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A Multiplicity Result for Periodic Solutions of Higher Order Ordinary Differential Equations via the Method of Upper and Lower Solutions*

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We prove a multiplicity result of Ambrosetti–Prodi type problems of higher order. Proofs are based on upper and lower solutions method for higher order periodic boundary value problems and coincidence degree arguments.

Keywords: Existence; Multiplicity; Periodic solution; Higher order ordinary differential equation; Ambrosett–Prodi type problem; Upper solution; Lower solution; *A priori* estimate; Coincidence degree

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1. INTRODUCTION

This paper is devoted to the study of a multiplicity result for higher order ordinary differential equations of the form

$$\begin{aligned} &x^{(n)}(t) + f(t, x(t)) = s \quad \text{on } J = [0, 2\pi], \\ &x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, \ldots, n-1, \end{aligned} \tag{1_s}$$

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where s is a real parameter and $f: J \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function. Throughout this paper, we assume that f is 2π-periodic in the first variable. Assuming the following coerciveness condition

$$\lim_{|x| \to \infty} f(t, x) = \infty \text{ uniformly in } t \in J, \tag{H}$$

we may consider the existence of multiple solutions of (1_s), the so-called Ambrosetti–Prodi type problem. In 1988, among their general set-up of differential operator L, Ding and Mawhin [3] have proved under the assumption $f(t,x) = g(x) + e(t,x)$, where g is continuous with the coerciveness condition and e is of Carathéodory type, uniformly bounded by an L^1-function, that there exist s_o and $\bar{s}$ with $s_o \le \bar{s}$ such that (1_s) has no, at least one or at least two solutions according to $s < s_o$, $s = \bar{s}$ or $s > \bar{s}$. When n is even, they require an additional growth restriction on g. i.e. there exists $\gamma \in (0, 1)$ such that

$$(g(x) - g(y))(x - y) \ge -\gamma (x - y)^2, \quad x, y \in \mathbf{R}.$$

In this case, assuming $e(t,x) = e(t)$ has zero mean value, they also prove that there exists s_o such that (1_s) has no, at least one or at least two solutions according to $s < s_o$, $s = s_o$ or $s > s_o$.

Allowing joint dependence of (t, x) in the nonlinear terms, Ramos and Sanchez [6] deal with a number of situations in which one of the above results can be established. Among others, when n is even and f is continuous and coercive and the following condition holds: there exists $\gamma \in (0, 1)$ such that

$$(f(t,x) - f(t,y))(x - y) \ge -\gamma (x - y)^2, \quad \text{for all } t \in J \text{ and } x, y \in \mathbf{R},$$

they prove the second result in [3].

In this paper, we give a similar result as Ramos and Sanchez [6] with no restriction on the order n. More precisely, if f is continuous satisfying (H) and the following condition holds: there exists $M \in (0, A(n))$ such that

$$(f(t,x) - f(t,y))(x - y) \ge -M (x - y)^2, \quad \text{for all } t \in J \text{ and } x, y \in \mathbf{R}, \tag{H1}$$

where $A(n)=n!/\pi^n(n-1)^{n-1}$, then (1_s) satisfies the conclusion of the second result in [3].

The proof is based on the method of upper and lower solutions for higher order ordinary differential equations introduced in [2] and an application of coincidence degree.

In what follows, $J=[0,2\pi]$. Mean value $\bar{x}$ of x and the function $\tilde{x}$ of mean value 0 will be respectively defined by $\bar{x}=1/2\pi\int_0^{2\pi}x(t)\,\mathrm{d}t$ and $\tilde{x}(t)=x(t)-\bar{x}$. $C^k(J)$ will denote the space of continuous functions defined on J into R whose derivative through order k are continuous, $C^k_{2\pi}(J)$ the space of 2π-periodic functions of $C^k(J)$, $L^p(J)$ the classical real Lebesgue space with the usual norm $\|\cdot\|_p$. $W^{k,1}(J)$ denotes the Sobolev space of all functions x of C^{k-1}, with $x^{(k-1)}$ absolutely continuous and $W^{k,1}_{2\pi}(J)$ the space of 2π-periodic functions of $W^{k,1}$.

