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Research Article

Some Weighted Hardy-Type Inequalities on Anisotropic Heisenberg Groups

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We prove some weighted Hardy type inequalities associated with a class of nonisotropic Greiner-type vector fields on anisotropic Heisenberg groups. As an application, we get some new Hardy type inequalities on anisotropic Heisenberg groups which generalize a result of Yongyang Jin and Yazhou Han.

1. Introduction

The Hardy inequality in $\mathbb{R}^N$ states that, for all $u \in C_0^\infty(\mathbb{R}^N)$ and $N \geq 3$,

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx. \tag{1.1}$$

In the case of the Heisenberg group $\mathbb{H}_n$, Garofalo and Lanconelli (cf. [1]) firstly proved the following Hardy inequality:

$$\int_{\mathbb{H}^n} |\nabla_H u|^2 \geq \frac{(Q-2)^2}{4} \int_{\mathbb{H}^n} \frac{u^2}{d^2} |\nabla_H d|^2, \quad u \in C_0^\infty(\mathbb{H}^n \setminus \{e\}), \tag{1.2}$$

where e is the neutral element of $\mathbb{H}^n$, $d = (|z|^4 + t^2)^{1/4}$ is the Korányi-Folland nonisotropic gauge induced by the fundamental solution, and $Q = 2n + 2$ is the homogenous dimension of $\mathbb{H}^n$ (see also [2]). Inequality (1.2) was generalized by Niu et al. [3] (see also [4]) using the

Picone-type identify. For more Hardy-Sobolev inequalities on nilpotent groups, we refer the reader to [5–19].

More recently, Jin and Han (cf. [20, 21]), using the method by Niu et al. [3], have proved the following Hardy inequalities on anisotropic Heisenberg groups $\mathbb{H}_a^n$:

$$\int_{\mathbb{H}_a^n} |\nabla_L u|^p \geq \left(\frac{2\sum_{j=1}^n a_j + 2 - p}{p}\right)^p \int_{\mathbb{H}_a^n} \frac{\left(\sum_{j=1}^n a_j^2 |z_j|^2\right)^{p/2} \left(\sum_{j=1}^n a_j |z_j|^2\right)^{(k-1)p}}{N(z,t)^{2kp}} |u|^p, \tag{1.3}$$

where ∇_L are the nonisotropic Greiner-type vector fields, k is a positive integer,

$$N(z,t)^{4k} = \left(\sum_{j=1}^n a_j |z_j|^2\right)^{2k} + t^2, \tag{1.4}$$

and $2 \leq p < 2\sum_{j=1}^n a_j + 2$. However, the inequalities above do not cover the case of $1 < p < 2$ and $2\sum_{j=1}^n a_j + 2k \leq p < 2n + 2k$. So, it is an interesting problem to study a Hardy-type inequality related to $N(z,t)$ for $1 < p < 2$ on $\mathbb{H}_a^n$ and $2\sum_{j=1}^n a_j + 2k \leq p < 2n + 2k$. In this note, we will consider some Hardy inequalities on $\mathbb{H}_a^n$ for $1 < p < 2n + 2k$. In fact, we prove a representation formula associated with $N(z,t)$, which is analogous to the Korányi-Folland nonisotropic gauge on Heisenberg group (cf. [22]). Using this representation formula, we prove some new Hardy inequalities on $\mathbb{H}_a^n$, which include the case of $1 < p < 2$ and $2\sum_{j=1}^n a_j + 2k \leq p < 2n + 2k$.

This paper is organized as follows. We start in Section 2 with the necessary background on anisotropic Heisenberg groups $\mathbb{H}_a^n$. In Section 3, we prove a representation formula and use it to obtain some Hardy-type inequalities.

