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Some Embeddings of Weighted Sobolev Spaces on Finite Measure and Quasibounded Domains

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We show that several of the classical Sobolev embedding theorems extend in the case of weighted Sobolev spaces to a class of quasibounded domains which properly include all bounded or finite measure domains when the weights have an arbitrarily weak singularity or degeneracy at the boundary. Sharper results are also shown to hold when the domain satisfies an integrability condition which is equivalent to the Minkowski dimension of the boundary being less than n. We apply these results to derive a class of weighted Poincaré inequalities which are similar to those recently discovered by Edmunds and Hurri. We also point out a formal analogy between one of our results and an interpolation theorem of Cwikel.

Keywords: Weighted Sobolev spaces; Continuous and compact embeddings; Regularity conditions; Minkowski dimension; Weighted Poincaré inequalities

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1 INTRODUCTION

Let Ω be a domain, in other words a nonempty open subset of $\mathbb{R}^N$, and let $W^{1,p}(\Omega)$, $p \in [1, \infty]$, denote the Sobolev space consisting of complex-valued measurable functions defined on Ω whose partial derivatives

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exist in the sense of distributions and which is endowed with the norm

$$\|u\|_{\Omega;1,p} := \|u\|_{\Omega;p} + \|\nabla u\|_{\Omega;p}. \tag{1.1}$$

Here the notation "$\|\nabla u\|_{\Omega;p}$" signifies $\sum_{i=1}^{N} \|\partial u/\partial x_i\|_{\Omega;p}$ and

$$\|u\|_{\Omega;p} := \left(\int_0^1 |u|^p \,\mathrm{d}x\right)^{1/p}. \tag{1.2}$$

If $C_0^\infty(\Omega)$ consists of the infinitely differentiable (or "smooth") functions on Ω, we define $W_0^{1,p}(\Omega) \subset W^{1,p}$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (1.1).

$W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ are classic and well-known function spaces with many important properties and applications to almost all areas of analysis. Among the most fundamental results concerning Sobolev spaces are *embedding* theorems. We say that $W \equiv W^{1,p}(\Omega)$ or $W_0 \equiv W_0^p(\Omega)$ is embedded in $L^q(\Omega)$ if W or $W_0 \subseteq L^q(\Omega)$ and the natural maps $i\colon W \to L^q(\Omega)$ or $i_0\colon W_0 \to L^q(\Omega)$ are continuous. We express this by the notation

$$W \text{ or } W_0 \hookrightarrow L^q(\Omega).$$

Likewise, when the embedding map i or i_0 is compact we write

$$W \text{ or } W_0 \hookrightarrow\hookrightarrow L^q(\Omega). \tag{1.3}$$

Since our own results will parallel them, it is convenient to review some of the most basic embedding theorems. These results are standard; proofs and detailed discussion may be found in the books [1,12, or 24].

THEOREM A *For any bounded domain Ω, $1 < p < \infty$ and $q \in [1,p)$ then W satisfies the embedding* (1.3).

Remark 1.1 Theorem A is true even on domains of finite measure as has been recently shown by Carnavati and Fontes [10].

THEOREM B *If $1 \le p < N$, $p^* := pN/(N-p)$, $q \in [p,p^*]$, and Ω is a bounded domain, then*

$$W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega).$$

Further if $q \in [p,p^*)$ then the embedding is compact.

Let $C(\bar{\Omega})$ denote the space of bounded continuous functions on Ω which have continuous and bounded extensions to $\bar{\Omega}$. We also need spaces of Hölder continuous functions on $\bar{\Omega}$. If $\lambda \in (0,1)$ and $D_i(u) := \partial u/\partial x_i$ set

$$C^\lambda(\bar{\Omega}) := \{u \in C(\bar{\Omega}) \colon |D_i(u(x)) - D_j(u(y))| \leq |x-y|^\lambda\}.$$

These spaces are Banach spaces under the norms

$$\begin{aligned} \|u\|_{\Omega;\infty} &:= \sup_{x\in\Omega} |u(x)|, \\ |||u|||_{\Omega;\lambda} &:= \|u\|_{\Omega;\infty} + \max_{1\leq i\leq N} \sup_{\substack{x,y\in\Omega \\ x\neq y}} |D_i(u(x)) - D_j(u(y))|. \end{aligned}$$

THEOREM C *If* $\lambda \in (0,1)$, $N \leq (1-\lambda)p < \infty$, *and* Ω *is a bounded domain, then*

$$W_0 \hookrightarrow C^\lambda(\bar{\Omega}).$$

If $N < (1-\lambda)p$ *then the embedding is compact.*

The attempt to extend Theorems B and C to the Sobolev spaces $W^{1,p}(\Omega)$ has been a continuing project in Sobolev space theory. Only Theorem A holds on $W^{1,p}(\Omega)$ for *arbitrary* (finite measure) Ω, the remaining theorems are *not* true unless additional conditions are imposed on Ω. There are a bewildering array of possibilities including those satisfying various cone conditions [1,15] or twisted cone conditions [3], being star shaped or convex [16,24], having a "minimally smooth" boundary $\partial\Omega$ [11,27], satisfying the segment condition [15], being a Hölder domain [26] or generalized ridge domain [14], etc. Many of these conditions are quite technical, apparently mutually independent, and the proofs of the embedding theorems using them are not easy. One of the weakest, for at least the continuity of the embedding in Theorem B, is due to Bojarski [5]; he requires that Ω satisfy the "Boman chain condition" (see [6]). The Boman condition in turn implies that Ω is a "John" domain and satisfies a "quasihyperbolic" boundary condition. For a survey of the situation in 1979 which is still informative (see [15]). More recent information can be found in [18].

In this paper we are going to show that Sobolev-like embedding theorems which mimic the Theorems A and B stated above hold on *all* bounded as well as on a wide class of unbounded domains Ω if we introduce weighted Sobolev spaces or Lebesgue spaces where the weights have an arbitrarily weak singularity or degeneracy on $\partial\Omega$. We omit, however, consideration of Theorem C since an exhaustive treatment of weighted extensions of that theorem may be found in [9]. Many of our results resemble the weighted embeddings given in the book of Opic and Kufner [25] but with weaker conditions on $\partial\Omega$.

The following are some examples of our general approach. Suppose that $d(x) = \operatorname{dist}(x, \partial\Omega)$. Then we can show for a wide range of $1 \leq p, q < \infty$ that

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega; d^\beta) \tag{1.4}$$

for any $\beta > 0$ and $p \in [1, \infty)$ where $L^p(\Omega; d^\beta)$ denotes the weighted L^p space with norm

$$\|u\|_{\Omega;d^\beta,p} := \left(\int_\Omega d^\beta |u|^p \, \mathrm{d}x \right)^{1/p}.$$

Similarly if $W^{1,p}(\Omega; d^{-\beta}, 1)$, $W^{1,p}(\Omega; 1, d^{-\beta})$ denotes the spaces having the respective norms

$$\|u\|_{\Omega;d^{-\beta},1,p} := \|u\|_{\Omega;d^{-\beta}} + \|\nabla u\|_{\Omega;p},$$
$$\|u\|_{\Omega;1,d^{-\beta},p} := \|u\|_\Omega + \|\nabla u\|_{\Omega;d^{-\beta},p},$$

we shall see that it is often the case that

$$W^{1,p}(\Omega; d^{-\beta}, 1) \hookrightarrow\hookrightarrow L^q(\Omega), \tag{1.5}$$
$$W^{1,p}(\Omega; 1, d^{-\beta}) \hookrightarrow\hookrightarrow L^q(\Omega). \tag{1.6}$$

Both (1.4) and (1.5) can hold on a class of quasibounded domains which need not even be of finite measure. Furthermore they may be generalized in many ways to include weights which go to zero or infinity more slowly near the boundary than d^β or $d^{-\beta}$. In still other cases an arbitrary weight can be attached to the gradient term or (e.g., (1.7) below) the sign of β can be reversed.

Embeddings like (1.4)–(1.6) and their generalizations should have applications to the problem of proving the existence of weak solutions to degenerate partial differential equations. We intend to explore this kind of employment elsewhere. But there is another aspect of weighted embeddings, particularly concerning those like (1.5) which we wish to point out here. The spaces $W^{1,p}(\Omega; d^{-\beta}, 1)$ share at least one property in common with the interpolation spaces $(A_0, A_1)_{\theta,q}$ for a Banach couple given by the real method of interpolation. We know that $W_0(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ while the embedding of W into $L^p(\Omega)$ is merely continuous. But since (1.5) happens to hold when $p = q$, there is a chain of spaces "interpolating" between W_0 and W which embed compactly into $L^p(\Omega)$. This is analogous to a similar phenomenon shown to hold for $(A_0, A_1)_{\theta,q}$ by Cwikel [11].

The main contents of the paper are presented in Sections 3 and 4. Sections 3 considers the case where no regularity condition at all is imposed on $\partial\Omega$. Here we introduce an integrability condition "I_μ^+" which is satisfied by all finite measure domains as well as by certain quasibounded domains of infinite volume. We show that many weighted embeddings (including (1.4) and (1.5)) which mimic Theorems A–C exist on these domains and explore further the analogy between some of these embeddings and interpolation. Section 4 introduces another integrability condition "I_μ^-" that is weaker than the usual Sobolev conditions. For bounded domains it is equivalent to the Minkowski dimension of $\partial\Omega < N$. It is too weak for the ordinary Sobolev embeddings (the well known "rooms and passages" example (cf. [14]) satisfies it), but it permits the embedding

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega; d^\beta) \tag{1.7}$$

for sufficiently small *negative* β and $q < p$. Thus this is a strengthening of Theorem A, but at the price of a mild condition on Ω.