2. MAXIMUM PRINCIPLES AND THE METHOD OF UPPER AND LOWER SOLUTIONS

Let $L_n: F^n_{2\pi}\to L^1(J)$ be defined by $L_n\equiv D^n+MI$, where $D=\mathrm{d}/\mathrm{d}t$, I is the identity operator, M is a nonzero real constant, and

$$F^n_{2\pi}=\{x\in W^{n,1}(J):\ x^{(i)}(0)=x^{(i)}(2\pi),\ i=0,\ldots,n-2,\\ x^{(n-1)}(0)\geq x^{(n-1)}(2\pi)\}$$

Definition 1 We say that L_n *is inverse positive* in $F^n_{2\pi}$ if $L_nx\geq 0$ implies $x\geq 0$, for all $x\in F^n_{2\pi}$ and L_n is *inverse negative* if $L_nx\geq 0$ implies $x\leq 0$, for all $x\in F^n_{2\pi}$.

We present some maximum principles for the operator L_n.

Lemma 1 (Cabada [1]) *Let* $A(n)=n!/\pi^n(n-1)^{n-1}$. *Then the operator* L_n *is inverse positive in* $F^n_{2\pi}$ for $M\in(0,A(n))$, *and* L_n *is inverse negative in* $F^n_{2\pi}$ *for* $M\in(-A(n),0)$.

We notice that the second statement of Lemma 1 can be restated as follows; D^n-MI is inverse negative in $F^n_{2\pi}$ for $M\in(0,A(n))$.

Remark 1 By Lemmas 2.1 and 2.2 in [1], we have a strict inequality version of Lemma 1 as follows; If $M\in(0,A(n))$ $(M\in(-A(n),0))$, then $L_nx>0$ implies $x>0$ $(x<0)$ in $F^n_{2\pi}$.

Consider the periodic boundary value problem of higher order

$$\begin{aligned} &x^{(n)}(t) + f(t, x(t)) = 0 \quad \text{a.e. on } J, \\ &x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, \ldots, n-1, \end{aligned} \tag{2}$$

where $f: J \times \mathbf{R} \to \mathbf{R}$ is a *Carathéodory function*, i.e. $f(\cdot, x)$ is measurable for each $x \in \mathbf{R}$, $f(t, \cdot)$ is continuous for a.e. $t \in J$, and for every $r > 0$ there exists $h_r \in L^1(J)$ such that

$$|f(t, x)| \le h_r(t),$$

for a.e. $t \in J$ and all $x \in \mathbf{R}$ with $|x| \le r$. We define lower and upper solutions of Eq. (2);

DEFINITION 2 $\alpha \in W^{n,1}(J)$ is called a *lower solution* of (2) if

$$\begin{aligned} &\alpha^{(n)}(t) + f(t, \alpha(t)) \ge 0 \quad \text{a.e. } t \in J, \\ &\alpha^{(i)}(0) = \alpha^{(i)}(2\pi), \qquad i = 0, 1, \ldots, n-2, \\ &\alpha^{(n-1)}(0) \ge \alpha^{(n-1)}(2\pi). \end{aligned}$$

Similarly, $\beta \in W^{n,1}(J)$ is called an *upper solution* of (2) if

$$\begin{aligned} &\beta^{(n)}(t) + f(t, \beta(t)) \le 0 \quad \text{a.e. } t \in J, \\ &\beta^{(i)}(0) = \beta^{(i)}(2\pi), \qquad i = 0, 1, \ldots, n-2, \\ &\beta^{(n-1)}(0) \le \beta^{(n-1)}(2\pi). \end{aligned}$$

The following theorem is proved by Cabada [2], but here we give a different proof for reader's convenience, since part of the proof is useful to continue arguments in the proof of Theorem 3 in Section 3. The proof essentially follows Theorem 1.1 in [4].

THEOREM 1 *Assume that α and β are lower and upper solutions of* (2) *respectively with $\alpha(t) \le \beta(t)$, for all $t \in J$. Also assume that f satisfies that there exists $M \in (0, A(n))$ such that*

$$f(t, \alpha(t)) + M\alpha(t) \le f(t, x) + Mx \le f(t, \beta(t)) + M\beta(t), \tag{H2}$$

for a.e. $t \in J$ with $\alpha(t) \le x \le \beta(t)$. Then (2) *has a solution x such that $\alpha(t) \le x(t) \le \beta(t)$, for all $t \in J$.*

Proof Let us consider the modified problem

$$\begin{aligned} &x^{(n)}(t) + F(t, x(t)) = 0 \quad \text{a.e. on } J, \\ &x^{(i)}(0) = x^{(i)}(2\pi), \qquad i = 0, 1, \ldots, n-1, \end{aligned} \tag{3}$$

where $F: J \times \mathbf{R} \to \mathbf{R}$ is defined by

$$F(t,x) = \begin{cases} f(t,\beta(t)) - M(x - \beta(t)), & \textit{if } x > \beta(t), \\ f(t,x), & \textit{if } \alpha(t) \le x \le \beta(t), \\ f(t,\alpha(t)) - M(x - \alpha(t)), & \textit{if } x < \alpha(t), \end{cases}$$