2. Notations and Preliminaries

Recall that the anisotropic Heisenberg groups $\mathbb{H}_a^n$ are the Carnot group of step two whose group structure is given by (cf. [23])

$$(z,t) \circ (z',t') = \left(z + z', t + t' + 2\sum_{j=1}^n a_j z\overline{z'}\right), \tag{2.1}$$

where $z = (z_1, \ldots, z_n)$, $z_j = x_j + iy_j$ $(x_j, y_j \in \mathbb{R})$, and $a_1, \ldots, a_n$ are positive constants, numbered so that

$$0 < a_1 \leq a_2 \leq \cdots \leq a_n. \tag{2.2}$$

We consider the following nonisotropic Greiner-type vector fields which are introduced by Jin and Han [21]:

$$X_j = \frac{\partial}{\partial x_j} + 2ka_j y_j \left(\sum_{j=1}^{n} a_j |z_j|^2 \right)^{k-1} \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2ka_j x_j \left(\sum_{j=1}^{n} a_j |z_j|^2 \right)^{k-1} \frac{\partial}{\partial t}, \tag{2.3}$$

$(j = 1, \ldots, n)$. These vector fields are not left or right invariant when $k \geq 2$. The horizontal gradient is the $(2n$-$)$ dimensional vector given by

$$\nabla_L = (X_1, \ldots, X_n, Y_1, \ldots, Y_n). \tag{2.4}$$

A natural family of anisotropic dilations related to ∇_L is

$$\delta_\lambda(z,t) = \left(\lambda z, \lambda^{2k} t\right). \tag{2.5}$$

For simplicity, we denote by $\lambda(z,t) = (\lambda z, \lambda^{2k} t)$. The Jacobian determinant of δ_λ is λ^Q, where $Q = 2n + 2k$ is the homogenous dimension. The anisotropic norm on $\mathbb{H}_a^n$ is

$$N(z,t) = \left(\left(\sum_{j=1}^{n} a_j |z_j|^2 \right)^{2k} + t^2 \right)^{1/4k}. \tag{2.6}$$

For simplicity, we use the notation $|z|^2 = \sum_{j=1}^{n} |z_j|^2$ and $|z|_a^2 = \sum_{j=1}^{n} a_j |z_j|^2$. Then,

$$N(z,t) = \left(|z|_a^{4k} + t^2 \right)^{1/4k}, \tag{2.7}$$

and $a_1 |z|^2 \leq |z|_a^2 \leq a_n |z|^2$. With this norm, we can define the metric ball centered at neutral element and with radius ρ by

$$B(e,\rho) = \left\{ (z,t) \in \mathbb{H}_a^n : N(z,t) < \rho \right\}, \tag{2.8}$$

and the unit sphere $\Sigma = \partial B(e,1)$. Furthermore, we have the following polar coordinates for all $f \in L^1(\mathbb{H}_a^n)$ (cf. [24]):

$$\int_{\mathbb{H}_a^n} f(z,t) dz\, dt = \int_0^\infty \int_\Sigma f(r(z^*, t^*)) r^{Q-1} d\sigma\, dr, \tag{2.9}$$

where $z^* = z/N(z,t)$ and $t^* = t/N^{2k}(z,t)$.

Let $\beta > -2n$ and set $C_\beta = \int_\Sigma |z^*|_a^\beta d\sigma$. We will explicitly calculate the constant C_β to show $C_\beta < \infty$ when $\beta > -2n$. The method of calculation is similar to that used in [22].

Lemma 2.1. *For $\beta > -2n$,*

$$C_\beta = \frac{\omega_{2n-1}\Gamma(1/2)\Gamma\big((\beta+Q-2k)/4k\big)}{\Gamma\big((\beta+Q)/4k\big)\prod_{j=1}^{n}a_j}, \tag{2.10}$$

where ω_{2n-1} is the volume of S^{2n-1}, that is, the unit sphere in $\mathbb{R}^{2n}$.

