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A Refinement of Various Mean Inequalities*

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A new refinement of the classical arithmetic mean and geometric mean inequality is given. Moreover, a new interpretation of the classical mean is given and this refinement theorem is generalized.

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1 INTRODUCTION

Faiziev [3] obtained a refinement of the classical arithmetic mean and geometric mean inequality. Also Alzer [1] obtained a continuous version of Faiziev's refinement and Pečarić [4] gave a simple proof of the above Alzer–Faiziev inequality. Recently Takahasi and Miura [5] obtained a generalization of the Alzer–Faiziev inequality.

Our main purpose of this paper is to give a new refinement of the classical arithmetic mean and geometric mean inequality (Theorem 2.1).

* The original concept of this research was inspired by the discussion held during the second author's visit to the Faculty of Engineering of Yamagata University.

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Furthermore we give a new interpretation of the classical mean and generalize this refinement theorem (Theorem 3.2).

2 A REFINEMENT OF THE CLASSICAL MEAN INEQUALITY

Let $\mathbb{R}_+$ denote the set of all positive real numbers and $\mathbb{R}_+^n$ its n-product. Recall the arithmetic mean, geometric mean, and harmonic mean;

$$\begin{aligned} A_n(x_1,\dots,x_n) &\equiv \frac{x_1+\cdots+x_n}{n}, \\ G_n(x_1,\dots,x_n) &\equiv (x_1\cdots x_n)^{1/n}, \\ H_n(x_1,\dots,x_n) &\equiv \frac{1}{(1/n)(1/x_1+\cdots+1/x_n)}, \end{aligned}$$

where $n\in\mathbb{N}$ and $(x_1,\dots,x_n)\in\mathbb{R}_+^n$. The order relation among these means is well-known;

$$H_n(x_1,\dots,x_n) \le G_n(x_1,\dots,x_n) \le A_n(x_1,\dots,x_n), \tag{1}$$

and the equality holds if and only if $x_1=x_2=\cdots=x_n$ (see for instance [2]).

Given any $\mathbf{x}=(x_1,\dots,x_n)\in\mathbb{R}_+^n$ and k with $1\le k\le n$ we first take the geometric means of any k terms and then consider the arithmetic mean of these ${}_nC_k$ numbers. So we obtain

$$u(A,G,\mathbf{x};k) \equiv \frac{1}{{}_nC_k}\sum_{1\le i_1<\cdots<i_k\le n}(x_{i_1}\cdots x_{i_k})^{1/k}, \tag{2}$$

and by the similar procedure

$$u(G,A,\mathbf{x};k) \equiv \left\{\prod_{1\le i_1<\cdots<i_k\le n}\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right\}^{1/{}_nC_k}. \tag{3}$$

By the definitions (2) and (3), we have

$$\begin{aligned} u(A,G,\mathbf{x};1) &= u(G,A,\mathbf{x};n) = A_n(x_1,\dots,x_n), \\ u(A,G,\mathbf{x};n) &= u(G,A,\mathbf{x};1) = G_n(x_1,\dots,x_n), \end{aligned}$$

so $u(A,G,\mathbf{x};1)\ge u(A,G,\mathbf{x};n)$ and $u(G,A,\mathbf{x};1)\le u(G,A,\mathbf{x};n)$. We will prove that $u(A,G,\mathbf{x};k)$ and $u(G,A,\mathbf{x};k)$ monotonously lie between A_n and G_n.

THEOREM 2.1 *Fix* $n \in \mathbb{N}$ *and* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$. *The refinement* $u(A, G, \mathbf{x}; k)$ *is nonincreasing and* $u(G, A, \mathbf{x}; k)$ *is nondecreasing with respect to* k $(1 \le k \le n)$, *that is,*

$$\begin{aligned} A_n = u(A, G, \mathbf{x}; 1) &\ge u(A, G, \mathbf{x}; 2) \ge \cdots \ge u(A, G, \mathbf{x}; n-1) \\ &\ge u(A, G, \mathbf{x}; n) = G_n, \end{aligned} \tag{4}$$

$$\begin{aligned} G_n = u(G, A, \mathbf{x}; 1) &\le u(G, A, \mathbf{x}; 2) \le \cdots \le u(G, A, \mathbf{x}; n-1) \\ &\le u(G, A, \mathbf{x}; n) = A_n. \end{aligned} \tag{5}$$

In the above inequalities one equality occurs only if $x_1 = x_2 = \cdots = x_n$.

Proof For any k with $2 \le k \le n$, by the inequality (1)

$$\begin{aligned} &\sum_{1 \le i_1 < \cdots < i_k \le n} (x_{i_1} \cdots x_{i_k})^{1/k} \\ &= \sum_{1 \le i_1 < \cdots < i_k \le n} \left\{ (x_{i_2} \cdots x_{i_k})^{1/(k-1)} \cdot (x_{i_1} x_{i_3} \cdots x_{i_k})^{1/(k-1)} \right. \\ &\qquad\qquad \left. \cdots (x_{i_1} \cdots x_{i_{k-1}})^{1/(k-1)} \right\}^{1/k} \\ &\le \sum_{1 \le i_1 < \cdots < i_k \le n} \frac{1}{k} \left\{ (x_{i_2} \cdots x_{i_k})^{1/(k-1)} + (x_{i_1} x_{i_3} \cdots x_{i_k})^{1/(k-1)} \right. \\ &\qquad\qquad \left. + \cdots + (x_{i_1} \cdots x_{i_{k-1}})^{1/(k-1)} \right\} \\ &= \frac{1}{k} \{n - (k-1)\} \sum_{1 \le i_1 < \cdots < i_{k-1} \le n} (x_{i_1} \cdots x_{i_{k-1}})^{1/(k-1)}, \end{aligned}$$