M is a real constant in $(0, A(n))$. We claim that any solution x of (3) satisfies $\alpha(t) \le x(t) \le \beta(t)$, for all $t \in J$ so that it is a solution of (2). Let $J_1 = \{t \in J\colon\ x(t) > \beta(t)\}$, $J_2 = \{t \in J\colon\ \alpha(t) \le x(t) \le \beta(t)\}$ and $J_3 = \{t \in J\colon$ $x(t) < \alpha(t)\}$. Then on J_1,

$$\begin{aligned} x^{(n)}(t) - \beta^{(n)}(t) &\ge -F(t,x(t)) + f(t,\beta(t)) \\ &= -f(t,\beta(t)) + M(x(t) - \beta(t)) + f(t,\beta(t)) \\ &= M(x(t) - \beta(t)) \quad \text{a.e.} \end{aligned}$$

On J_2,

$$\begin{aligned} x^{(n)}(t) - \beta^{(n)}(t) &\ge -f(t,x(t)) + f(t,\beta(t)) \\ &\ge M(x(t) - \beta(t)) \quad \text{a.e. by (H2).} \end{aligned}$$

On J_3,

$$\begin{aligned} x^{(n)}(t) - \beta^{(n)}(t) &\ge -F(t,x(t)) + f(t,\beta(t)) \\ &= -f(t,\alpha(t)) + M(x(t) - \alpha(t)) + f(t,\beta(t)) \\ &\ge M(x(t) - \alpha(t)) - M(\beta(t) - \alpha(t)) \text{ by (H2)} \\ &= M(x(t) - \beta(t)) \quad \text{a.e.} \end{aligned}$$

Thus by the above three cases, we get

$$x^{(n)}(t) - \beta^{(n)}(t) - M(x(t) - \beta(t)) \ge 0 \quad \text{a.e. } J.$$

It is not hard to check that $x - \beta \in F^n_{2\pi}$ and thus by Lemma 1,

$$x(t) \le \beta(t) \quad \text{for all } t \in J.$$

Obviously, a similar argument applies to show that

$$\alpha(t) \le x(t) \quad \text{for all } t \in J.$$

Therefore we get

$$\alpha(t) \le x(t) \le \beta(t), \quad \text{for all } t \in J.$$

It remains to prove that (3) has at least one solution. To this purpose, consider the homotopy

$$\begin{aligned} x^{(n)}(t) - (1-\lambda)Mx(t) + \lambda F(t, x(t)) = 0 \quad \text{a.e. on } J \\ x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, \ldots, n-1, \end{aligned} \tag{4}$$

where $\lambda \in [0, 1]$. First of all, we will obtain *a priori* estimate for all possible solutions of (4). Let x be a solution of (4). We do the case when n is odd first. Multiplying both sides of (4) by x' and integrating on J,

$$\|x^{(p)}\|_2^2 = (-1)^{(n+1)/2}\lambda \int_J F(t, x(t))x'(t)\,\mathrm{d}t,$$

where $p = (n+1)/2$. Now

$$\begin{aligned} &\int_J F(t, x(t))x'(t)\,\mathrm{d}t \\ &\quad = \int_{J_1} \big(f(t, \beta(t)) - M(x(t) - \beta(t))\big)x'(t)\,\mathrm{d}t \\ &\qquad + \int_{J_2} f(t, x(t))x'(t)\mathrm{d}t + \int_{J_3} \big(f(t, \alpha(t)) - M(x(t) - \alpha(t))\big)x'(t)\,\mathrm{d}t \\ &\quad = \int_{J_1} \big(f(t, \beta(t)) + M\beta(t)\big)x'(t)\mathrm{d}t + \int_{J_2} \big(f(t, x(t)) + Mx(t)\big)x'(t)\,\mathrm{d}t \\ &\qquad + \int_{J_3} \big(f(t, \alpha(t)) + M\alpha(t)\big)x'(t)\mathrm{d}t - M\int_J x(t)x'(t)\,\mathrm{d}t. \end{aligned}$$