Proof. To compute C_β, let $\beta > -Q$, then,

$$\begin{aligned}\int_\Sigma |z^*|_a^\beta d\sigma &= (Q+\beta)\int_0^1 r^{\beta+Q-1}dr\int_\Sigma |z^*|_a^\beta d\sigma\\ &= (Q+\beta)\int_\Sigma\int_0^1 |rz^*|_a^\beta r^{Q-1}dr\,d\sigma\\ &= (Q+\beta)\int_{N(z,t)<1}|z|_a^\beta d\sigma.\end{aligned} \tag{2.11}$$

Next, if $\beta > -2n$,

$$\begin{aligned}\int_{N(z,t)<1}|z|_a^\beta d\sigma &= \int_{|t|<1}\int_{|z|_a<(1-|t|^2)^{1/4k}}|z|_a^\beta dz\,dt\\ &= \frac{1}{\prod_{j=1}^{n}a_j}\int_{|t|<1}\int_{|z|<(1-|t|^2)^{1/4k}}|z|^\beta dz\,dt.\end{aligned} \tag{2.12}$$

Therefore,

$$\begin{aligned}\int_{N(z,t)<1}|z|_a^\beta d\sigma &= \frac{\omega_{2n-1}}{\prod_{j=1}^{n}a_j}\int_{|t|<1}\int_0^{(1-|t|^2)^{1/4k}} r^{\beta+2n-1}dr\,dt\\ &= \frac{\omega_{2n-1}}{(2n+\beta)\prod_{j=1}^{n}a_j}\int_{|t|<1}\left(1-|t|^2\right)^{(\beta+2n)/4k}dt\\ &= \frac{\omega_{2n-1}}{(2n+\beta)\prod_{j=1}^{n}a_j}\int_0^1 (1-s)^{(\beta+2n)/4k}s^{-1/2}ds\\ &= \frac{\omega_{2n-1}}{(2n+\beta)\prod_{j=1}^{n}a_j}B\left(\frac{\beta+2n}{4k}+1,\frac{1}{2}\right)\\ &= \frac{\omega_{2n-1}}{(2n+\beta)\prod_{j=1}^{n}a_j}\cdot\frac{\Gamma\big((\beta+2n)/4k+1\big)\Gamma(1/2)}{\Gamma\big((\beta+Q)/4k+1\big)}.\end{aligned} \tag{2.13}$$

Thus, if $\beta > -2n$,

$$
\begin{aligned}
C_\beta &= (Q+\beta)\int_{N(z,t)<1} |z|_a^\beta d\sigma \\
&= \frac{\omega_{2n-1}\Gamma(1/2)\Gamma((\beta+2n)/4k)}{\Gamma((\beta+Q)/4k)\prod_{j=1}^n a_j} \\
&= \frac{\omega_{2n-1}\Gamma(1/2)\Gamma((\beta+Q-2k)/4k)}{\Gamma((\beta+Q)/4k)\prod_{j=1}^n a_j}.
\end{aligned}
\tag{2.14}
$$

□

3. Hardy-Type Inequality

Firstly, we prove the following representation formula on $\mathbb{H}_a^n$, which is of its independent interest.

Lemma 3.1. *Let $\beta > -2n + 2k - 1$ and $f \in C_0^\infty(\mathbb{H}_a^n)$. Then,*

$$
-C_\beta f(0) = \frac{1}{4k}\int_{\mathbb{H}_a^n} \frac{|z|_a^{\beta+2-4k}}{N(z,t)^{Q+\beta}} \left\langle \nabla_L f(z,t), \Lambda_a \nabla_L N(z,t)^{4k} \right\rangle dz\, dt, \tag{3.1}
$$

where Λ_a is a diagonal matrix given by

$$
\Lambda_a = \operatorname{diag}\left\{ \frac{1}{a_1}, \dots, \frac{1}{a_n}, \frac{1}{a_1}, \dots, \frac{1}{a_n} \right\}. \tag{3.2}
$$

