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A Characterization of Chaotic Order and a Problem

MASATOSHI FUJII [a,*], JIAN FEI JIANG [b], EIZABURO KAMEI [c] and KOTARO TANAHASHI [d]

[a] *Department of Mathematics, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582, Japan;* [b] *Department of Basic Science and Technology, China Textile University, Shanghai, Postal code 200051, China;* [c] *Maebashi Institute of Technology, Kamisadori, Maebashi, Gunma 371, Japan;* [d] *Department of Mathematics, Tohoku College of Pharmacy, Komatsushima, Aoba-ku, Sendai 981, Japan*

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In our previous notes, we give a useful characterization of the chaotic order, i.e., $\log A \geq \log B$ for positive invertible operators A and B. In this note, we present a short proof to the characterization of the chaotic order and give an answer to a related problem on it. Moreover we consider the orders defined by $A^\delta \geq B^\delta$ $(0 < \delta < 1)$ as an interpolation between the chaotic order and the usual order via the Furuta inequality.

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1. INTRODUCTION

A (bounded linear) operator A on a Hilbert space H is positive, in symbol $A \geq 0$ if $(Ax, x) \geq 0$ for all $x \in H$. And $A > 0$ means that A is positive invertible. First of all, we recall the Furuta inequality [9], cf. [2,11] and [10] for an elementary one-page proof.

THE FURUTA INEQUALITY *If $A \geq B \geq 0$, then for each $r \geq 0$*

$$A^{(p+2r)/q} \geq (A^r B^p A^r)^{1/q} \tag{0}$$

* Corresponding author.

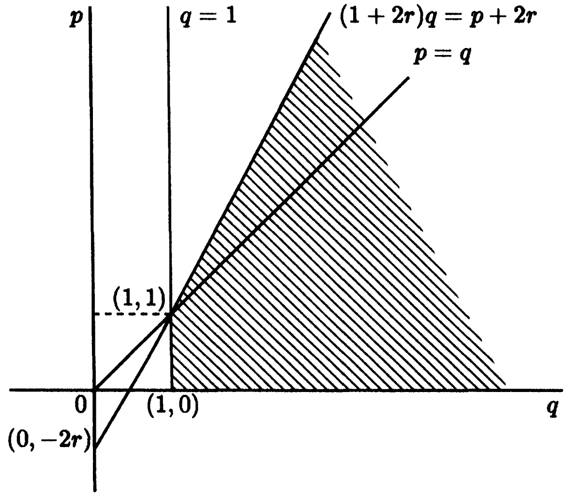

FIGURE 1

holds for $p \geq 0$ *and* $q \geq 1$ *with*

$$(1 + 2r)q \geq p + 2r. \tag{0'}$$

The domain of (0′) is expressed in Figure 1 above.

Motivated by the Furuta inequality, Ando [1] gave a nice operator inequality, by which the usual order $A \geq B$ is characterized.

THEOREM A *For self-adjoint operators A and B,* $A \geq B$ *if and only if the following inequality holds for all* $p \geq 0$:

$$(e^{pA/2}e^{pB}e^{pA/2})^{1/2} \leq e^{pA}. \tag{1}$$

Based on the well-known fact that $\log t$ is operator monotone on $(0, \infty)$, we introduced the chaotic order $A \gg B$ among positive invertible operators which is weaker than the usual order, i.e., $A \gg B$ if $\log A \geq \log B$, see [7]. Thereby Theorem A is interpreted as a characterization of the chaotic order.

THEOREM B *For* $A, B > 0$, $A \gg B$ *if and only if*

$$(A^{p/2}B^pA^{p/2})^{1/2} \leq A^p \tag{1'}$$

holds for all $p \geq 0$.

Afterwards, we generalized Theorem B in order to discuss the monotonicity of an operator function associated with the Furuta inequality under the chaotic order [8], cf. [3].

THEOREM C *For* $A, B > 0$, $A \gg B$ *if and only if*

$$(A^r B^p A^r)^{2r/(p+2r)} \leq A^{2r} \tag{2}$$

holds for all $p, r \geq 0$.

We here note that the original proof of Theorem C in [8] depends on Theorem B and the Furuta inequality.

Very recently, we obtained the following simple characterization of the chaotic order which easily links up the Furuta inequality to Theorem C [4,5].

THEOREM 1 *For* $A, B > 0$, $A \gg B$ *if and only if for any* $\delta \in (0, 1]$ *there exists an* $\alpha = \alpha_\delta > 0$ *such that*

$$(e^\delta A)^\alpha > B^\alpha.$$

The essence of the theorem is as follows.

THEOREM 2 *If* $\log A > \log B$ *for* $A, B > 0$, *then there exists an* $\alpha \in (0, 1]$ *such that* $A^\alpha > B^\alpha$.

Now the most serious problem arising from Theorem 2 is whether the assumption $\log A > \log B$ can be relaxed to $\log A \geq \log B$, i.e., $A \gg B$, or not. More presicely, the problem is whether $A \gg B$ implies that $A^\alpha \geq B^\alpha$ for some $\alpha > 0$ or not.