We remark finally that several of the weighted embeddings presented in this paper, especially in the case $q \leq p$, are quite easy to prove; nevertheless, they appear to be new.

2 PRELIMINARIES

In this section we fix notation and present certain technical lemmas required in the main body of the paper.

Suppose v_0, v_1, w are positive a.e. measurable functions (i.e. "weights" defined on a domain $\Omega \subseteq \mathbb{R}^N$. Canonical examples of weights are powers or monotone functions of "$d(t)$", the distance to the boundary function introduced in Section 1. For $1 \le p, q < \infty$ we consider the spaces of complex-valued measurable functions $L^q(\Omega; w)$ and $W^{1,p}(\Omega; v_0, v_1)$ defined on Ω and equipped with the respective norms

$$\|u\|_{\Omega;w,q} := \left(\int_\Omega w|u|^q\right)^{1/q}, \qquad (2.1)$$
$$\|u\|_{\Omega;v_0,v_1,1,p} := (\|u\|^p_{\Omega;v_0,p} + \|\nabla u\|^p_{\Omega;v_1,p})^{1/p}.$$

In all cases the derivatives in the gradient ∇u are understood in the distributional sense. In unweighted case (that is, when $v_0 = v_1 = w = 1$) we use the notation (1.1), (1.2). If $v_0^{-1/p}, v_1^{-1/p}$ are locally $L^{p'}$ integrable where $1 < p' \le \infty$ is the conjugate exponent of p defined by $1/p + 1/p' = 1$ it is not difficult to show (see [22]) that $W^{1,p}(\Omega; v_0, v_1)$ is a Banach space. Likewise, if v_0, v_1 are locally integrable it is routine to prove that C_0^∞ is dense in $W^{1,p}(\Omega; v_0, v_1)$. This implies that we can define $W_0^{1,p}(\Omega; v_0, v_1)$ (the analogue of $W_0^{1,p}(\Omega)$) as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.1).

The notations $B_{t,R}, B(t, R)$ or simply "B_t" will denote the open ball with center t and radius R. A domain Ω is said to be *quasibounded* if

$$\lim_{|t|\to\infty} d(t) = 0$$

and *quasicylindrical* if

$$\sup_{t\in\Omega} d(t) < \infty.$$

The volume of a finite measure domain Ω will be denoted by $|\Omega|$. Given $\epsilon > 0$, we set $\Omega_{(\epsilon)} := \{t \in \Omega: d(t) < \epsilon\}$ and $\Omega^{(\epsilon)} := \Omega \backslash \bar{\Omega}_{(\epsilon)}$. If $\Omega = \mathbb{R}^N$ we interpret $\Omega_{(\epsilon)}$ as the complement of $\overline{B(0, 1/\epsilon)}$.

Constants will be denoted by capital or small letters such as K, C, c, etc., and may change their value from line to line. If we wish to emphasize a change in the value of a certain constant we use subscript notation and write K_1, K_2, etc. If $F(f), G(f)$ are two expressions defined on some underlying space of functions such that $F(f) \le KG(f)$ or $KG(f) \le F(f) \le K_1 G(f)$ for fixed constants K, K_1 whose precise value is immaterial to the argument, it will be often convenient to write $F(f) \preceq G(f)$ or $F(f) \approx G(f)$; in the particular case $F(f) \leq K\epsilon G(f)$ for a small ϵ we write $F(f) = O(\epsilon)G(f)$.

The following lemma follows from invoking one of Theorems A, B, or C on the unit ball followed by a change of variables.

LEMMA 2.1 *Let* $1 \leq q, p < \infty$, *Then the inequality*

$$\int_{B_t} |u|^q \leq K\left\{ R^{-q(N/p-N/q)}\left(\int_{B_t} |u|^p\right)^{q/p} + R^{q(1-N/p+N/q)}\left(\int_{B_t} |\nabla u|^p\right)^{q/p}\right\} \tag{2.1a}$$

holds for all $u \in W^{1,p}(B_t)$ *with constant K depending only on N if*

(i) $p > N$;
(ii) $p \leq N$ *and* $p \leq q \leq p^*$;
(iii) $q < p$.

Also if $p > N$ *the inequality* (*which implies* (2.1a))

$$\sup_{t\in B_t} |u(t)| \leq K\left\{ R^{-N/p}\left(\int_{B_t} |u|^p\right)^{1/p} + R^{1-N/p}\left(\int_{B_t} |\nabla u|^p\right)^{1/p}\right\} \tag{2.1b}$$

is true. Finally, the mappings from $W^{1,p}(B_t)$ *to* $L^q(B_t)$ *or to* $L^\infty(B_t)$ *defined by* (2.1a) *or* (2.1b) *are compact except when* $p = p^*$ *in* (ii).

The next lemma gives a well-known necessary and sufficient abstract condition for there to be a compact embedding of $W^{1,p}(\Omega; v_0, v_1)$ into $L^q(\Omega; w)$. Given a domain Ω, suppose $\mathcal{C} := \{\Omega_n\}_{i=1}^\infty$ is an infinite nested chain of bounded domains, i.e.,

$$\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \Omega_{n+1} \subset \cdots \subset \Omega,$$

such that

$$\Omega = \bigcup_{i=1}^\infty \Omega_n.$$

For proofs of the following fundamental result, see [8, Theorem 4.1] or [25, Chapter 3, §17]. For the original unweighted prototype with $p = 2$, see [2].

Lemma 2.2 *Suppose for all* $\Omega_n \in \mathcal{C}$,

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega_n, w).$$

Then

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega, w)$$

if, and only if, given $\epsilon > 0$ *there exists an integer* $n(\epsilon)$, *and all* $u \in W^{1,p}(\Omega; v_0, v_1)$, *such that for all* $n > n(\epsilon)$

$$\|u\|_{\Omega \sim \Omega_n} = \mathrm{O}(\epsilon)\|u\|_{\Omega; v_0, v_1, 1, p}$$

holds.

The final tool we shall require is the Besicovitch Covering Lemma. For the proof see [17, Theorem 1.1, p. 2].

Lemma 2.3 *If* $\mathcal{S}$ *is a system of cubes or balls covering a bounded domain* Ω *such that every* $t \in \Omega$ *is the center of an element* $B \in \mathcal{S}$, *then there exists a subcovering* $\mathcal{T} \subset \mathcal{S}$ *consisting of finitely many subfamilies* Γ_i, $i = 1, \ldots, \chi(N)$, *of* $\mathcal{S}$ *such that*

(i) *each* Γ_i *consists of mutually disjoint members of* $\mathcal{S}$;
(ii) *each* $B \in \mathcal{T}$ *intersects at most* $\chi(N)$ *members of* $\mathcal{T}$;
(iii) *the number* $\chi(N)$ *depends only on the dimension* N.

Additionally if Ω *is unbounded a subcover* $\mathcal{T}$ *exists satisfying* (i)–(iii) *if the members of* $\mathcal{S}$ *are uniformly bounded.*

3 WEIGHTED EMBEDDINGS ON A GENERAL CLASS OF DOMAINS WITH NO REGULARITY CONDITIONS ON THE BOUNDARY

In this section we prove some embedding results for weighted Sobolev and Lebesgue spaces where the weights mainly are powers of $d(t)$ and are defined either on domains of finite measure or on a more general type of quasibounded domain.

DEFINITION 3.1 A domain Ω is *quasicylindrical* if

$$\sup_{t\in\Omega} d(t) = D < \infty,$$

and *quasibounded* if

$$\lim_{t\in\Omega,|t|\to\infty} d(t) = 0.$$

Clearly every domain with finite measure is quasibounded but the reverse implication is not true.

DEFINITION 3.2 Ω is an (I_μ^+) domain or $\Omega \in (\mathrm{I}_\mu^+)$ if

(I_μ^+) there is a number $\mu \geq 0$ such that $\int_\Omega d(t)^\mu \, \mathrm{d}t = M < \infty$.

If this integrability condition holds for arbitrarily small μ, we say that Ω is an $(\mathrm{I}_0^{+\prime})$ domain or $\Omega \in (\mathrm{I}_0^{+\prime})$. This is to be distinguished from $\Omega \in (\mathrm{I}_0^+)$ which is equivalent to Ω being of finite measure.

Remark 3.1 The following properties of the (I_μ^+) condition are easily verified.

(i) $\Omega \in (\mathrm{I}_\mu^+) \Rightarrow \Omega$ is quasibounded.

(ii) If $\Omega \in \mathrm{I}_\mu^+$ then $\Omega \in \mathrm{I}_{\mu'}^+$ if $\mu' > \mu$.

(iii) $(\mathrm{I}_0^{+\prime})$ includes non-finite measure quasibounded domains. To see this in $\mathbb{R}^2$ for example, let Ω be the union of open adjacent squares S_n, $n = 1, 2, \ldots$, of edge length $1/\sqrt{n}$ erected on the x-axis. Then on S_n $d(t) \leq 1/(2\sqrt{n})$ and for every $\mu > 0$

$$\int_\Omega d(t)^\mu \, \mathrm{d}t \leq (1/2)^\mu \sum_{n=1}^{\infty} \frac{1}{n^{1+\mu}} < \infty.$$

(iv) On the other hand, there are Ω which are in (I_μ^+) for some $\mu > 0$ but not in $(\mathrm{I}_0^{+\prime})$. Again in $\mathbb{R}^2$ we can let Ω be the union of adjacent squares S_n with edge length $1/n^{1/3}$. Then $d(t) \leq (1/2n^{1/3})$ if $t \in S_n$ and $\Omega \in (\mathrm{I}_\mu^+)$ if $\mu > 1$.