Proof. We argue as in the proof of Theorem 1.2 in [22]. Since $f \in C_0^\infty(\mathbb{H}_a^n)$,

$$
\begin{aligned}
-f(0) &= \int_0^\infty \frac{d}{dr} f(r(z^*, t^*))dr \\
&= \int_0^\infty \sum_{j=1}^n \left(\frac{x_j}{r}\frac{\partial f}{\partial x_j}(r(z^*,t^*)) + \frac{y_j}{r}\frac{\partial f}{\partial y_j}(r(z^*,t^*)) \right) + \frac{2kt}{r}\frac{\partial f}{\partial t}(r(z^*,t^*))dr \\
&= \int_0^\infty \sum_{j=1}^n \left(\frac{x_j}{r}\frac{\partial f}{\partial x_j}(z,t) + \frac{y_j}{r}\frac{\partial f}{\partial y_j}(z,t) \right) + \frac{2kt}{r}\frac{\partial f}{\partial t}(z,t)dr \\
&= \int_0^\infty \sum_{j=1}^n \left(\frac{x_j}{r}\frac{\partial f}{\partial x_j}(z,t) + \frac{y_j}{r}\frac{\partial f}{\partial y_j}(z,t) + \frac{a_j x_j^2 + a_j y_j^2}{|z|_a^{2k}} \cdot \frac{2k|z|_a^{2k-2}t}{r}\frac{\partial f}{\partial t}(z,t) \right) dr.
\end{aligned}
\tag{3.3}
$$

Therefore,

$$
\begin{aligned}
-C_\beta f(0) &= -\left(\int_\Sigma |z^*|_a^\beta d\sigma\right) f(0) \\
&= \int_{\mathbb{H}_a^n} \frac{|z|_a^\beta}{N^{Q+\beta}} \sum_{j=1}^n \left(x_j \frac{\partial f}{\partial x_j} + y_j \frac{\partial f}{\partial y_j} + a_j \frac{x_j^2 + y_j^2}{|z|_a^{2k}} \cdot 2k|z|_a^{2k-2} t \frac{\partial f}{\partial t} \right) dz\, dt \\
&= \int_{\mathbb{H}_a^n} \frac{|z|_a^{\beta-2k}}{N^{Q+\beta}} \sum_{j=1}^n \left(X_j f \cdot \left(|z|_a^{2k} x_j + y_j t \right) + Y_j f \cdot \left(|z|_a^{2k} y_j - x_j t \right) \right) \\
&\quad - \int_{\mathbb{H}_a^n} \frac{t|z|_a^{\beta-2k}}{N^{Q+\beta}} \sum_{j=1}^n \left(y_j \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial y_j} \right) dz\, dt.
\end{aligned}
\tag{3.4}
$$

Notice that

$$
X_j N^{4k} = 4k a_j |z|_a^{2k-2}\left(|z|_a^{2k} x_j + y_j t\right), \qquad Y_j N^{4k} = 4k a_j |z|_a^{2k-2}\left(|z|_a^{2k} y_j - x_j t\right), \tag{3.5}
$$

we have, by (3.4),

$$
\begin{aligned}
-C_\beta f(0) =& \frac{1}{4k} \int_{\mathbb{H}_a^n} \frac{|z|_a^{\beta+2-4k}}{N(z,t)^{Q+\beta}} \left\langle \nabla_L f(z,t), \Lambda_a \nabla_L N(z,t)^{4k} \right\rangle dz\, dt \\
& - \int_{\mathbb{H}_a^n} \frac{t|z|_a^{\beta-2k}}{N^{Q+\beta}} \sum_{j=1}^n \left(y_j \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial y_j} \right) dz\, dt.
\end{aligned}
\tag{3.6}
$$

To finish the proof, it is enough to show that

$$
\int_{\mathbb{H}_a^n} \frac{t|z|_a^{\beta-2k}}{N^{Q+\beta}} \sum_{j=1}^n \left(y_j \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial y_j} \right) dz\, dt \tag{3.7}
$$

vanishes. Notice that the operator $y_j \partial_{x_j} - x_j \partial_{y_j}$ annihilates functions of $|z|_a$, and, for $\beta > -2n+2k-1$, the integrand above is absolutely integrable. We have, for any $\epsilon > 0$, though integration by parts,

$$
\int_{\mathbb{H}_a^n} \frac{t\left(|z|_a^2 + \epsilon\right)^{\beta/2-2}}{\left(N^4 + \epsilon\right)^{(Q+\beta)/4}} \sum_{j=1}^n \left(y_j \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial y_j} \right) dz\, dt = 0. \tag{3.8}
$$