The integral in the last term is 0 and by the Carathéodory condition, integrals $\int_{J_1} |f(t, \beta(t))||x'(t)|\,\mathrm{d}t$, $\int_{J_2} |f(t, x(t))||x'(t)|\,\mathrm{d}t$ and $\int_{J_3} |f(t, \alpha(t))||x'(t)|\,\mathrm{d}t$ are bounded by $\|h_1\|_1\|x'\|_\infty$, for $h_1 \in L^1(J)$ determined by $\max\{\|\alpha\|_\infty, \|\beta\|_\infty\}$ in the definition of Carathéodory function. Thus we get

$$\begin{aligned}\|x^{(p)}\|_2^2 &\le \int_J |F(t,x(t))||x'(t)|\,\mathrm{d}t\\ &\le 3(\bar{M}+\|h_1\|_1)\|x'\|_\infty\\ &\le \sqrt{\frac{3\pi}{2}}(\bar{M}+\|h_1\|_1)\|x''\|_2 \quad \text{by Sobolev inequality}\\ &\le \sqrt{\frac{3\pi}{2}}(\bar{M}+\|h_1\|_1)\|x^{(p)}\|_2 \quad \text{by Wirtinger inequality,}\end{aligned}$$

where $\bar{M} = 2\pi M \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}$. Therefore

$$\|x^{(p)}\|_2 \le \sqrt{\frac{3\pi}{2}}(\bar{M}+\|h_1\|_1),$$

and by Wirtinger inequality again,

$$\|x'\|_2 \le \sqrt{\frac{3\pi}{2}}(\bar{M}+\|h_1\|_1). \tag{2a}$$

When n is even, multiplying both sides of (4) by $\tilde{x}$ and integrating on J, we get for $p=n/2$,

$$(-1)^p\|x^{(p)}\|_2^2 = (1-\lambda)M\int_J x(t)\tilde{x}(t)\,\mathrm{d}t - \lambda\int_J F(t,x(t))\tilde{x}(t)\,\mathrm{d}t.$$

Now

$$\begin{aligned}&\int_J F(t,x(t))\tilde{x}(t)\,\mathrm{d}t\\ &\quad= \int_{J_1}\big(f(t,\beta(t)) - M(x(t)-\beta(t))\big)\tilde{x}(t)\,\mathrm{d}t\\ &\qquad+\int_{J_2} f(t,x(t))\tilde{x}(t)\,\mathrm{d}t + \int_{J_3}\big(f(t,\alpha(t)) - M(x(t)-\alpha(t))\big)\tilde{x}(t)\,\mathrm{d}t\\ &\quad= \int_{J_1}\big(f(t,\beta(t)+M\beta(t)\big)\tilde{x}(t)\,\mathrm{d}t + \int_{J_2}\big(f(t,x(t))+Mx(t)\big)\tilde{x}(t)\,\mathrm{d}t\\ &\qquad+\int_{J_3}\big(f(t,\alpha(t)+M\alpha(t)\big)\tilde{x}(t)\,\mathrm{d}t - M\int_J x(t)\tilde{x}(t)\,\mathrm{d}t.\end{aligned}$$

Thus

$$\begin{aligned}(-1)^p\|x^{(p)}\|_2^2 = M\int_J x(t)\tilde{x}(t)\,\mathrm{d}t - \lambda\bigg[\int_{J_1}\big(f(t,\beta(t)) + M\beta(t)\big)\tilde{x}(t)\,\mathrm{d}t \\ + \int_{J_2}\big(f(t,x(t)) + Mx(t)\big)\tilde{x}(t)\,\mathrm{d}t \\ + \int_{J_3}\big(f(t,\alpha(t)) + M\alpha(t)\big)\tilde{x}(t)\,\mathrm{d}t\bigg],\end{aligned}$$

and we get

$$\begin{aligned}\|x^{(p)}\|_2^2 &\le M\|\tilde{x}\|_2^2 + 3(\bar{M} + \|h_1\|_1)\|\tilde{x}\|_\infty \\ &\le M\|\tilde{x}\|_2^2 + \sqrt{\frac{3\pi}{2}}(\bar{M} + \|h_1\|_1)\|x'\|_2 \quad \text{by Sobolev inequality} \\ &\le M\|x^{(p)}\|_2^2 + \sqrt{\frac{3\pi}{2}}(\bar{M} + \|h_1\|_1)\|x^{(p)}\|_2 \\ &\qquad \text{by Wirtinger inequality.}\end{aligned}$$

Since $0 < M < 1$, we get

$$\|x^{(p)}\|_2 \le \frac{\sqrt{3\pi}(\bar{M} + \|h_1\|_1)}{\sqrt{2}(1 - M)},$$

and by Wirtinger inequality,

$$\|x'\|_2 \le \frac{\sqrt{3\pi}(\bar{M} + \|h_1\|_1)}{\sqrt{2}(1 - M)}. \tag{2b}$$

Therefore both (2a) and (2b) imply that

$$\|x'\|_2 \le \frac{\sqrt{3\pi}(\bar{M} + \|h_1\|_1)}{\sqrt{2}(1 - M)},$$

for all possible solutions x of (4).