(v) Let Ω be the union of progressively thin adjacent rectangles aligned on the x-axis of length e^n and thickness $1/n$. Then

$$\int_\Omega d(t)^\mu \, \mathrm{d}t \geq \frac{1}{4^\mu \cdot 2} \sum_{n=1}^{\infty} \frac{\mathrm{e}^n - 2/n}{n^{\mu+1}} = \infty.$$

Thus there are quasibounded domains not in (I_μ^+) for any $\mu > 0$.

THEOREM 3.1 *Suppose* $1 \le q, p < \infty$, $\alpha, \beta, \gamma \in \mathbb{R}$, v_1 *is a weight*, Ω *is quasibounded, and* $\delta \ge 1$. *Then the embedding*

$$W^{1,p}(\Omega; d^\gamma, v_1) \hookrightarrow\hookrightarrow L^q(\Omega; d^\beta) \tag{3.1}$$

holds if

(i) $q < p$, $\Omega \in (\mathrm{I}_\mu^+)$, $\mu \ge 0$, *and*

$$\beta \ge (q/p)\gamma + \mu(1 - q/p); \tag{3.2a}$$

(ii) $q = p$ *and* $\beta > \gamma$;
(iii) $q > p$, $p > N$ *or* $p \le N$ *and* $p < q < p^*$, $v_1 = d^\alpha$, *and*

$$\begin{aligned} \beta &> (q/p)\gamma + \delta q(N/p - N/q) \\ &\ge (q/p)\alpha - \delta q(1 - N/p + N/q); \end{aligned} \tag{3.2b}$$

(iv) $p > N$, Ω *is quasibounded*, $q = \infty$, $v_1 = d^\alpha$, *and*

$$\beta > \begin{cases} \gamma/p + \delta(N/p), \\ \alpha/p - \delta(1 - N/p). \end{cases} \tag{3.2c}$$

Further, in Cases (ii)–(iv) *the* continuous *embedding*

$$W^{1,p}(\Omega; d^\gamma, v_1) \hookrightarrow L^q(\Omega; d^\beta) \tag{3.3}$$

holds if equality is substituted for strict inequality in the inequalities $\beta > \gamma$ *of* (ii) *or in* (3.2b) *and* (3.2c) *of* (iii) *and* (iv). *The embedding is also continuous in Cases* (ii)–(iv) *if* Ω *is quasicylindrical or* $q = p^*$ *in Case* (iii).

Proof *Case* (i): On $\Omega_{(1/n)}$ we have the chain of estimates

$$\begin{aligned}
&\left(\int_{\Omega_{(1/n)}} d^\beta |u|^q\right)^{1/q} \\
&= \left(\int_{\Omega_{(1/n)}} d^{\beta+(\mu-\mu)(1-q/p)} |u|^q\right)^{1/q} \\
&\le \left(\int_{\Omega_{(1/n)}} d^\mu\right)^{1-q/p} \left(\int_{\Omega_{(1/n)}} d^{\beta(p/q)-\mu(1-q/p)(p/q)-\gamma} d^\gamma |u|^p\right)^{1/p} \\
&\le M^{1/p} \left(\int_{\Omega_{(1/n)}} d^\mu\right)^{1-q/p} \left\{\left(\int_\Omega d^\gamma |u|^p\right)^{1/p} + \left(\int_\Omega v_1 |\nabla u|^p\right)^{1/p}\right\},
\end{aligned} \tag{3.4}$$

where

$$M = \sup_{t \in \Omega} d^{\beta(p/q)-\mu(1-q/p)(p/q)-\gamma} < \infty$$

by (3.2a) and the quasiboundedness of Ω. Finally, we take n large enough that $\int_{\Omega_{(1/n)}} d^{\mu} < \epsilon^{p(p-q)^{-1}}$ and obtain that

$$\left(\int_{\Omega_{(1/n)}} d^{\beta}|u|^{q}\right)^{1/q} = \mathrm{O}(\epsilon)\left\{\left(\int_{\Omega} d^{\gamma}|u|^{p}\right)^{1/p} + \left(\int_{\Omega} v_1|\nabla u|^{p}\right)^{1/p}\right\}. \quad (3.5)$$

Since $d(t)$ is bounded above and below on $\Omega^{(1/2n)}$ and this set is bounded, we have as an immediate consequence of Theorem A that

$$W^{1,p}(\Omega; d^{\gamma}, v_1) \hookrightarrow\hookrightarrow L^{q}(\Omega^{(1/2n)}; d^{\beta}); \quad (3.6)$$

(3.5) and (3.6) together with Lemma 2.2 now give (3.1).

Case (ii): We have since $\beta > \gamma$

$$\left(\int_{\Omega_{(1/n)}} d^{\beta}|u|^{p}\right)^{1/p} = \left(\int_{\Omega_{(1/n)}} d^{\beta-\gamma+\gamma}|u|^{p}\right)^{1/p}$$

$$\leq (1/n)^{\beta-\gamma}\left\{\left(\int_{\Omega} d^{\gamma}|u|^{p}\right)^{1/p} + \left(\int_{\Omega} v_1|\nabla u|^{p}\right)^{1/p}\right\}.$$

The proof is completed in a way similar to Case (i) with n chosen greater than $1/\epsilon$. If we repeat these estimates on Ω instead of $\Omega_{(1/n)}$ and Ω is quasicylindrical or $\beta = \gamma$ the embedding is obviously continuous.

Case (iii): For $R > 0$ let $A := B(0, R) \cap \Omega$ and for fixed $\epsilon \in (0, 1)$ and a $\delta \geq 1$ consider the cover

$$C_A := \{B_t \equiv B(t, \epsilon d(t)^{\delta}) \colon t \in A\}.$$

By Lemma 2.1 we have the inequality

$$\int_{B_t} |u|^{q} \leq K\left\{(\epsilon d(t)^{\delta})^{-q(N/p-N/q)}\left(\int_{B_t} |u|^{p}\right)^{q/p} + (\epsilon d(t)^{\delta})^{q(1-N/p+N/q)}\left(\int_{B_t} |\nabla^{m} u|^{p}\right)^{q/p}\right\}.$$

Moreover $B_t \subset \Omega$ and if $s \in B_t$, then

$$d(t) - \epsilon d(t)^{\delta} \leq d(s) \leq d(t) + \epsilon d(t)^{\delta}.$$

Hence if $\epsilon < 1/2$

$$\frac{1}{2} \leq \frac{d(s)}{d(t)} \leq \frac{3}{2}; \tag{3.7}$$

and so

$$\int_{B_t} d(s)^{\beta}|u|^q \leq \left\{K_1(\epsilon)d(t)^{\phi}\left(\int_{B_t} d(s)^{\gamma}|u|^p\right)^{q/p} + K_2(\epsilon)d(t)^{\theta}\left(\int_{B_t} d(s)^{\alpha}|\nabla^m u|^p\right)^{q/p}\right\}, \tag{3.8}$$

where

$$\begin{aligned} K_1(\epsilon) &:= K\epsilon^{-q(N/p-N/q)}, \\ K_2(\epsilon) &:= K\epsilon^{q(1-N/p+N/q)}, \\ \phi &:= \beta - (q/p)\gamma - \delta q(N/p - N/q), \\ \theta &:= \beta - (q/p)\alpha + \delta q(1 - N/p + N/q). \end{aligned}$$

By (3.2b) and the fact that $d(t)$ is bounded $d(t)^{\phi}$ and $d(t)^{\theta}$ are uniformly bounded by a positive constant M on Ω. Thus (3.8) has the form

$$\int_{B_t} d(s)^{\beta}|u|^q \leq K(\epsilon)M\left\{\left(\int_{B_t} d(s)^{\gamma}|u|^p\right)^{q/p} + \left(\int_{B_t} d(s)^{\alpha}|\nabla^m u|^p\right)^{q/p}\right\} \tag{3.9}$$

where $K(\epsilon) := \max\{K_1(\epsilon), K_2(\epsilon)\}$.

Since A is bounded it follows from Lemma 2.3 that $\mathcal{C}_A$ may be decomposed into finitely many families $\Gamma_1, \ldots, \Gamma_{\chi(N)}$ of disjoint balls where the number $\chi(N)$ depends only on the dimension N and not on A. Addition of (3.9) over Γ_i yields the inequality

$$\int_{\Gamma_i} d(s)^{\beta}|u|^q \leq KM\left\{\left(\int_{\Gamma_i} d(s)^{\gamma}|u|^p\right)^{q/p} + \left(\int_{\Gamma_i} d(s)^{\alpha}|\nabla u|^p\right)^{q/p}\right\}.$$

Hence

$$\int_A d(s)^\beta |u|^q \leq \int_{\cup \Gamma_i} d(s)^\beta |u|^q \leq \chi(N) KM \left\{ \left(\int_\Omega d^\gamma(s)|u|^p \right)^{q/p} + \left(\int_\Omega d^\alpha(s)|\nabla u|^p \right)^{q/p} \right\}. \tag{3.10}$$

Here we use the elementary inequality $\sum A_i^c \leq (\sum A_i)^c$ for $c > 1$ (where of course $c = q/p$) on each of the right-hand integral terms. Since (3.10) is independent of A, it remains true if Ω is substituted for A in the left-hand integral.

By these arguments for Case (iii) we have shown that there is a continuous embedding $\mathcal{J}$ of $W^{1,p}(\Omega; d^\gamma, d^\alpha)$ into $L^q(\Omega; d^\beta)$ if Ω is quasicylindrical. It is also clear that the proof did not require strict inequality in the first inequality of (3.2c) or that $q < p^*$.