Let $\epsilon \to 0$. By dominated convergence theorem,

$$
\int_{\mathbb{H}_a^n} \frac{t|z|_a^{\beta-2k}}{N^{Q+\beta}} \sum_{j=1}^n \left(y_j \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial y_j} \right) dz\, dt = 0. \tag{3.9}
$$

The proof is therefore completed. □

We now prove the following Hardy inequalities on $\mathbb{H}_a^n$.

Theorem 3.2. *Let* $1 < p < Q - \alpha$ *and* $\gamma > -2n - (p-1)(2k-1)$. *There holds, for all* $u \in C_0^\infty(\mathbb{H}_a^n)$,

$$\int_{\mathbb{H}_a^n} \frac{|\nabla_L u|^p}{N^\alpha} \frac{|z|_a^\gamma}{N^\gamma} \left(\frac{|z|}{|z|_a}\right)^p \geq \left(\frac{Q-p-\alpha}{p}\right)^p \int_{\mathbb{H}_a^n} \frac{|u|^p}{N^{\alpha+p}} \frac{|z|_a^{\gamma+p(2k-1)}}{N^{\gamma+p(2k-1)}}. \tag{3.10}$$

Proof. Set $u_\epsilon := (|u|^2+\epsilon^2)^{p/2}-\epsilon^p$ with $\epsilon > 0$. Replacing f by $u_\epsilon N^{Q-p-\alpha}$ in Lemma 3.1, we obtain, for any $\beta > -2n + 2k - 1$,

$$\begin{aligned} 0 = {} & \frac{1}{4k} \int_{\mathbb{H}_a^n} \frac{|z|_a^{\beta+2-4k}}{N(z,t)^{Q+\beta}} \left\langle \nabla_L u_\epsilon, \Lambda_a \nabla_L N^{4k} \right\rangle N^{Q-p-\alpha} \\ & + \frac{1}{4k} \int_{\mathbb{H}_a^n} \frac{|z|_a^{\beta+2-4k}}{N(z,t)^{Q+\beta}} \left\langle \nabla_L N^{Q-p-\alpha}, \Lambda_a \nabla_L N^{4k} \right\rangle u_\epsilon. \end{aligned} \tag{3.11}$$

It is easy to check that the following equations hold

$$\begin{aligned} \left\langle \nabla_L N^{4k}, \Lambda_a \nabla_L N^{4k} \right\rangle &= 16k^2 |z|_a^{4k-4} \sum_{j=1}^n a_j \left(\left(|z|_a^{2k} x_j + y_j t \right)^2 + \left(|z|_a^{2k} y_j - x_j t \right)^2 \right) \\ &= 16k^2 |z|_a^{4k-2} N^{4k}; \\ \left\langle \Lambda_a \nabla_L N^4, \Lambda_a \nabla_L N^4 \right\rangle &= 16k^2 |z|_a^{4k-4} \sum_{j=1}^n \left(\left(|z|_a^{2k} x_j + y_j t \right)^2 + \left(|z|_a^{2k} y_j - x_j t \right)^2 \right) \\ &= 16 N^{4k} |z|_a^{4k-4} |z|^2. \end{aligned} \tag{3.12}$$