Claim that there is $\tau \in J$ such that $|x(\tau)| < m + 1$, where $m = \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}$. Suppose that the claim is not true, so let $x(t) \ge m + 1$ for all $t \in J$. Then $x(t) > \beta(t)$ for all $t \in J$ and by the fact that β is an upper solution of (2), Eq. (4) becomes

$$\begin{aligned} x^{(n)}(t) &= (1-\lambda)Mx(t) - \lambda F(t, x(t)) \\ &= Mx(t) - \lambda f(t, \beta(t)) - \lambda M\beta(t) \\ &\geq Mx(t) + \lambda\beta^{(n)}(t) - \lambda M\beta(t) \quad \text{a.e.} \end{aligned}$$

Thus

$$(x - \lambda\beta)^{(n)}(t) - M(x - \lambda\beta)(t) \geq 0, \quad \text{a.e. } t \in J.$$

Since $x - \lambda\beta \in F^n_{2\pi}$ for all $\lambda \in [0, 1]$, it follows from Lemma 1 that

$$x(t) \leq \lambda\beta(t), \quad \text{for all } t \in J.$$

Thus

$$|\beta(t)| < x(t) \leq \lambda\beta(t),$$

for all $t \in J$ and $\lambda \in [0, 1]$ and this is a contradiction. We may get a contradiction by a similar argument for the case $x(t) \leq -m-1$, for all $t \in J$, and the claim is verified. Now

$$\begin{aligned} |x(t)| &\leq |x(\tau)| + \int_\tau^t |x'(s)| \, \mathrm{d}s \\ &\leq |x(\tau)| + 2\pi \|x'\|_2 \\ &< m + 1 + \frac{\sqrt{6\pi^3}(\bar{M} + \|h_1\|_1)}{(1 - M)} \equiv M(h_1), \end{aligned}$$

and the *a priori* estimate is complete. For degree computations, we reduce problem (4) to an equivalent operator form. Define $L : D(L) \subset C^0_{2\pi}(J) \to L^1(J)$ by $x \mapsto x^{(n)}$, where $D(L) = W^{n,1}_{2\pi}$ and $N_\lambda : C^0_{2\pi}(J) \to L^1(J)$ by

$$N_\lambda x(\cdot) = -(1 - \lambda)Mx(\cdot) + \lambda F(\cdot, x(\cdot))$$

so that (4) can be written as

$$Lx + N_\lambda x = 0.$$

By the standard argument [5], we can easily check that L is a Fredholm operator of index 0 and N_λ is L-compact on $\bar{\Omega}$ for any bounded open subset Ω in $C^0_{2\pi}(J)$. Let Ω_0 be an open bounded subset in $C^0_{2\pi}(J)$ with

$$\Omega_0 \supset \{x \in C^0_{2\pi}(J) \colon \|x\|_\infty < M(h_1)\}.$$

Then by the *a priori* estimate, $Lx + N_\lambda x \neq 0$ for $x \in D(L) \cap \partial\Omega_0$, and thus the coincidence degree $D_L(L + N_\lambda, \Omega_0)$ is well-defined. Since the linear problem

$$x^{(n)}(t) - Mx(t) = 0$$

does not have any nontrivial 2π-periodic solutions, by homotopy invariance property and Proposition II.16 [5], we obtain

$$\pm 1 = D_L(L - MI, \Omega_0) = D_L(L + N_0, \Omega_0) = D_L(L + N_1, \Omega_0).$$

This implies that (3) has at least one solution in $D(L) \cap \Omega_0$ and the proof is complete.

3. MULTIPLICITY RESULTS

In this section, we shall apply Theorem 1 of Section 2 to get multiplicity results of 2π-periodic solutions for higher order Ambrosetti–Prodi type problems. Let us consider Eq. (1_s):

$$\begin{aligned} &x^{(n)}(t) + f(t, x(t)) = s \quad \text{on } J, \\ &x^{(i)}(0) = x^{(i)}(2\pi), \qquad i = 0, 1, \ldots, n-1, \end{aligned}$$

where s is a real parameter and $f\colon J \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function. Throughout the remainder of this paper, sometimes without further comment, we shall assume the following condition; there exists $M \in (0, A(n))$ such that

$$(f(t,x) - f(t,y))(x - y) \geq -M(x-y)^2, \quad \text{for a.e. } t \in J \text{ and } x, y \in \mathbf{R}, \tag{H1$'$}$$

where $A(n)$ is given in Lemma 1. We notice that (H1′) implies the condition (H2) in Theorem 1 and if f is continuous, then (H1′) is equivalent to (H1).