We next demonstrate that $\mathcal{J}$ is compact in Case (iii) if Ω is quasibounded, $p \leq q < p^*$ when $p \leq N$, and the strict inequality in (3.2b) holds. Let $\epsilon > 0$ be chosen and take n large enough that $d(t)^\phi < \epsilon^{1 + q(N/p - N/q)}$ and $d(t)^\theta < M < \infty$ on $\Omega_{(1/n)}$. (This is possible by (3.2b).) Let "Ω" $= \Omega_{(1/n)}$, and apply the reasoning leading to (3.7)–(3.10) using a bounded set $A \subset \Omega_{(1/n)}$ for the Besicovitch part of the argument. We will end up with the equivalent of (3.5). By [12, Theorem 4.20] we can always construct a nested chain $C \equiv \{\Omega_n\}_{i=1}^\infty$ of bounded domains such that each $\Omega_n \in C$ has an analytic boundary and $\Omega \sim \Omega_n \in \Omega_{(1/n)}$. The compactness of $\mathcal{J}$ now follows by application of Lemma 2.2.

Case (iv): Let $t \in \Omega^{1/n}$ where $n > 1/\epsilon$. Then (2.1b) of Lemma 2.1, (3.2c), (3.7), and extension of the right-hand integrals to all of Ω imply that

$$\sup_{s \in B_t} |d(s)^\beta u(s)| \leq K\epsilon^\eta \|u\|_{\Omega;|m|,d^\gamma,d^\alpha,r,p}$$

where $\eta = \min\{\beta - (\delta N + \gamma)/p, \beta + \delta(1 - N/p) - \alpha/p\} > 0$. Consequently, we may conclude again that

$$\|u\|_{\Omega_\epsilon; d^\beta, \infty} = \mathrm{O}(\epsilon)(\|u\|_{\Omega;|m|,d^\gamma,d^\alpha,p}).$$

The remaining details to establish continuity and compactness are left to the reader.

Remark 3.2 Note that the case $\mu=\alpha=\beta=\gamma=0$ in Theorem 3.1(i) gives the extension of Theorem A to finite measure domains due to Carnavati and Fontes.

By eliminating δ by equating the right-hand expressions in the inequality pairs (3.2b) and (3.2c) we obtain

COROLLARY 3.1 *The compact embedding* (3.1) *is true if* $(\alpha-\gamma)/p\geq 1$ *and*

(i) $q>p$, Ω *is quasibounded*, $p>N$ *or* $p\leq N$ *and* $p\leq q<p^*$, *and*

$$\beta/q>(\gamma/p)(1+N/q-N/p)+(\alpha/p)(N/p-N/q);$$

(ii) $q=\infty$, Ω *is quasibounded*, $p>N$, *and*

$$\beta>\gamma/p+N(\alpha-\gamma)/p^2.$$

COROLLARY 3.2 *Suppose* $1\leq q,p<\infty$. *Then the embeddings*

$$W^{1,p}(\Omega)\hookrightarrow\hookrightarrow L^q(\Omega;d^{\beta}),$$
$$W^{1,p}(\Omega;d^{-\beta},1)\hookrightarrow\hookrightarrow L^q(\Omega)$$

hold if

(i) $q<p$, $\Omega\in(\mathrm{I}_0^{+\prime})$, *and* $\beta>0$ *or* $\Omega\in(\mathrm{I}_0^+)$ *and* $\beta\geq 0$;

(ii) $\beta>p(N/p-N/q)$; Ω *is quasibounded*, $q\geq p$, *and either* $p>N$ *or* $p\leq N$ *and* $q<p^*$.

One interpretation of Corollary 3.2 is that *any* bounded domain Ω "nearly" satisfies the standard Rellich–Kondrachov compact embedding theorems if we are willing to replace a regularity assumption on $\partial\Omega$ by a mildly singular or degenerate weight on the left or right side of the embedding. However it is not essential that the weight be a power of $d(t)$. It is possible to generalize both Theorem 3.1 and Corollary 3.2 by considering an arbitrary weight w. We begin by replacing the integrability condition (I_μ^+) by:

$(\mathrm{I}_{\mu,w}^+)$ there is a number $\mu>0$ such that $\int_\Omega w^\mu\,\mathrm{d}t=M<\infty$.

Also the following conditions will replace the inequality possibilities in (3.2b) and (3.2c):

(H1) $\lim_{t\to\partial\Omega} w(t)^{\beta-(q/p)\gamma}d(t)^{-q\delta(N/p-N/q)}=0$;

(H1′) $\limsup_{t\to\partial\Omega} w(t)^{\beta-(q/p)\gamma}d(t)^{-q\delta(N/p-N/q)}<\infty$;

(H2) $\limsup_{t\to\partial\Omega} w(t)^{\beta-(q/p)\alpha}d(t)^{q\delta(1-N/p+N/q)}<\infty$;

(H3) $\lim_{t\to\partial\Omega} w(t)^{\beta-\gamma/p} d(t)^{-\delta(N/p)} = 0$;
(H3′) $\lim_{t\to\partial\Omega} w(t)^{\beta-\gamma/p} d(t)^{-\delta(N/p)} < \infty$;
(H4) $\limsup_{t\to\partial\Omega} w(t)^{\beta-(\alpha/p)} d(t)^{\delta(1-N/p)} < \infty$;
(H5) There exist positive constants $C, D > 0$ such that

$$C \le \frac{w(s)}{w(t)} \le D$$

on $B_t \equiv B(t, \epsilon d(t)^\delta)$ for t sufficiently near $\partial\Omega$.

With these conditions we can obtain the following extension of Theorem 3.1.

THEOREM 3.2 *Suppose* $1 \le q, p < \infty$, *and* $\delta \ge 1$. *Then the embedding*

$$W^{1,p}(\Omega; w^\gamma, v_1) \hookrightarrow\hookrightarrow L^q(\Omega; w^\beta)$$

holds if

(i) $q < p$, $\Omega \in (\mathrm{I}^+_{\mu,w})$, *and* β, γ *satisfy* (3.2a);
(ii) $q = p$ *and* $\beta > \gamma$;
(iii) $q > p$, Ω *is quasibounded*, $v_1 = d^\alpha$, $p > N$ *or* $p \le N$ *and* $p \le q < p^*$, *and the conditions* (H1), (H2), (H5) *hold*;
(iv) $p > N$, Ω *is quasibounded*, $v_1 = d^\alpha$, $q = \infty$, *and the conditions* (H3), (H4), (H5) *hold.*

In Cases (iii)–(iv) *the* continuous *embedding*

$$W^{1,p}(\Omega; w^\gamma, v_1) \hookrightarrow L^q(\Omega; d^\beta)$$

holds if (H1′) *is substituted for* (H1) *in* (iii) *and* (H3′) *is substituted for* (H3) *in* (iv). *The embedding is also continuous in Cases* (iii) *and* (iv) *if* Ω *is quasicylindrical or in Case* (iii) *if* $q = p^*$.

Proof We just retrace the argument of Theorem 3.1. For (i) and (ii) we substitute "w" for the distance function throughout. The condition $(\mathrm{I}^+_{\mu,w})$ replaces (I^+_μ). In the remaining parts we choose the same balls $B_t \equiv B(t, \epsilon d(t)^\delta)$ in the cover and use w as the weight. In (iii), (H1) and (H2) replace (3.2b); similarly (H3) and (H4) replace (3.4c), and so forth.

It is easy to give examples of weights w with much weaker (or stronger) degenerate or singular behavior at $\partial\Omega$ than $d(t)$ or $d(t)^{-1}$. For instance, we can define

$$u_0(x) := \log(1 + (d(x))^{1/2})$$
$$u_j(x) := \log(1 + (u_{j-1}(x))^{1/2}), \quad j = 1, 2, \dots$$

Then for all j

$$\lim_{t\to\partial\Omega} u_{j+1}(x)/u_j(x) = \infty.$$

Set $w := u_k$ for some (large!) k. Since $1/2 \le \mathrm{d}s/\mathrm{d}t \le 3/2$ if $s \in B_t$ with $\epsilon < 1/2$, we see (provided $d(t)$) is small enough that

$$\frac{u_0(d(s))}{u_0(d(t))} \equiv \frac{\log(1 + (d(s))^{1/2})}{\log(1 + (d(t))^{1/2})} \le \frac{\log(1 + \sqrt{3/2}(d(t))^{1/2})}{\log(1 + (d(t))^{1/2})} \le 2\sqrt{\frac{3}{2}},$$
$$\frac{u_0(d(t))}{u_0(d(s))} \equiv \frac{\log(1 + (d(t))^{1/2})}{\log(1 + (d(s))^{1/2})} \le \frac{\log(1 + \sqrt{2}(d(s))^{1/2})}{\log(1 + (d(s))^{1/2})} \le 2\sqrt{2},$$

so that

$$\frac{1}{2\sqrt{2}} \le \frac{u_0(d(s))}{u_0(d(t))} \le 2\sqrt{\frac{3}{2}}.$$

An inductive argument will show that the same bounds hold for $u_{[k]}(d(s))/u_{[k]}(d(t))$, $k = 1, 2, \dots$

In this case it will follow that:

COROLLARY 3.3 *For $q \le p$, any positive integer k, and $\beta > 0$*

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega; u_k^{\beta}),$$
$$W^{1,p}(\Omega; u_k^{-\beta}, 1) \hookrightarrow\hookrightarrow L^q(\Omega) \tag{3.11}$$

provided also $\Omega \in (\mathrm{I}^+_{\mu,w})$ for any $\mu > 0$ unless $q = p$ in which case Ω may be quasibounded.

Notice that Corollary 3.3 does not apply if $q > p$ because (H1) is not satisfied. However, if $w(t)$ goes to zero more rapidly than $d(t)$ (e.g., take $w = e^{-1/d(t)}$) and $q > p$ then the embedding will hold when $\beta > (q/p)\gamma$ and $\beta \geq (q/p)\alpha$ and $p > N$ or $q \leq p^*$ and $p \leq N$.