Therefore, by (3.11),

$$\begin{aligned} & \frac{1}{4k} \int_{\mathbb{H}_a^n} \frac{|z|_a^{\beta+2-4k}}{N(z,t)^{Q+\beta}} \left\langle \nabla_L N^{Q-p-\alpha}, \Lambda_a \nabla_L N^{4k} \right\rangle u_\epsilon \\ & = \frac{(Q-p-\alpha)}{16k^2} \int_{\mathbb{H}_a^n} \frac{|z|_a^{\beta+2-4k} \langle \nabla_L N^{4k}, \Lambda_a \nabla_L N^{4k} \rangle}{N(z,t)^{p+\alpha+\beta-4k}} u_\epsilon \\ & = (Q-p-\alpha) \int_{\mathbb{H}_a^n} \frac{|z|_a^\beta}{N(z,t)^{p+\alpha+\beta}} u_\epsilon \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{4k}\int_{\mathbb{H}_a^n} \frac{|z|_a^{\beta+2-4k}}{N(z,t)^{Q+\beta}} \left\langle \nabla_L u_\epsilon, \Lambda_a \nabla_L N^{4k} \right\rangle N^{Q-p-\alpha} \\
&= -\frac{p}{4k}\int_{\mathbb{H}_a^n} \frac{|z|_a^{\beta-2}}{N(z,t)^{p+\alpha+\beta}} \left(|u|^2+\epsilon^2\right)^{(p-2)/2} u \left\langle \nabla_L u, \Lambda_a \nabla_L N^4 \right\rangle \\
&\le \frac{p}{4k}\int_{\mathbb{H}_a^n} \frac{|z|_a^{\beta+2-4k}}{N(z,t)^{p+\alpha+\beta}} \left(|u|^2+\epsilon^2\right)^{(p-2)/2} |u|\cdot|\nabla_L u|\cdot\left|\Lambda_a \nabla_L N^{4k}\right| \\
&\le \frac{p}{4k}\int_{\mathbb{H}_a^n} \frac{|z|_a^{\beta+2-4k}}{N(z,t)^{p+\alpha+\beta}} \left(|u|^2+\epsilon^2\right)^{(p-1)/2} |\nabla_L u|\cdot\left|\Lambda_a \nabla_L N^{4k}\right| \\
&= p\int_{\mathbb{H}_a^n} \frac{|z|_a^{\beta-2k}|z|}{N(z,t)^{p+\alpha+\beta-2k}} \left(|u|^2+\epsilon^2\right)^{(p-1)/2} |\nabla_L u|.
\end{aligned}
\tag{3.13}
$$

By dominated convergence, letting $\epsilon \to 0+$, we have

$$
(Q-p-\alpha)\int_{\mathbb{H}_a^n} \frac{|z|_a^{\beta}}{N(z,t)^{p+\alpha+\beta}}|u|^p \le p\int_{\mathbb{H}_a^n} \frac{|z|_a^{\beta-2k}|z|}{N(z,t)^{p+\alpha+\beta-2k}}|u|^{p-1}|\nabla_L u| \ . \tag{3.14}
$$

By Hölder's inequality,

$$
\begin{aligned}
&(Q-p-\alpha)\int_{\mathbb{H}_a^n} \frac{|z|_a^{\beta}}{N(z,t)^{p+\alpha+\beta}}|u|^p \\
&\le p\left(\int_{\mathbb{H}_a^n} \frac{|z|_a^{\beta}}{N(z,t)^{p+\alpha+\beta}}|u|^p\right)^{(p-1)/p} \left(\int_{\mathbb{H}_a^n} \frac{|\nabla_L u|^p}{N^\alpha}\frac{|z|_a^{\beta-p(2k-1)}}{N^{\beta-p(2k-1)}}\left(\frac{|z|}{|z|_a}\right)^p\right)^{1/p}.
\end{aligned}
\tag{3.15}
$$

Canceling and raising both sides to the power p, we obtain

$$
\left(\frac{Q-p-\alpha}{p}\right)^p \int_{\mathbb{H}_a^n} \frac{|z|_a^{\beta}}{N(z,t)^{p+\alpha+\beta}}|u|^p \le \int_{\mathbb{H}_a^n} \frac{|\nabla_L u|^p}{N^\alpha}\frac{|z|_a^{\beta-p(2k-1)}}{N^{\beta-p(2k-1)}}\left(\frac{|z|}{|z|_a}\right)^p. \tag{3.16}
$$

