepsilon := \max_{1\le j\le m_\varepsilon} n_\varepsilon(j)$, we estimate

$$\left|\sum_{j=1}^{\infty} z_\infty(j) - \sum_{j=1}^{n} z_n(j)\right| \le \sum_{j=1}^{m_\varepsilon} |z_\infty(j) - z_n(j)| + \sum_{j=m_\varepsilon+1}^{\infty} |z_\infty(j)| + \sum_{j=m_\varepsilon+1}^{n} |z_n(j)|$$
$$< m_\varepsilon \cdot \frac{\varepsilon}{3m_\varepsilon} + \sum_{j=m_\varepsilon+1}^{\infty} M_j + \sum_{j=m_\varepsilon+1}^{n} M_j < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \tag{2.3}$$

□

Now, using (2.1) and putting $z_n(j) = (1 + -j/n)^n$ and $z_\infty(j) = e^{-j}$ into Tannery's Lemma, we obtain

$$\lim_{n\to\infty} S(n) = 1 + \sum_{j=1}^{\infty} e^{-j} = \frac{e}{e-1}. \tag{2.4}$$

3. The Rate of Convergence

Referring to Figure 1, the convergence of the sequence $(S(n))_{n\in\mathbb{N}}$ appears to be rather slow. The difference

$$\Delta(n) := \frac{e}{e-1} - S(n) \tag{3.1}$$

determines the sequence $n \mapsto n\Delta(n)$. Its graph, shown in Figure 2, suggests it is monotonic increasing, which we will prove first.

Indeed, according to (3.1) and (2.1), we have

$$\Delta(n) = \sum_{j=0}^{\infty} e^{-j} - \sum_{j=0}^{n}\left(1 - \frac{j}{n}\right)^n$$
$$= \sum_{j=1}^{n} f_n(j) + \sum_{j=n+1}^{\infty} e^{-j}$$
$$= \sum_{j=1}^{n} f_n(j) + \frac{e^{-n}}{e-1}, \tag{3.2}$$

where

$$f_n(x) := e^{-x} - \left(1 - \frac{x}{n}\right)^n \quad (x \in \mathbb{R}) \tag{3.3}$$

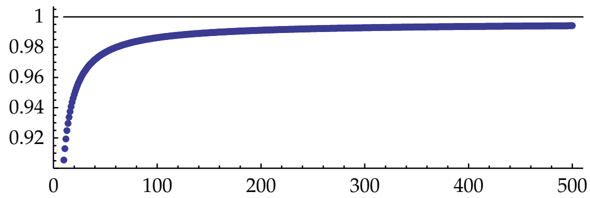

Figure 2: The graph of the sequence $n \mapsto n\Delta(n)$.

and, for $x \neq 0$, the sequence $n \mapsto f_n(x)$ is strictly decreasing and converges to zero [4, (4)]. Thus, we have

$$n\Delta(n) = \sum_{j=1}^{n} g_n(j) + n\,\frac{e^{-n}}{e-1} = \sum_{j=1}^{n-1} g_n(j) + Cne^{-n} \tag{3.4}$$

with

$$g_n(x) := nf_n(x), \qquad C = \frac{e}{e-1}. \tag{3.5}$$

To examine the monotonicity of the sequence $n \mapsto n\Delta(n)$, we study, using (3.3), (3.4) and (3.5), the difference $(n+1)\Delta(n+1) - n\Delta(n)$, which is equal to

$$\begin{aligned}
&\left(\sum_{j=1}^{n-1} g_{n+1}(j) + g_{n+1}(n)\right) + C\cdot(n+1)e^{-n-1} - \sum_{j=1}^{n-1} g_n(j) - Cne^{-n} \\
&\quad= \sum_{j=1}^{n-1}\left(g_{n+1}(j) - g_n(j)\right) + (n+1)e^{-n} - \frac{1}{(n+1)^n} + \frac{n+1}{e-1}e^{-n} - \frac{en}{e-1}e^{-n} \\
&\quad= \sum_{j=1}^{n-1}\left((n+1)f_{n+1}(j) - nf_n(j)\right) + \left(\frac{e}{e-1}e^{-n} - (n+1)^{-n}\right) \\
&\quad> \sum_{j=1}^{n-1}\left(nf_{n+1}(j) - nf_n(j)\right) + 0 > 0.
\end{aligned} \tag{3.6}$$