In these results we have not only replaced a cone condition on Ω or a regularity condition on $\partial\Omega$ by a weakly singular or degenerate weight but also Ω need not be bounded. In the unweighted case the embedding is almost never compact on unbounded domains; an old result of Adams and Fournier (see [1]) shows that a necessary condition for compactness of the embedding of $W^{1,p}(\Omega)$ into $L^q(\Omega), q \geq p$, is that

$$\lim_{R\to\infty} e^{\lambda R}|\Omega\backslash B(0,R)| = 0$$

for all $\lambda > 0$. (I_μ^+) or quasiboundedness are less restrictive conditions.

From an "interpolation" point of view in the case $m = 1$ Corollaries 3.2 and 3.3 state that there exist (many!) nested chains of weighted Sobolev spaces between $W^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ enjoying the Rellich–Kondrachov embedding property. For example, if we set $w_{k,\beta}^{1,p}(\Omega) := W^{1,p}(\Omega; u_k^{-\beta}, 1)$ and take $\beta > \beta'$ we have

$$\begin{aligned} W_0^{1,p}(\Omega) \subset W_{0,\beta'}^{1,p} \subset W_{0,\beta}^{1,p} \subset \cdots \subset W_{k,\beta'}^{1,p}(\Omega) \subset W_{k,\beta}^{1,p}(\Omega) \\ \subset W_{k+1,\beta'}^{1,p}(\Omega) \subset W_{k+1,\beta}^{1,p}(\Omega) \subset \cdots W^{1,p}(\Omega), \end{aligned}$$

with all the embeddings below $W^{1,p}(\Omega)$ being compact. We can construct similar nested chains in the target spaces $L^p(\Omega, u_k^\beta)$. Clearly many other examples of the same kind using other families of weights can be constructed.

The fact that the intermediate spaces $W_{k,\beta}^{1,p}(\Omega)$ compactly embed into $L^q(\Omega)$ has a strong analogy to an abstract result of Cwikel [11] concerning the real interpolation method. To understand the parallelism we sketch a few of the essential ideas of the K-functional method of interpolation. For a complete treatment of the theory see [3 or 28]. Suppose A_0, A_1 are a pair of Banach spaces with norms $\|(\cdot)\|_{A_0}, \|(\cdot)\|_{A_1}$ or "Banach couple" which are each continuously embedded in a

Hausdorff topological vector space $\mathcal{A}$. We form the sum

$$A_0 + A_1 = \{a\colon a_0 + a_1;\ a_0 \in A_0,\ a_1 \in A_1\}$$

and endow it with the norm

$$\|a\|_{A_0+A_1} := \inf_{\substack{a_0,a_1 \\ a=a_0+a_1}} \{\|a_0\|_{A_0} + \|a_1\|_{A_1}\},$$

and the intersection $A_0 \cap A_1$ with norm

$$\|a\|_{A_0\cap A_1} := \max\{\|a\|_{A_0}, \|a\|_{A_1}\}.$$

Under these norms $A_0 + A_1$ and $A_0 \cap A_1$ become Banach spaces. For $a \in A_0 + A_1$ and $t > 0$ we define the functional

$$K(t,a) := \inf_{\substack{a_0,a_1 \\ a=a_0+a_1}} \{\|a_0\|_{A_0} + t\|a_1\|_{A_1}\}$$

and the interpolation space

$$(A_0, A_1)_{\theta,q} := \left\{ a \in A_0 + A_1\colon \|a\|_{\theta,q} := \left(\int_0^\infty \left(\frac{K(t,a)}{t^\theta} \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} < \infty \right\}.$$

The main result of the theory is that $(A_0, A_1)_{\theta,q}$ equipped with the norm $\|(\cdot)\|_{\theta,q}$ becomes a Banach space such that

$$A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{\theta,q} \hookrightarrow A_0 + A_1. \tag{3.12}$$

Cwikel [11] proved the following result concerning interpolation spaces.

THEOREM D *Let $\bar{A}, \bar{B}$ be Banach couples and suppose that the linear operator $T\colon A_0 + A_1 \to B_0 + B_1$ is bounded. Suppose that $T\colon A_0 \to B_0$ is compact. Then $T\colon (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$ compactly for $\theta \in (0,1)$ and $p \in [1, \theta]$.*

The prototype of this theorem is a result of Krasnosel'ski [21] who proved that if T mapped $L^{p_i}(\mathbb{R})$ continuously into $L^{q_i}(\mathbb{R})$, $1 \le p_0, p_1, q_1 \le \infty$, $1 \le q_0 < \infty$, and additionally $T: L^{p_0}(\mathbb{R}) \to L^{q_0}(\mathbb{R})$ is compact, then $T: L^p(\mathbb{R}) \to L^q(\mathbb{R})$ where

$$1/p = (1-\theta)1/p_0 + \theta/p_1,$$
$$1/q = (1-\theta)1/q_0 + \theta/q_1,$$

where $0 < \theta < 1$.

The resemblance between (3.11) and Cwikel's theorem is clear if we take $\bar{A} = (W_0^p(\Omega), W^{1,p}(\Omega))$ and $\bar{B}$ the pair of images of $W_0^p(\Omega)$, $W^{1,p}(\Omega)$ in $L^q(\Omega)$ under the embedding map T. Then $W^{1,p}(\Omega; u_{[k]}^{\beta}, 1)$ "corresponds" to $(A_0, A_1)_{\theta,q}$. However this analogy is only formal; for (as in the present case) when A_0 is a topological *subspace* of A_1, the K-functional method is vacuous and gives the same intermediate space for all θ, q – namely A_0. This follows because (see, e.g., [3, Theorem 2.9]) $A_0 \equiv A_0 \cap A_1$ is dense in $(A_0, A_1)_{\theta,q}$ for $\theta \in (0,1)$, $q \in [1,\infty)$ and the embeddings (3.12) entail that the norms $\|(\cdot)\|_{\theta,q}$ and $\|(\cdot)\|_{A_0}$ are equivalent on A_0. Nevertheless, it may be of some interest that we can exhibit chains of spaces intermediate between W_0 and W which are not given by interpolation and yet have the behavior of Theorem D.

4 EMBEDDINGS WITH A WEAK INTEGRABILITY CONDITION ON Ω

We now change the (I_μ^+) condition to allow μ to be negative. We say that Ω is an (I_μ^-) domain or "$\Omega \in (\mathrm{I}_\mu^-)$" if

(I_μ^-) there is a number $-N < \mu < 0$ such that $\int_\Omega d(t)^\mu \, \mathrm{d}t = M < \infty$.

PROPOSITION 4.1 *If Ω is an (I_μ^-) domain, then Ω has finite measure.*

Proof Suppose that Ω is not quasicylindrical. Then for each $n \in \mathbb{Z}$ we can find $t_n \in \Omega$ such that $d(t_n) \to \infty$. Consider the ball $B_n \equiv B(t_n, d(t_n)) \subset \Omega$. If $s \in B_n$ then $d(s) \leq 2d(t_n)$. Hence

$$\begin{aligned} \infty > \int_\Omega d(t)^\mu \geq \int_{B_n} d(t)^\mu \\ \geq 2^\mu d(t_n)^\mu |B_n| \\ \geq |B(0,1)| 2^\mu d(t_n)^{N+\mu} \to \infty, \end{aligned}$$

which is a contradiction.

On the other hand, if Ω is quasicylindrical, then and (I_μ^-) is true then

$$\infty > \int_\Omega d(t)^\mu > D^\mu |\Omega|$$

where $D = \sup_{t\in\Omega} d(t)$ so that Ω has finite measure.

An (I_μ^-) domain, however, need not be bounded. Suppose $f\colon \mathbb{Z} \to \mathbb{R}^+$ is any function such that (a) $\sum_1^\infty f(n)$ diverges (b) $\sum_1^\infty f(n)^{2+\mu}$ converges. Let $N = 2$ and C_n be a family of pairwise tangent circles of radii $r_n := f(n)$ and centers c_n with coordinates $(f(n) + 2\sum_1^{n-1} f(k), 0)$. Define

$$\Omega := \bigcup_{n=1}^\infty \Omega_n,$$

where Ω_n is the interior of the circle with radius $(1+\epsilon)r_n$ and center c_n and $0 < \epsilon \ll f(n)$. Ω is unbounded since $c_n \to \infty$. Further if $t \in \Omega_n$ then $d(t) \geq f(n) - |t - c_n|$. Introducing polar coordinates (with $r := |t - c_n|$) we find that

$$\begin{aligned}\int_{\Omega_n} d(t)^{-\mu} &\leq \int_0^{2\pi}\int_0^{(1+\epsilon)f(n)} (f(n) - r)^{-\mu} r \,\mathrm{d}r\,\mathrm{d}\theta \\ &\leq 2\pi(1+\epsilon)^{1-\mu}(1/(1-\mu) - 1/(2-\mu))f(n)^{2-\mu}.\end{aligned}$$

Thus Ω satisfies (I_μ^-).

There is a connection between the (I_μ^-) condition and the notion of Minkowski dimension.