In this note, we give a short and elementary proof of Theorem 2, and a negative answer to the problem stated above by posing a counter example of 2×2 matrices. Inspired by this characterization, we moreover consider the orders defined by $A^\delta \geq B^\delta$ $(0 < \delta < 1)$ as an interpolation between the chaotic order and the usual order, and point out that they exactly correspond to Furuta's type operator inequalities. Of course, the case $\delta = 1$ does to the Furuta inequality.

2. SHORT PROOF AND PROBLEM

Inspired by Theorem A, we could prove Theorem 2; actually we showed that if $A > B$, then there exists an $\alpha > 0$ such that $e^{\alpha A} > e^{\alpha B}$, by using the Taylor expansion of the function e^t. We now give another short and direct proof by paying our attention to the fact

$$\log x = \lim_{h\to+0} \frac{x^h - 1}{h} \quad \text{decreasingly,} \tag{3}$$

so that the convergence of (3) is uniform on any bounded interval in $(0, \infty)$.

Proof of Theorem 2 Suppose that $\log A - \log B \geq 2s > 0$. By the above remark, there exists an $\alpha > 0$ such that

$$\left\| \frac{x^h - 1}{h} - \log x \right\|_I < s \quad \text{for } 0 < h \leq \alpha,$$

where I is a bounded interval including the spectra of A and B. Since

$$0 \leq \frac{A^\alpha - 1}{\alpha} - \log A \leq \left\| \frac{x^\alpha - 1}{\alpha} - \log x \right\|_I < s$$

and also

$$0 \leq \frac{B^\alpha - 1}{\alpha} - \log B \leq \left\| \frac{x^\alpha - 1}{\alpha} - \log x \right\|_I < s$$

by the spectral theorem, we have

$$\begin{aligned}
\frac{A^\alpha - B^\alpha}{\alpha} &= \left(\frac{A^\alpha - 1}{\alpha} - \log A \right) + \log A - \log B - \left(\frac{B^\alpha - 1}{\alpha} - \log B \right) \\
&\geq \log A - \log B - \left(\frac{B^\alpha - 1}{\alpha} - \log B \right) \\
&\geq \log A - \log B - \left\| \frac{B^\alpha - 1}{\alpha} - \log B \right\| \\
&\geq 2s - s = s,
\end{aligned}$$

so that $A^\alpha - B^\alpha \geq \alpha s > 0$, i.e., $A^\alpha > B^\alpha$.

Next we consider the problem: Does $A \gg B$ imply that there exists an $\alpha > 0$ such that $A^\alpha \geq B^\alpha$? As a matter of fact, we pose a counter example of a pair of positive invertible 2×2 matrices.

Example Take A and B as follows:

$$\log A = \begin{pmatrix} 2 & \sqrt{6} \\ \sqrt{6} & 1 \end{pmatrix} \quad \text{and} \quad \log B = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}.$$

Then $A \gg B$ clearly but $A^{\alpha} \geq B^{\alpha}$ does not hold for any $\alpha > 0$.

Actually, $\log A$ is diagonalized by U;

$$U(\log A)U = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{where } U = \frac{1}{\sqrt{5}}\begin{pmatrix} \sqrt{3} & \sqrt{2} \\ \sqrt{2} & -\sqrt{3} \end{pmatrix},$$

so that

$$A^{\alpha} = U\begin{pmatrix} e^{4\alpha} & 0 \\ 0 & e^{-\alpha} \end{pmatrix}U \quad \text{and} \quad B^{\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\alpha} \end{pmatrix}.$$

Replacing by $x = e^{\alpha} \geq 1$, we have

$$\begin{aligned} 5\det(A^{\alpha} - B^{\alpha}) &= \begin{vmatrix} 3x^4 + 2x^{-1} - 5 & \sqrt{6}(x^4 - x^{-1}) \\ \sqrt{6}(x^4 - x^{-1}) & 2x^4 + 3x^{-1} - 5x^{-2} \end{vmatrix} \\ &= (3x^4 + 2x^{-1} - 5)(2x^4 + 3x^{-1} - 5x^{-2}) - 6(x^4 - x^{-1})^2 \\ &= -5x^{-3}(x+1)(x-1)^4(2x^2 + x + 2), \end{aligned}$$

so that it is negative for all $x > 1$. Hence it means that $A^{\alpha} \geq B^{\alpha}$ does not hold for any $\alpha > 0$.

3. PATH BETWEEN THE FURUTA INEQUALITY AND THEOREM C

For the sake of convenience, we rewrite the Furuta inequality.

If $A \geq B \geq 0$, then for each $r \geq 0$

$$A^{(p+2r)/q} \geq (A^r B^p A^r)^{1/q} \tag{4}$$

holds for $p \geq 0$ and $q \geq 1$ with

$$(1 + 2r)q \geq p + 2r. \tag{4'}$$

As a connection with the Furuta inequality and Theorem C, we present the following theorem, cf. [6; Theorem 9].