Hence:

$$\text{The sequence } n \longmapsto n\Delta(n) \text{ is strictly increasing.} \tag{3.7}$$

Next, we examine also the question of convergence of the above sequence. First, referring to (3.3), (3.5), and [4, page 29, equation (16)], there exists the limit

$$g_\infty(j) := \lim_{n\to\infty} g_n(j) = \frac{e^{-j}j^2}{2} \quad (j \in \mathbb{N}). \tag{3.8}$$

Moreover, according to (3.3), (3.5), and [4, (15)], the estimates

$$g_n(j) < \frac{e^{-j}j^2}{2} \cdot \frac{n}{n-j} \leq \frac{e^{-j}j^2}{2} \cdot (1+j) \tag{3.9}$$

hold true for $j \leq n-1$. Additionally, $g_n(n) = ne^{-n}$, due to (3.3) and (3.5). Thus, the estimate

$$g_n(j) \leq M_j := \frac{(j+1)j^2}{2} \cdot e^{-j} \tag{3.10}$$

is being valid for $n \in \mathbb{N}$ and $j \leq n$ with

$$\sum_{j=1}^{\infty} M_j = \sum_{j=1}^{\infty} \frac{(j+1)j^2}{2} \cdot e^{-j} < \infty. \tag{3.11}$$

According to (3.8) and differentiating the appropriate power series resulting from the geometric series, we obtain

$$\sum_{j=1}^{\infty} g_\infty(j) = \sum_{j=1}^{\infty} \frac{e^{-j}j^2}{2} = \frac{e^2+e}{2(e-1)^3}. \tag{3.12}$$

Now, referring to (3.4) and (3.8)–(3.12), and applying Tannery's Lemma—equation (2.2), with $z_n(j) \equiv g_n(j)$, we obtain the result

$$\lim_{n\to\infty} n\Delta(n) = \sum_{j=1}^{\infty} g_\infty(j) + 0 = \frac{e(e+1)}{2(e-1)^3}. \tag{3.13}$$

Therefore, using (3.1) and (3.7), we find the following sharp inequality

$$\frac{e}{e-1} - S(n) < \frac{e(e+1)}{2(e-1)^3} \cdot \frac{1}{n}, \tag{3.14}$$

true for every $n \in \mathbb{N}$. In addition, we have also the estimate

$$\frac{e}{e-1} - S(n) \geq m\left(\frac{e}{e-1} - S(m)\right) \cdot \frac{1}{n}, \tag{3.15}$$

valid for every $m, n \in \mathbb{N}$ such that $n \geq m$.

We have $e(e+1)/2(e-1)^3 = 0.996147\ldots$, and for the function $P(m) := m\Delta(m) = m(e/(e-1) - S(m))$ we calculate $P(1) = 0.581976\ldots$, and $P(999) = 0.995149\ldots$. This way we obtain simple and rather accurate estimates

$$\begin{aligned} 0.581 \cdot \frac{1}{n} < \frac{e}{e-1} - S(n) < 0.996 \cdot \frac{1}{n}, \quad \text{for } n \geq 1, \\ 0.995 \cdot \frac{1}{n} < \frac{e}{e-1} - S(n) < 0.997 \cdot \frac{1}{n}, \quad \text{for } n \geq 1000. \end{aligned} \tag{3.16}$$

Consequently, we get, for example, a simple double inequality

$$\frac{e}{e-1} - \frac{1}{n} < S(n) < \frac{e}{e-1} - \frac{1}{2n}, \quad \text{for } n \geq 1. \tag{3.17}$$

Open Question. Are the sequences $n \mapsto S(n)$ and $n \mapsto n\Delta(n)$ strictly concave?

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