THEOREM 2 *Assume that there exist* s_1 *and* $r(s_1) > 0$ *such that*

$$\operatorname*{ess\,sup}_{t \in J} f(t, 0) < s_1 < f(t, x)$$

for a.e. $t \in J$ *and* $x \in \mathbf{R}$ *with* $x \leq -r(s_1)$. *Then there exits* $s_0 < s_1$, *possibly* $s_0 = -\infty$ *such that* (1_s) *has no solution for* $s < s_0$ *and at least one solution for* $s \in (s_0, s_1]$.

Proof Let $s^* = \operatorname{ess\,sup}_{t \in J} f(t, 0)$, then constant functions $\alpha \equiv -r(s_1)$ and $\beta \equiv 0$ are lower and upper solutions of (1_{s^*}), respectively. Thus by Theorem 1, Eq. (1_s) has a solution for $s = s^*$. We also see that if $(1_{\bar{s}})$ has a solution $\bar{x}$ for $\bar{s} < s_1$, then (1_s) also has a solution for $s \in [\bar{s}, s_1]$, since $\bar{x}$ and $-r(s_1)$ are upper solution and lower solution of (1_s) for $s \in [\bar{s}, s_1]$, and $-r(s_1) \leq \bar{x}(t)$ for all $t \in J$ by necessary adjustment for $r(s_1)$. We complete the proof by taking $s_0 = \inf\{s \in \mathbf{R}$: (1_s) has at least one solution$\}$.

For multiplicity results, we shall employ coincidence degree arguments. Define $L : D(L) \subset C^0_{2\pi}(J) \to L^1(J)$ by $x \mapsto x^{(n)}$, where $D(L) = W^{n,1}_{2\pi}(J)$, and $N_s : C^0_{2\pi}(J) \to L^1(J)$ by

$$N_s x(\cdot) = f(\cdot, x(\cdot)) - s.$$

Then (1_s) can be equivalently written as

$$Lx + N_s x = 0,$$

and it is well-known that L is a Fredholm operator of index 0 and N_λ is L-compact on $\bar{\Omega}$ for any bounded open subset Ω in $C^0_{2\pi}(J)$.

THEOREM 3 *Assume that there exist* s_1 *and* $r(s_1) > 0$ *such that*

$$\operatorname*{ess\,sup}_{t \in J} f(t, 0) < s_1 < f(t, x) \tag{3a}$$

for a.e. $t \in J$ *and* $x \in \mathbf{R}$ *with* $|x| \geq r(s_1)$. *Also assume that there exists* $R = R(s_1, f) > 0$ *such that every possible solution* x *of* (1_s), *for* $s \leq s_1$, *satisfies*

$$\|x\|_\infty < R. \tag{3b}$$

Then there exists a real number $s_0 < s_1$ *such that* (1_s) *has*

(i) *no solution for* $s < s_0$,
(ii) *at least one solution for* $s = s_0$,
(iii) *at least two solutions for* $s \in (s_0, s_1]$.

Proof We know that for s_0 given in Theorem 2, (1_s) has no solution for $s < s_0$ and at least one solution for $s \in (s_0, s_1]$.

First, we show that s_0 is finite. It follows from (3a) and f Carathéodory that

$$f(t,x) \geq -|s_1| - h_r(t),$$

for a.e. t and all $x \in \mathbf{R}$, where h_r is the L^1-function determined by $r(s_1)$ in the definition of Carathéodory function. If (1_s) has a solution x, then

$$s = \frac{1}{2\pi}\int_J f(t,x(t))\,\mathrm{d}t \geq -|s_1| - \frac{1}{2\pi}\|h_r\|_1.$$

Thus $s_0 \geq -|s_1| - 1/2\pi\|h_r\|_1 > -\infty$.