Set $\gamma = \beta - p(2k-1)$. Then, $\gamma > -2n-(p-1)(2k-1)$, and we get (3.11). □

Remark 3.3. Notice that $a_1|z|^2 \le |z|_a^2 \le a_n|z|^2$, we have, by Theorem 3.2, for all $u \in C_0^\infty(\mathbb{H}_a^n)$,

$$
\int_{\mathbb{H}_a^n} \frac{|\nabla_L u|^p}{N^\alpha}\frac{|z|_a^{\gamma}}{N^{\gamma}} \ge \left(\frac{\sqrt{a_1}(Q-p-\alpha)}{p}\right)^p \int_{\mathbb{H}_a^n} \frac{|u|^p}{N^{\alpha+p}}\frac{|z|_a^{\gamma+p(2k-1)}}{N^{\gamma+p(2k-1)}}. \tag{3.17}
$$

From inequality (3.17), we have the following corollary which generalizes the result of [21] when $1 < p < 2$ and $2\sum_{j=1}^n a_j + 2 - \alpha \le p < Q - \alpha$.

Corollary 3.4. *Let* $0 < a_1 \leq a_2 \leq \cdots \leq a_n \leq 1$, $1 < p < Q - \alpha$ *and* $\gamma + p \geq 0$. *There holds, for all* $u \in C_0^\infty(\mathbb{H}_a^n)$,

$$\int_{\mathbb{H}_a^n} \frac{|\nabla_L u|^p}{N^\alpha} \frac{|z|_a^\gamma}{N^\gamma} \geq \left(\frac{\sqrt{a_1}(Q-p-\alpha)}{p} \right)^p \int_{\mathbb{H}_a^n} \frac{|u|^p}{N^{\alpha+p}} \frac{|z|_a^{2p(k-1)} \left(\sum_{j=1}^n a_j^2 |z_j|^2 \right)^{(\gamma+p)/2}}{N^{\gamma+p(2k-1)}}. \tag{3.18}$$

Proof. Since $a_1 \leq a_2 \leq \cdots \leq a_n \leq 1$,

$$|z|_a^2 = \sum_{j=1}^n a_j |z_j|^2 \geq \sum_{j=1}^n a_j^2 |z_j|^2. \tag{3.19}$$

We have, by inequality (3.17),

$$\int_{\mathbb{H}_a^n} \frac{|\nabla_L u|^p}{N^\alpha} \frac{|z|_a^\gamma}{N^\gamma} \geq \left(\frac{\sqrt{a_1}(Q-p-\alpha)}{p} \right)^p \int_{\mathbb{H}_a^n} \frac{|u|^p}{N^{\alpha+p}} \frac{|z|_a^{2p(k-1)} \left(\sum_{j=1}^n a_j^2 |z_j|^2 \right)^{(\gamma+p)/2}}{N^{\gamma+p(2k-1)}}. \tag{3.20}$$

□

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References

[1] N. Garofalo and E. Lanconelli, "Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation," *Annales de l'Institut Fourier (Grenoble)*, vol. 40, no. 2, pp. 313–356, 1990.

[2] J. A. Goldstein and Q. S. Zhang, "On a degenerate heat equation with a singular potential," *Journal of Functional Analysis*, vol. 186, no. 2, pp. 342–359, 2001.

[3] P. Niu, H. Zhang, and Y. Wang, "Hardy type and Rellich type inequalities on the Heisenberg group," *Proceedings of the American Mathematical Society*, vol. 129, no. 12, pp. 3623–3630, 2001.

[4] L. D'Ambrozio, "Some Hardy inequalities on the Heisenberg group," *Differential Equations*, vol. 40, no. 4, pp. 552–564, 2004.

[5] J. Dou, "Picone inequalities for p-sub-Laplacian on the Heisenberg group and its applications," *Communications in Contemporary Mathematics*, vol. 12, no. 2, pp. 295–307, 2010.