Definition 4.1 Let $0 < \lambda \leq N$ and $r > 0$ and set

$$\begin{aligned} M_\Omega^\lambda(\partial\Omega; r) &:= \frac{|(\partial\Omega + B(0,r)) \cap \Omega|}{r^{N-\lambda}}, \\ M_\Omega^\lambda(\partial\Omega) &:= \limsup_{r\to 0+} M_\Omega^\lambda(\partial\Omega; r), \\ \dim_{M,\Omega}(\partial\Omega) &:= \inf\{\lambda\colon M_\Omega^\lambda(\partial\Omega) < \infty\}. \end{aligned}$$

The last two of these quantities are called respectively the λ-*dimensional Minkowski precontent* and *Minkowski dimension* of $\partial\Omega$. A consequence of the definition is that $\dim_{M,\Omega}(\partial\Omega) \leq N$. However this dimension need not be strictly less than N. There exists Ω such that $M_\Omega^\lambda(\partial\Omega) = \infty$ for all $\lambda \in (0, N)$ (see [13, Remark 4.3] and the associated reference).

Next, recall that a Whitney covering $\mathcal{W}$ of Ω is a family of cubes Q each of edge length $L_Q = 2^{-k}$, $k \in \mathbb{N}$ and diameter $D_Q = L_Q\sqrt{N}$ such that the following five properties hold:

(i) $\Omega = \cup_{Q \in \mathcal{W}} Q$;
(ii) the interiors of distinct cubes are disjoint;
(iii) $1 \leq \operatorname{dist}(Q, \partial\Omega)/D_Q \leq 4$;
(iv) $\frac{1}{4} \leq \operatorname{diam}(Q_1)/\operatorname{diam}(Q_2)$ if $Q_1 \cap Q_2 \neq \emptyset$;
(v) at most 12^N other cubes in $\mathcal{W}$ can touch a fixed $Q \in \mathcal{W}$; further for fixed $t \in (1, \frac{5}{4})$ each $x \in \Omega$ lies in at most 12^N of the dilated cubes tQ, $Q \in \mathcal{W}$.

It is known [27] that such a covering exists for any bounded Ω. Condition (iii) in particular means that there are fixed constants c_1, c_2 such that

$$c_1 L_Q \leq d(s) \leq c_2 L_Q \tag{4.1}$$

for any $s \in Q$. Note that we can take in (4.1) $c_1 = \sqrt{N}$ and $c_2 = 5\sqrt{N}$.

Now let $n(k)$ denote the number of cubes in $\mathcal{W}_k$ where

$$\mathcal{W}_k := \{Q \in \mathcal{W} \colon L_Q = 2^{-k}\}$$

and k is a positive integer. The domain Ω is said to satisfy a Whitney cube #-condition if there is a continuous increasing function $h : (0, \infty) \to (0, \infty)$ such that $n(k) \leq h(k)$ for all $k \geq k_0 \geq 1$.

The following lemma as well as (i) and (ii) of Lemma 4.2 were proved in [7, Lemma 6.1 and Proposition 6.1] (also see [23, Theorem 3.11; 13, Lemma 2.2]).

LEMMA 4.1 *Let Ω be bounded and $\lambda \in (0, N]$. Then $M_\Omega^\lambda(\partial\Omega) < \infty$ if, and only if, $n(k) \leq K2^{\lambda k}$ for all $k \geq k_0$ where K and k_0 are finite positive constants.*

LEMMA 4.2 *Suppose Ω is bounded and let $\dim_{M,\Omega}(\partial\Omega) = \lambda$. Then the following conditions are equivalent:*

(i) $\lambda < N$.
(ii) $\Omega \in \mathrm{I}_\mu^-$ *for some* $\mu < 0$.

Moreover, if $\lambda < N$, then the set of μ for which (ii) *is true is exactly the open interval* $(-(N - \lambda), 0)$.

Proof We give here only the proof of the final part of the lemma. By Lemma 4.1 and the definition of $\dim_{M,\Omega}(\partial\Omega)$ if $\dim_{M,\Omega}(\partial\Omega) \equiv \lambda < N$, then λ must be the least number such that $n(k) \leq K2^{\lambda k}$. Therefore there exists a subsequence $\{n(k_j)\}$, $n(k_j) = K2^{\lambda k_j}$. Then as in Proposition 6.1 of [7] we have that

$$\int_\Omega d(s)^{-(N-\lambda)} \geq K\left\{\sum_{j=1}^{\infty} 2^{k_j(\lambda+(N-\lambda)-N}\right\} = \infty.$$

On the other hand if $0 > \mu > -(N-\lambda)$, we find that

$$\int_\Omega d(s)^{\mu} \preceq \left\{K\sum_{j=1}^{\infty} 2^{k(\lambda-N-\mu)}\right\} < \infty.$$

Thus μ may be an arbitrary element of $(-(N-\lambda), 0)$.

Lemma 4.2 shows that (I_μ^-) can be a very weak condition. For instance, since it can be shown that John domains, domains satisfying a quasihyperbolic boundary condition, domains satisfying a cone condition or minimal smoothness conditions on their boundary or convex domains have the property that $\partial\Omega$ has Minkowski dimension $<N$ we can conclude that the class of (I_μ^-) domains includes all of these if $|\mu|$ is small enough. At the same time, however, the condition is not strong enough to yield the classical embedding theorems for it is routine to show that it applies to domains of the "rooms and passages" type which are counterexamples to these theorems.

We are now in a position to improve Theorem 3.1 when $q < p$.

THEOREM 4.1 *Suppose* $\Omega \in (\mathrm{I}_\mu^-)$ *or equivalently that* $\dim_{M,\Omega}(\partial\Omega) < N + \mu$, *and* $q < p$. *Then the compact embedding* (3.1) *of Theorem* 3.1 *holds if*

$$\beta > \gamma - |\mu|(1 - q/p). \tag{4.2}$$

Proof The argument is exactly the same as for Theorem 3.1(i) except that now μ is *negative* and (I_μ^-) is substituted for (I_μ^+).

COROLLARY 4.1 *Suppose* Ω *is* (I_μ^-), *and* $q < p$ *then*

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega; d^\beta),$$
$$W^{1,p}(\Omega; d^\gamma, 1) \hookrightarrow\hookrightarrow L^q(\Omega)$$

if $0 > \beta > -|\mu|(1 - q/p)$ *and* $\gamma < |\mu|(1 - q/p)$.

Remark 4.1 This corollary which is an immediate consequence of (4.2) says that W embeds compactly into $L^q(\Omega; d^\beta)$ for some *negative* powers of $d(t)$ or into $L^q(\Omega)$ for sufficiently small positive γ provided $q<p$, Ω has finite measure, and satisfies the (I_μ^-) condition. Thus (I_μ^-) gives a better embedding than Theorem A or the extension of it to finite measure domains given in [10].

The next result includes the case $q \geq p$ but depends on much more complicated arguments than those given for $q<p$. We first present some preliminary definitions and lemmas.

DEFINITION 4.2 A domain Ω is said to be a *(q,p)-Poincaré domain*, for $1 \leq p, q < \infty$ if the inequality

$$\left(\int_\Omega |u - u_\Omega|^q\right)^{1/q} \leq K(p,q,\Omega)\left(\int_\Omega |\nabla u|^p\right)^{1/p}$$

holds. Here

$$u_\Omega := |\Omega|^{-1}\int_\Omega u.$$

The constant $K(p,q,\Omega)$ which is independent of u is called the Poincaré constant of Ω. If $p=q$ we call Ω a *p-Poincaré* domain.

The question of exactly which domains are p-Poincaré or (q,p)-Poincaré has been much investigated see e.g. [18–20]. The only fact we need here, however, is that cubes are p-Poincaré and hence have a Poincaré constant. We can then prove

LEMMA 4.3 *If $q \in [1,p^*]$ when $p \leq N$ or $q \in [1,\infty)$ when $p>N$ and Q is a cube, then*

$$\left(\int_\Omega |u - u_Q|^q\right)^{1/q} \leq K(q,N)|Q|^{1/N+1/q-1/p}\left(\int_\Omega |\nabla u|^p\right)^{1/p}$$

for all $u \in W^{1,p}(Q)$.

Proof If $q<p$ we obtain from the q-Poincaré inequality and Hölder's inequality that

$$\left(\int_\Omega |u - u_Q|^q\right)^{1/q} \leq K(q,N)|Q|^{1/N}\left(\int_\Omega |\nabla u|^q\right)^{1/q}$$

$$\leq K(q,N)|Q|^{1/N+1/q-1/p}\left(\int_\Omega |\nabla u|^p\right)^{1/p}$$

Under the other conditions on q we have by Lemma 2.1 and the p-Poincaré inequality that

$$\left(\int_\Omega |u-u_Q|^q\right)^{1/q} \le K(p,q,N)\left\{|Q|^{1/q-1/p}\left(\int_\Omega |u-u_Q|^p\right)^{1/p} + |Q|^{1/N-1/q-1/p}\left(\int_Q |\nabla u|^p\right)^{1/p}\right\}$$
$$\le K_1(p,q,N)|Q|^{1/N-1/q-1/p}\left(\int_Q |\nabla u|^p\right)^{1/p}.$$

LEMMA 4.4 *If $1 \le q \le p < \infty$, w is a weight on Ω and A is a measurable subset of Ω such that $\int_\Omega w < \infty$. Let*

$$u_{A,w} := \frac{\int_A wu}{\int_A w}.$$

Then for all $u \in L^p(\Omega; w)$ and $c \in \mathbb{R}$,

$$\left(\int_\Omega w|u-u_A|^q\right)^{1/q} \le K\left(\int_\Omega w|u-c|^p\right)^{1/p},$$

where

$$K = \left(\int_\Omega w\right)^{1-q/p} + \left(\int_A w\right)^{-1/p}\left(\int_\Omega w\right)^{1/q}.$$

Proof The Lemma extends [13, Lemma 2.3] and the proof is similar. By the triangle inequality

$$\left(\int_\Omega w|u-u_{A,w}|^q\right)^{1/q} \le \left(\int_\Omega w|u-c|^q\right)^{1/q} + \left(\int_\Omega w|c-u_{A,w}|^q\right)^{1/q}.$$

But by Hölder's inequality

$$\left(\int_\Omega w|c-u_{A,w}|^q\right)^{1/q} \le |c-u_{A,w}|\left(\int_\Omega w\right)^{1/q}$$
$$\le \left|\int_A w(c-u)\right|\left(\int_A w\right)^{-1}\left(\int_\Omega w\right)^{1/q}$$
$$\le \left(\int_A w|c-u|^p\right)^{1/p}\left(\int_A w\right)^{-1/p}\left(\int_\Omega w\right)^{1/q}.$$

Also by Hölder's inequality

$$\left(\int_\Omega w|u-c|^q\right)^{1/q} \le \left(\int_\Omega w|u-c|^p\right)^{1/p}\left(\int_\Omega w\right)^{1-q/p}.$$

The result follows by combination of these estimates.