THEOREM 3 *For a fixed* $\delta > 0$, $A^\delta \geq B^\delta$ *for* $A, B \geq 0$ *if and only if for each* $r \geq 0$

$$A^{(p+2r)/q} \geq (A^r B^p A^r)^{1/q} \tag{5}$$

holds for $p \geq 0$ *and* $q \geq 1$ *with*

$$(\delta + 2r)q \geq p + 2r. \tag{5'}$$

Proof Put $p_1 = p/\delta$, $r_1 = r/\delta$, $A_1 = A^\delta$ and $B_1 = B^\delta$. Since $A_1 \geq B_1$ and $(1 + 2r_1)q \geq p_1 + 2r_1$ by the assumption, the Furuta inequality implies that

$$A^{(p_1+2r_1)/q} \geq (A_1^{r_1} B_1^{p_1} A_1^{r_1})^{1/q}.$$

Hence we have the conclusion (5).

We note that Theorem C exactly corresponds to the case $\delta = 0$ in Theorem 3. In fact, Theorem C is represented as follows.

For $A, B > 0$, $A \gg B$ if and only if for each $r \geq 0$

$$(A^r B^p A^r)^{1/q} \leq A^{(p+2r)/q} \tag{6}$$

holds for $p \geq 0$ and $q \geq 1$ with

$$2rq \geq p + 2r. \tag{6'}$$

The point of the proof is that if p, q and r as in above, then they satisfy $(\delta + 2r)q \geq p + 2r$ for all $\delta > 0$, which suggests Theorem 3.

Finally we remark that the boundaries of (4′), (5′) and (6′) in which the inequalities (4), (5) and (6) hold respectively are the lines through the points $(0, -2r)$ and $(1, \delta)$ for $0 \leq \delta \leq 1$. The case $\delta = 1$ (resp. $\delta = 0$) is just the Furuta inequality (resp. Theorem C), and the cases $\delta \in (0, 1)$ interpolate between them. One of the authors [12] showed that the boundary for the Furuta inequality is the best possible. Anyway, one will be able to recognize the decrease of the domains when δ moves from 1 to 0; we present Figure 2 unifying them.

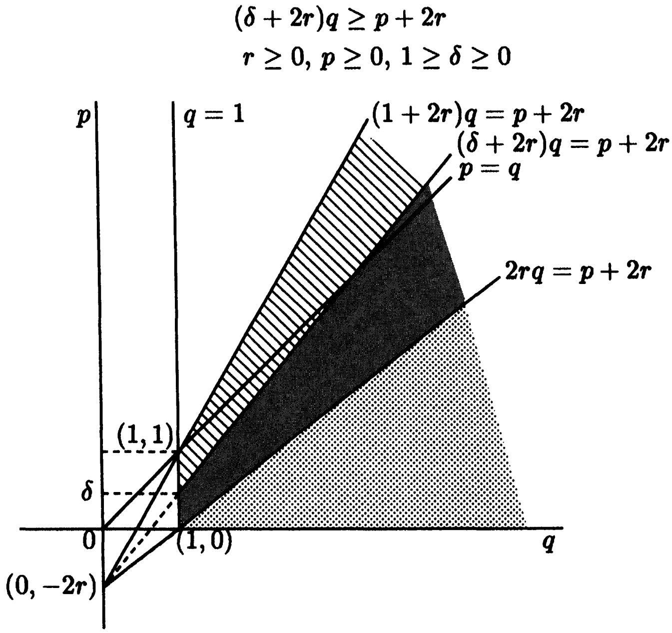

FIGURE 2

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References

[1] T. Ando, On some operator inequalities, *Math. Ann.*, **279** (1987), 157–159.
[2] M. Fujii, Furuta's inequality and its mean theoretic approach, *J. Operator Theory*, **23** (1990), 67–72.
[3] M. Fujii, T. Furuta and E. Kamei, Operator functions associated with Furuta's inequality, *Linear Alg. and its Appl.*, **179** (1993), 161–169.
[4] M. Fujii, J.-F. Jiang and E. Kamei, Characterization of chaotic order and its application to Furuta inequality, *Proc. Amer. Math. Soc.*, to appear.
[5] M. Fujii, J.-F. Jiang and E. Kamei, Characterization of chaotic order and its applications to Furuta's type operator inequalities, *Linear Multilinear Alg.*, to appear.
[6] M. Fujii, J.-F. Jiang and E. Kamei, A characterization of orders defined by $A^\delta \geq B^\delta$ via Furuta inequality, *Math. Japon.*, **45** (1997), 519–525.
[7] M. Fujii and E. Kamei, Furuta's inequality for the chaotic order, I and II, *Math. Japon.*, **36** (1991), 603–606 and 717–722.

[8] M. Fujii and E. Kamei, Furuta's inequality and a generalization of Ando's Theorem, *Proc. Amer. Math. Soc.*, **115** (1992), 409–413.
[9] T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$, *Proc. Amer. Math. Soc.*, **101** (1987), 85–88.
[10] T. Furuta, Elementary proof of an order preserving inequality, *Proc. Japan Acad.*, **65** (1989), 126.
[11] E. Kamei, A satellite to Furuta's inequality, *Math. Japon.*, **33** (1988), 883–886.
[12] K. Tanahashi, Best possibility of the Furuta inequality, *Proc. Amer. Math. Soc.*, **124** (1996), 141–146.