Second, we show existence of the second solution of (1_s) for $s \in (s_0, s_1]$. Without loss of generality, let us assume that $R > r(s_1)$. Let Ω be an open bounded subset in $C^0_{2\pi}(J)$ such that $\Omega \supset \{x \in C^0_{2\pi}(J)\colon\ \|x\|_\infty < R\}$; then by (3b), the coincidence degree $D_L(L + N_s, \Omega)$ is well-defined. Since (1_s) does not have solution for $s > s_0$, by the common argument of Ambrosetti-Prodi type problems [3,6], we get

$$D_L(L + N_s, \Omega) = 0, \quad \text{for } s \leq s_1. \tag{3c}$$

Let $s \in (s_0, s_1]$, $\tilde{s} \in (s_0, s)$ and let $\tilde{x}$ be a solution of $(1_{\tilde{s}})$ known to exist by Theorem 2. Then $-R$ and $\tilde{x}$ are lower and upper solutions of (1_s) with $-R < \tilde{x}(t)$, for all $t \in J$. Let $\Omega_1 = \{x \in C^0_{2\pi}(J) : -R < x(t) < \tilde{x}(t),\ t \in J\}$, then $\Omega_1 \subset \Omega$. By (3b) and Remark 1, solutions of (1_s) never lie on $\partial\Omega_1$. Thus $D_L(L + N_s, \Omega_1)$ is well-defined. To compute the degree, let us consider a modified problem:

$$\begin{aligned} &x^{(n)}(t) + F(t, x(t)) = 0 \quad \text{a.e. on } J, \\ &x^{(i)}(0) = x^{(i)}(2\pi), \qquad i = 0, 1, \dots, n-1, \end{aligned} \tag{5}$$

where

$$F(t,x) = \begin{cases} f(t, \tilde{x}(t)) - s - M(x - \tilde{x}(t)), & \text{if } x > \tilde{x}(t), \\ f(t,x) - s, & \text{if } -R \le x \le \tilde{x}(t), \\ f(t, -R) - s - M(x + R), & \text{if } x < -R, \end{cases}$$

and M is given in (H1′). By a similar argument as in the proof of Theorem 1, we get

$$D_L(L + N_F, \Omega_0) = \pm 1,$$

for certain open bounded open subset Ω_0 in $C^0_{2\pi}(J)$, where N_F is defined by $N_F x(\cdot) = F(\cdot, x(\cdot))$, and we also know that all solution x of (5) must satisfy

$$-R < x(t) < \tilde{x}(t), \quad \text{for all } t \in J$$

so that (1_s) is equivalent to (5) in Ω_1. Therefore, by the excision and the additive properties of the coincidence degree together with (3c), we get

$$\pm 1 = D_L(L + N_F, \Omega_0) = D_L(L + N_F, \Omega_1) = D_L(L + N_s, \Omega_1)$$

and

$$D_L(L + N_s, \Omega \setminus \overline{\Omega_1}) = \mp 1.$$

Consequently, (1_s) has one solution in Ω_1 and another in $\Omega \setminus \overline{\Omega}_1$. Since $s \in (s_0, s_1]$ is arbitrary, the second part of the proof is complete.

Finally, the existence of at least one solution at $s = s_1$ can be proved through a limiting process based upon *a priori* boundedness of possible solutions as in [3].

THEOREM 4 *Assume that f is a Carathéodory function and satisfies* (H) *and* (H1′). *Moreover, assume that*

$$\left| \operatorname*{ess\,sup}_{t \in J} f(t, 0) \right| < \infty. \tag{3d}$$

Then there exists a real number s_0 *such that* (1_s) *has*

(i) *no solution for* $s < s_0$,
(ii) *at least one solution for* $s = s_0$,
(iii) *at least two solutions for* $s > s_0$.

Proof If f satisfies (H), then (3a) in Theorem 3 is valid for arbitrary large s_1. Thus it suffices to show that all possible solutions of (1_s) for $s \le s_1$ are uniformly bounded. Let $s \le s_1$ and x a solution of (1_s). Integrating both sides of (1_s) on J, we get

$$\int_J f(t, x(t))\, \mathrm{d}t = 2\pi s.$$

From the proof of Theorem 3, we know that

$$f(t, x) + |s_1| + h_r(t) \ge 0,$$

for a.e. t and for all $x \in \mathbf{R}$. When n is odd, Multiplying both sides of (1_s) by x' and integrating on the period, we get for $p = (n+1)/2$,

$$\begin{aligned}(-1)^p \|x^{(p)}\|_2^2 &= \int_J f(t, x(t)) x'(t)\, \mathrm{d}t \\ &= \int_J \big(f(t, x(t)) + |s_1| + h_r(t)\big) x'(t)\, \mathrm{d}t - \int_J h_r(t) x'(t)\, \mathrm{d}t.\end{aligned}$$