DEFINITION 4.3 Suppose we are given a covering $\mathcal{F}$ by cubes of a domain Ω. Let Q_0, Q be respectively a fixed and arbitrary member of $\mathcal{F}$. We call a finite sequence of cubes $\{Q_i\}_0^{n_Q}$ joining Q_0 and $Q_{n_Q} \equiv Q$ a *Poincaré chain* if

(i) $Q_i \cap Q_j \neq \emptyset$, $|i-j| \le 1$, $0 \le i, j \le n_Q - 1$.
(ii) $\max\{|Q_i|, |Q_{i+1}|\} \le K|Q_i \cap Q_{i+1}|$, $i = 0, \dots, n_Q - 1$ where

$$\sup_{Q\in\mathcal{F}} K(p,Q) \le K < \infty.$$

Further the number n_Q is called the *length* of the chain.

THEOREM 4.2 *Suppose* $\Omega \in (\mathrm{I}_\mu^-)$ *and bounded*, $1 \le p, q < \infty$,

$$\beta \ge \begin{cases} \mu, \\ \alpha/p + N(1/p - 1/q) - 1, \end{cases}$$

$\gamma < (p-1)|\mu|$, $\alpha < (p-1)(|\mu| - N) + p - N$, *and that* $q \in [p, p^*]$ *when* $p \le N$. *Then*

$$W^{1,p}(\Omega; d^\gamma, d^\alpha) \hookrightarrow L^q(\Omega; d^\beta). \tag{4.3}$$

Further if additionally $q \in [p, p^*)$ *when* $p < N$ *and the conditions*

(i) $\gamma/p < \beta/q + N(1/q - 1/p)$,
(ii) $\beta/q > \max\{\alpha/p + N(1/p - 1) - 1, \mu\}$

are satisfied, then the embedding (4.3) *is compact.*

Proof Our strategy will be to first obtain a Poincaré-type inequality and then derive (4.3) by means of a triangle inequality argument. Our methods are similar to those of Bojarski [5] and Edmunds and Hurri [13].

Let $\mathcal{W}$ be a Whitney covering of Ω. For $Q \in \mathcal{W}$ define $\tilde{Q} := (9/8)Q$. It is known (see [18, Proposition 6.1]) that there exists a Poincaré chain $\mathcal{C}(\tilde{Q}) := \{\tilde{Q}_0, \tilde{Q}_1, \ldots, \tilde{Q}_{n_Q-1}, \tilde{Q}\}$ joining a fixed cube $\tilde{Q}_0 \in \mathcal{W}$ and an arbitrary $Q \in \mathcal{W}$. Let n_Q be the length of the chain and set $Q_{n_Q} = Q$. We are going to derive the inequality

$$I \equiv \left(\int_\Omega d^\beta |u - u_{\tilde{Q}_0}|^q\right)^{1/q} \leq K\left(\int_\Omega d^\alpha |\nabla u|^p\right)^{1/p}. \tag{4.4}$$

We begin estimating the left side of (4.4). By the triangle inequality

$$I^q \preceq \sum_{Q\in\mathcal{W}} \int_{\tilde{Q}} |u - u_{\tilde{Q}}|^q d^\beta + \sum_{Q\in\mathcal{W}} \int_{\tilde{Q}} |u_{\tilde{Q}} - u_{\tilde{Q}_0}|^q d^\beta = I_1 + I_2.$$

By the properties of Whitney cubes and the Poincaré inequality of Lemma 4.3 which holds under the stated conditions on p, q and N we obtain successively that

$$\begin{aligned}
I_1 &\preceq \sum_{Q\in\mathcal{W}} d(t_Q)^\beta \int_{\tilde{Q}} |u - u_{\tilde{Q}}|^q \\
&\preceq \sum_{Q\in\mathcal{W}} d(t_Q)^{\beta+q+N-qN/p} \left(\int_{\tilde{Q}} |\nabla u|^p\right)^{q/p} \\
&\preceq \sum_{Q\in\mathcal{W}} d(t_Q)^{\beta+q+N-qN/p-q\alpha/p} \left(\int_{\tilde{Q}} |d^\alpha \nabla u|^p\right)^{q/p} \\
&\preceq \left(\int_\Omega \sum_{Q\in\mathcal{W}} \chi_{\tilde{Q}}(t) d^\alpha |\nabla u|^p\right)^{q/p} \\
&\preceq 12^N \left(\int_\Omega d^\alpha |\nabla u|^p\right)^{q/p}.
\end{aligned} \tag{4.5}$$

Next using the triangle inequality and the chain $\mathcal{C}(\tilde{Q})$ we have that

$$I_2 \preceq \sum_{Q\in\mathcal{W}} \int_{\tilde{Q}} \sum_{j=1}^{k_Q} |u_{\tilde{Q}_j} - u_{\tilde{Q}_{j-1}}|^q d^\beta.$$

By property (ii) of Poincaré chains we find that

$$\begin{aligned}
&\sum_{j=1}^{k_Q} |u_{\tilde{Q}_j} - u_{\tilde{\Omega}_{j-1}}| \\
&= \sum_{j=1}^{k_Q} |\tilde{Q}_j \cap \tilde{Q}_{j-1}|^{-1} \int_{\tilde{Q}_j \cap \tilde{Q}_{j-1}} |u_{\tilde{Q}_j} - u_{\tilde{Q}_{j-1}}| \\
&\preceq \sum_{j=1}^{k_Q} |\tilde{Q}_j \cap \tilde{Q}_{j-1}|^{-1} \left(\int_{\tilde{Q}_j} |u - u_{\tilde{Q}_j}| + \int_{\tilde{Q}_{j-1}} |u - u_{\tilde{Q}_{j-1}}| \right) \\
&\preceq \left(\sum_{j=0}^{k_Q} |\tilde{Q}_j|^{-1+1/N} \int_{\tilde{Q}_j} |\nabla u| \right) + \left(\sum_{j=1}^{k_Q} |\tilde{Q}_{j-1}|^{-1+1/N} \int_{\tilde{Q}_{j-1}} |\nabla u| \right) \\
&\preceq \sum_{j=0}^{k_Q} |\tilde{Q}_j|^{-1+1/p'+1/N-\alpha/Np} \left(\int_{\tilde{Q}_j} d^\alpha |\nabla u|^p \right)^{1/p},
\end{aligned}$$

the last inequality following from an application of Hölder's inequality. Thus

$$I_2 \preceq \sum_{Q \in \mathcal{W}} \int_{\tilde{Q}} \left[\sum_{j=0}^{k_Q} |\tilde{Q}_j|^{-1/p+1/N-\alpha/Np} \left(\int_{\tilde{Q}_j} d^\alpha |\nabla u|^p \right)^{1/p} \right]^q d^\beta.$$

Minkowski's inequality for sums (for $p \neq 1$) then yields that

$$\begin{aligned}
I_2 &\preceq \sum_{Q \in \mathcal{W}} \int_{\tilde{Q}} \left\{ \left[\sum_{j=0}^{k_Q} |\tilde{Q}_j|^{(-1/p+1/N-\alpha/Np)p'} \right]^{q/p'} \left[\sum_{j=0}^{k_Q} \int_{\tilde{Q}_j} d^\alpha |\nabla u|^p \right]^{q/p} \right\} d^\beta \\
&\preceq \left(\int_\Omega d^\alpha |\nabla u|^p \right)^{q/p} \sum_{Q \in \mathcal{W}} \int_{\tilde{Q}} \left(\sum_{j=0}^{k_Q} d(t_{Q_j})^{Np'(-1/p+1/N-\alpha/Np)} \right)^{q/p'} d^\beta \\
&\preceq \int_\Omega d^\alpha |\nabla u|^p,
\end{aligned}$$

provided the sum involving the cubes in the Poincaré chain is finite. However

$$\sum_{j=0}^{k_Q} d(t_{Q_j})^{Np'(-1/p+1/N-\alpha/Np)} \preceq \sum_{k=0}^{\infty} H(k) 2^{-kNp'(-1/p+1/N-\alpha/Np)} \tag{4.6}$$

where $H(k)$ is the number of cubes whose volume $\approx 2^{-k}|\Omega|$. Since Ω is (I_μ^-), by Lemmas 4.1 and 4.2 we can conclude that $H(k) \approx 2^{k(N+\mu)}$. A calculation shows that the sum in (4.6) is finite if and only if $\alpha < (p-1)(-\mu - N) + p - N$.