[6] J. A. Goldstein and I. Kombe, "The Hardy inequality and nonlinear parabolic equations on Carnot groups," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 12, pp. 4643–4653, 2008.

[7] J. Han, "A class of improved Sobolev-Hardy inequality on Heisenberg groups," *Southeast Asian Bulletin of Mathematics*, vol. 32, no. 3, pp. 437–444, 2008.

[8] J. Q. Han, P. C. Niu, and Y. Z. Han, "Some Hardy-type inequalities on groups of Heisenberg type," *Journal of Systems Science and Mathematical Sciences*, vol. 25, no. 5, pp. 588–598, 2005.

[9] Y. Han, P. Niu, and X. Luo, "A Hardy type inequality and indefinite eigenvalue problems on the homogeneous group," *Journal of Partial Differential Equations*, vol. 15, no. 4, pp. 28–38, 2002.

[10] J. Han, P. Niu, and W. Qin, "Hardy inequalities in half spaces of the Heisenberg group," *Bulletin of the Korean Mathematical Society*, vol. 45, no. 3, pp. 405–417, 2008.

[11] Y. Jin and G. Zhang, "Degenerate p-Laplacian operators on H-type groups and applications to Hardy type inequalities," to appear in *Canadian Journal of Mathematics*.

[12] J.-W. Luan and Q.-H. Yang, "A Hardy type inequality in the half-space on $\mathbb{R}^n$ and Heisenberg group," *Journal of Mathematical Analysis and Applications*, vol. 347, no. 2, pp. 645–651, 2008.
[13] I. Kombe, "Sharp weighted Rellich and uncertainty principle inequalities on Carnot groups," *Communications in Applied Analysis*, vol. 14, no. 2, pp. 251–271, 2010.
[14] W.-C. Wang and Q.-H. Yang, "Improved Hardy-Sobolev inequalities for radial derivative," *Mathematical Inequalities and Applications*, vol. 14, no. 1, pp. 203–210, 2011.
[15] Y.-X. Xiao and Q.-H. Yang, "An improved Hardy-Rellich inequality with optimal constant," *Journal of Inequalities and Applications*, vol. 2009, Article ID 610530, 10 pages, 2009.
[16] Y.-X. Xiao and Q.-H. Yang, "Some Hardy and Rellich type inequalities on anisotropic Heisenberg groups," preprint.
[17] Q. Yang, "Best constants in the Hardy-Rellich type inequalities on the Heisenberg group," *Journal of Mathematical Analysis and Applications*, vol. 342, no. 1, pp. 423–431, 2008.
[18] Q. Yang, "Improved Sobolev inequalities on groups of Iwasawa type in presence of symmetry," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 998–1006, 2008.
[19] Q. H. Yang and B. S. Lian, "On the best constant of weighted Poincaré inequalities," *Journal of Mathematical Analysis and Applications*, vol. 377, no. 1, pp. 207–215, 2011.
[20] Y. Jin, "Hardy-type inequalities on H-type groups and anisotropic Heisenberg groups," *Chinese Annals of Mathematics Series B*, vol. 29, no. 5, pp. 567–574, 2008.
[21] Y. Jin and Y. Han, "Weighted Rellich inequality on H-type groups and nonisotropic Heisenberg groups," *Journal of Inequalities and Applications*, vol. 2010, Article ID 158281, 17 pages, 2010.
[22] W. S. Cohn and G. Lu, "Best constants for Moser-Trudinger inequalities on the Heisenberg group," *Indiana University Mathematics Journal*, vol. 50, no. 4, pp. 1567–1591, 2001.
[23] R. Beals, B. Gaveau, and P. C. Greiner, "Hamilton-Jacobi theory and the heat kernel on Heisenberg groups," *Journal de Mathématiques Pures et Appliquées. Neuvième Série*, vol. 79, no. 7, pp. 633–689, 2000.
[24] G. B. Folland and E. M. Stein, *Hardy spaces on Homogeneous Groups*, vol. 28 of *Mathematical Notes*, Princeton University Press, Princeton, NJ, USA, 1982.