Thus

$$\begin{aligned}\|x^{(p)}\|_2^2 &\le \|x'\|_\infty \int_J \big(f(t, x(t)) + |s_1| + h_r(t)\big)\, \mathrm{d}t + \|h_r\|_1 \|x'\|_\infty \\ &\le \|x'\|_\infty \big(2\pi(s + |s_1|) + 2\|h_r\|_1\big) \\ &\le \sqrt{\frac{2\pi}{3}} (2\pi|s_1| + \|h_r\|_1) \|x''\|_2, \quad \text{by Sobolev inequality} \\ &\le \sqrt{\frac{2\pi}{3}} M(s_1) \|x^{(p)}\|_2, \quad \text{by Wirtinger inequality,}\end{aligned}$$

where $M(s_1) = 2\pi|s_1| + \|h_r\|_1$. Thus

$$\|x^{(p)}\|_2 \le \sqrt{\frac{2\pi}{3}} M(s_1),$$

and by Wirtinger inequality,

$$\|x'\|_2 \leq \sqrt{\frac{2\pi}{3}} M(s_1).$$

When n is even, multiplying both sides of (1_s) by $\tilde{x}$, integrating on J and doing similar calculations, we get for $p = n/2$,

$$\begin{aligned}
\|x^{(p)}\|_2^2 &\leq \|\tilde{x}\|_\infty \int_J (f(t, x(t)) + |s_1| + h_r(t)) \, \mathrm{d}t + \|h\|_1 \|\tilde{x}\|_\infty \\
&\leq \|\tilde{x}\|_\infty (2\pi(s + |s_1|) + 2\|h_r\|_1) \\
&\leq \sqrt{\frac{2\pi}{3}} (2\pi|s_1| + \|h_r\|_1) \|x'\|_2, \quad \text{by Sobolev inequality} \\
&\leq \sqrt{\frac{2\pi}{3}} M(s_1) \|x^{(p)}\|_2, \quad \text{by Wirtinger inequality.}
\end{aligned}$$

Thus we also get

$$\|x'\|_2 \leq \sqrt{\frac{2\pi}{3}} M(s_1).$$

We claim that for each possible solution x of (1_s) and $s \in (s_0, s_1]$, there is $t_0 \in J$ such that $|x(t_0)| < r(s_1)$. Suppose the claim is not true, then there exists a solution x such that

$$|x(t)| \geq r(s_1), \quad \text{for all } t \in J.$$

So if $x(t) \geq r(s_1)$ for all $t \in J$, then by (3a),

$$f(t, x(t)) > s_1, \quad \text{a.e. in } t.$$

Thus

$$s = \frac{1}{2\pi} \int_J f(t, x(t)) \, \mathrm{d}t > s_1,$$

and this is a contradiction. Similarly, we may show a contradiction for the case $x(t) \leq -r(s_1)$ and the claim is verified. Consequently,

$$\begin{aligned}
|x(t)| &\leq |x(t_0)| + \int_{t_0}^{t} |x'(\tau)| \, \mathrm{d}\tau < r(s_1) + 2\pi\|x'\|_2 \\
&\leq r(s_1) + \frac{2\pi\sqrt{2\pi}}{\sqrt{3}} M(s_1).
\end{aligned}$$

The proof is complete.

Remark 2 If the function h_r determined by r in the definition of Carathéodory is of $L^\infty(J)$, then the condition (3d) in Theorem 4 is not necessary.

COROLLARY 1 *If f is continuous on $J \times \mathbf{R}$ and satisfies* (H) *and* (H1), *then there exists a real number s_0 such that* (1_s) *has*

(i) *no solution for $s < s_0$,*
(ii) *at least one solution for $s = s_0$.*
(iii) *at least two solutions for $s > s_0$.*

Consider the equation

$$\begin{aligned} &x^{(n)}(t) + g(x(t)) + h(t) = s \quad &&\text{on } J, \\ &x^{(i)}(0) = x^{(i)}(2\pi), &&i = 0, 1, \ldots, n-1, \end{aligned}$$

COROLLARY 2 *If $g : \mathbf{R} \to \mathbf{R}$ is continuous such that*

$$\lim_{|x| \to \infty} g(x) = \infty,$$

and g is also such that there exists $M \in (0, A(n))$ for which

$$\big(g(x) - g(y)\big)(x - y) \geq -M(x - y)^2,$$

for all $x, y \in \mathbf{R}$. Then for any given $h \in L^\infty(J)$, there exists a real number s_0 such that (6*s*) *has*

(i) *no solution for $s < s_0$,*
(ii) *at least one solution for $s = s_0$,*
(iii) *at least two solutions for $s > s_0$.*

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