If $p = 1$ we get from (4.5) and Minkowski's inequality that

$$
\begin{aligned}
I_2 &\preceq \sum_{Q \in \mathcal{W}} \int_{\tilde{Q}} \left\{ \left[\sup_{0 \le j \le k_Q} |\tilde{Q}_j|^{(-1+1/N-\alpha/N)} \right]^q \left[\sum_{j=0}^{k_Q} \int_{\tilde{Q}_j} d^\alpha |\nabla u| \right]^q \right\} d^\beta \\
&\preceq \left(\int_\Omega d^\alpha |\nabla u| \right)^q \int_{\tilde{Q}} \left(\sup_{Q \in \mathcal{W}} d(t_{Q_j})^{N(-1+1/N-\alpha/N)} \right)^q d^\beta \\
&\preceq \left(\int_\Omega d^\alpha |\nabla u|^p \right)^{q/p} \int_\Omega d^\beta,
\end{aligned}
$$

provided

$$\alpha < 1 - N \equiv \lim_{p \to 1^+} (p-1)(-\mu - N) + p - N.$$

This argument establishes the Poincaré inequality (4.4) when $p \le q \le p^*$ and $p \le N$, or $p > N$. By the triangle and Hölder's inequality (4.4) is equivalent to

$$
\begin{aligned}
\int_\Omega d^\beta |u|^q &\preceq \left\{ \left| \int_{\tilde{Q}_0} u \right|^q + \left(\int_\Omega d^\alpha |\nabla u|^p \right)^{q/p} \right\} \\
&\preceq \left\{ \left(\int_\Omega d^{-\gamma/(p-1)} \right)^{q/p'} \left(\int_\Omega d^\gamma |u|^p \right)^{q/p} + \left(\int_\Omega d^\alpha |\nabla u|^p \right)^{q/p} \right\}.
\end{aligned}
\tag{4.7}
$$

This establishes the continuity of the embedding (4.3).

We turn next to compactness. Suppose $q \in [p, p^*)$ if $p < N$ and $q \in [1, \infty)$ otherwise. Take $\varepsilon \ll 1$. If we repeat the previous argument on $\Omega_{(\epsilon)}$ then

$$I_1(\epsilon) := \sum_{Q \in \mathcal{W}_\epsilon} \int_{\tilde{Q}} |u - u_{\tilde{Q}}|^q d^\beta,$$

where now $\mathcal{W}_\epsilon$ is a Whitney cover of $\Omega_{(\epsilon)}$ and the cubes Q (including a fixed cube Q_0) are elements of $\mathcal{W}_\epsilon$. Since $Q \subset \Omega_{(\epsilon)}$ we see that

$|Q| \approx d(t_Q)^N \preceq \epsilon^N$. Therefore assumption (ii) guarantees in the chain of inequalities (4.5) that $d(t_Q)^{\beta+q+N-qN/p-q\alpha/p} \preceq \epsilon$ and so

$$I_1 \preceq \epsilon\left(\int_{\Omega_{(\epsilon)}} d^\alpha |\nabla u|^p\right)^{q/p}.$$

Similarly the sum (4.6) if finite is also $O(\epsilon)$. It follows that we have

$$\int_{\Omega(\epsilon)} |u - u_{Q_0}|^q \preceq \epsilon\left(\int_{\Omega_{(\epsilon)}} d^\alpha |\nabla u|^p\right)^{q/p}.$$

Therefore

$$\begin{aligned}
\int_{\Omega_{(\epsilon)}} d^\beta |u|^q &\preceq |Q_0|^{-q}\left(\int_{\Omega_{(\epsilon)}} d^\beta\right)\left|\int_{Q_0} u\right|^q + \epsilon\left(\int_\Omega d^\alpha |\nabla u|^p\right)^{q/p} \\
&\preceq \left[|Q_0|^{-1}\left(\int_{Q_0} d^{-\gamma p'/p}\right)^{1/p'}\left(\int_{\Omega_\epsilon} d^\beta\right)^{1/q}\right]^q \left(\int_{Q_0} d^\gamma |u|^p\right)^{q/p} \\
&\quad + \epsilon\left(\int_\Omega d^\alpha |\nabla u|^p\right)^{q/p} \preceq \left(\epsilon^{q(-\gamma/p+N/p'-N+\beta/q+1/q)}\right)\left(\int_{Q_0} d^\gamma |u|^r\right)^{q/p} \\
&\quad + \epsilon\left(\int_\Omega d^\alpha |\nabla u|^p\right)^{q/p} \preceq \epsilon\left\{\left(\int_\Omega d^\gamma |u|^p\right)^{q/p} + \left(\int_\Omega d^\alpha |\nabla u|^p\right)^{q/p}\right\}, \\
&\quad \text{(by Assumption (i)).} \qquad (4.8)
\end{aligned}$$

If we define $\{\Omega_n\}$ and $\{\Omega^n\}$ as in Theorem 3.1 and choose $n > 1/\epsilon$. Then by (4.8) $\|u\|_{\Omega^{1/n}; d^\beta, q} = O\|u\|_{\Omega; d^\gamma, d^\alpha, p}$ and also as in Theorem 3.1 we can show that $W^{1,p}(\Omega; d^\gamma, d^\alpha)$ embeds compactly into every $L^q(\Omega_n, d^\beta)$. By Lemma 2.2 the compactness of the embedding follows.

Remark 4.2 By Lemma 4.2 we can replace the conditions on α, β, γ involving μ in Theorem 4.2 by conditions involving $\dim_{M,\Omega}(\partial\Omega)$. A calculation shows that these are

$$\begin{aligned}
&\alpha < p(1 - \dim_{M,\Omega}(\partial\Omega)) + \dim_{M,\Omega}(\partial\Omega)) - N, \\
&\beta > -N + \dim_{M,\Omega}(\partial\Omega), \\
&\gamma < (N - \dim_{M,\Omega}(\partial\Omega))(p-1).
\end{aligned}$$

Remark 4.3 The arguments of Theorem 4.2 remain valid even if $\mu = 0$; equivalently if Ω is an arbitrary bounded domain, but the restrictions on α, β, γ are more severe than in Theorem 3.1. However since (4.7) will hold in this case when $\beta \geq 0$ and $\alpha < p(1-N)$, by substituting "u" $= u - u_Q$ in (4.7) we get the weighted Poincaré-type inequality

$$\left(\int_\Omega \left|u - u_{\tilde{Q}_0}\right|^q\right)^{1/q} \leq K\left(\int_\Omega d^\alpha |\nabla u|^p\right)^{1/p}.$$

We now apply Lemma 4.4 to the left side of this inequality, taking $w = d^\beta$, $A = \Omega$, $q = p$, $c = u_{\tilde{Q}_0}$ noting that $q < p$ is possible when $p > N$. This gives

$$\left(\int_\Omega \left|u - u_{\Omega, d^\beta}\right|^q\right)^{1/q} \leq K\left(\int_\Omega d^\alpha |\nabla u|^p\right)^{1/p}$$

for all nonnegative β. A similar observation was made in [13]. The same argument giving a (q,p)-Poincaré inequality of course will work for $0 > \mu > -N$. We also note here that when $\beta > 0$ it follows by the second part of Theorem 4.2 that the Poincaré inequality is associated with a compact embedding of $W^{1,p}(\Omega; 1, d^\alpha)$ into $L^q(\Omega; d^\beta)$.

COROLLARY 4.2 *If* $\dim_{M,\Omega}(\partial\Omega) < N$, $q \in [p, p^*]$ *when* $p \leq N$, *and*

$$\alpha < \min\{p(1 - \dim_{M,\Omega}(\partial\Omega)) + \dim_{M,\Omega}(\partial\Omega) - N, \; -N + Np/q\},$$

then

$$W^{1,p}(\Omega; 1, d^\alpha) \hookrightarrow L^q(\Omega).$$

COROLLARY 4.3 *If* $\Omega \in (\mathrm{I}_\mu^-)$, *and* $\alpha < (p-1)(-\mu - N) + p - N$ *or equivalently* $\alpha < p(1 - \dim_{M,\Omega}(\partial\Omega)) + \dim_{M,\Omega}(\partial\Omega) - N$ *where* $\dim_{M,\Omega}(\partial\Omega) < N$, *then*

$$W^{1,p}(\Omega; 1, d^\alpha) \hookrightarrow\hookrightarrow L^p(\Omega).$$

Proof Let p^- be chosen sufficiently close to but less than p so that $\alpha' := (p^-/p)\alpha$ satisfies the inequality assumed for α. Then if $\delta > 0$ is sufficiently small, application of Theorem 4.2 with $\beta = 0$ gives us the embedding

$$W^{1,p^-}(\Omega; d^{-\delta}, d^{\alpha'}) \hookrightarrow\hookrightarrow L^p(\Omega). \tag{4.9}$$

On the other hand, since we can choose $\delta < |\mu|(1 - p^-/p)$ Theorem 4.1 yields the inequality

$$\|u\|_{\Omega; d^{-\delta}, p^-} \leq K \|u\|_{\Omega; 1, p}. \tag{4.10}$$

But by Hölder's inequality

$$\|\nabla u\|_{\Omega; d^{\alpha'}, p^-} \leq |\Omega|^{1-p^-/p} \|\nabla u\|_{\Omega; d^{\alpha}, p}. \tag{4.11}$$

Substitution of (4.10) and (4.11) into (4.9) gives us the desired embedding.

Remark 4.4 Since it is known (see, e.g., [7]) that the compactness of the embedding $W^{1,p}(\Omega; v_0, v_1)$ into $L^p(\Omega; v_0)$ implies the existence of the Poincaré inequality, we have at once from Corollary 4.3 another proof of the inequality

$$\left(\int_\Omega |u - u_\Omega|^p\right)^{1/p} \leq K(p, \Omega) \left(\int_\Omega d^\alpha |\nabla u|^p\right)^{1/p}$$

where $u \in W^{1,p}(\Omega; 1, d^\alpha)$, Ω is a bounded (I_μ^-) domain, and $\alpha < (p-1)(|\mu| - N) + p - N$.

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