which implies

$$\begin{aligned} u(A, G, \mathbf{x}; k) &= \frac{1}{{}_nC_k} \sum_{1 \le i_1 < \cdots < i_k \le n} (x_{i_1} \cdots x_{i_k})^{1/k} \\ &\le \frac{1}{{}_nC_k} \cdot \frac{n-(k-1)}{k} \sum_{1 \le i_1 < \cdots < i_{k-1} \le n} (x_{i_1} \cdots x_{i_{k-1}})^{1/(k-1)} \\ &= \frac{1}{{}_nC_{k-1}} \sum_{1 \le i_1 < \cdots < i_{k-1} \le n} (x_{i_1} \cdots x_{i_{k-1}})^{1/(k-1)} \\ &= u(A, G, \mathbf{x}; k-1). \end{aligned}$$

Hence $u(A, G, \mathbf{x}; k)$ is nonincreasing, and (5) is proved similarly.

Next we consider the equality case. If $x_1 = x_2 = \cdots = x_n$ then $G_n = A_n$, so all values $u(A, G, \mathbf{x}; k)$ and $u(A, G, \mathbf{x}; k)$ are equal. Suppose that there exists k satisfying $u(A, G, \mathbf{x}; k) = u(A, G, \mathbf{x}; k-1)$. Then for any $i_1, \ldots, i_k$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$

$$x_{i_2} \cdots x_{i_k} = x_{i_1} x_{i_3} \cdots x_{i_k} = \cdots = x_{i_1} \cdots x_{i_{k-1}},$$

which implies $x_{i_1} = x_{i_2} = \cdots = x_{i_k}$. Hence $x_1 = x_2 = \cdots = x_n$.

Using the geometric mean and harmonic mean we obtain

$$u(G, H, \mathbf{x}; k) \equiv \left\{ \prod_{1 \leq i_1 < \cdots < i_k \leq n} \frac{1}{(1/k)(1/x_{i_1} + \cdots + 1/x_{i_k})} \right\}^{1/{}_nC_k}, \quad (6)$$

$$u(H, G, \mathbf{x}; k) \equiv \left\{ \frac{1}{{}_nC_k} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{1}{(x_{i_1} \cdots x_{i_k})^{1/k}} \right\}^{-1}. \quad (7)$$

As Theorem 2.1 we can prove that $u(G, H, \mathbf{x}; k)$ is nonincreasing and $u(H, G, \mathbf{x}; k)$ is nondecreasing.

3 A REFINEMENT OF A GENERALIZED MEAN

In order to generalize the previous inequalities we will regard the mean as the sequence of positive functions. Let f_k be a positive function on $\mathbb{R}_+^k$ $(k = 1, 2, 3, \ldots)$. The sequence of functions $\mathcal{F} = \{f_k\}$ is called *mean* if the following conditions (M-1)–(M-5) hold;

(M-1) $f_1(a) = a$ $(\forall a > 0)$,
(M-2) for any $k \in \mathbb{N}$

$$f_k(x_1, \ldots, x_k) \leq f_k(y_1, \ldots, y_k) \quad \text{if } 0 < x_i \leq y_i \ (i = 1, \ldots, k),$$

(M-3) for any $k \in \mathbb{N}$ and permutation σ of k elements

$$f_k(x_1, \ldots, x_k) = f_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}),$$

(M-4) for any $k, l \in \mathbb{N}$ and $(x_1, \ldots, x_k) \in \mathbb{R}_+^k$

$$f_k(x_1, \ldots, x_k) = f_{kl}(\overbrace{x_1, \ldots, x_1}^{l}, \overbrace{x_2, \ldots, x_2}^{l}, \ldots, \overbrace{x_k, \ldots, x_k}^{l}),$$

(M-5) for any $k, l \in \mathbb{N}$ with $1 \leq l \leq k$ and $(x_1, \ldots, x_k) \in \mathbb{R}_+^k$

$$f_k(x_1, \ldots, x_l, x_{l+1}, \ldots, x_k) = f_k\big(\overbrace{f_l(x_1, \ldots, x_l), \ldots, f_l(x_1, \ldots, x_l)}^{l}, x_{l+1}, \ldots, x_k\big).$$

The sequences generated by arithmetic, geometric, and harmonic means, $\{A_n\}$, $\{G_n\}$, and $\{H_n\}$, satisfy the above conditions (M-1)–(M-5). So the above mean is a generalization of well-known three means.

We first remark that by the condition (M-4) with $k = 1$

$$f_l(\overbrace{a, \ldots, a}^{l}) = f_1(a) = a \quad (\forall l \in \mathbb{N}, \forall a \in \mathbb{R}_+). \tag{8}$$

Consider another condition (M-6);

(M-6) for any $k, l \in \mathbb{N}$ and $(x_{11}, \ldots, x_{1l}, \ldots, x_{k1}, \ldots, x_{kl}) \in \mathbb{R}_+^{kl}$

$$f_k\big(f_l(x_{11}, \ldots, x_{1l}), \ldots, f_l(x_{k1}, \ldots, x_{kl})\big) = f_{kl}(x_{11}, \ldots, x_{1l}, \ldots, x_{k1}, \ldots, x_{kl}). \tag{9}$$

We will show that (M-4) and (M-5) are equivalent to (M-6) under the condition (M-3) and (8) above.

Proposition 3.1 *Let $\mathcal{F} = \{f_k\}$ be a sequence of positive functions. If $\mathcal{F} = \{f_k\}$ is a mean then $\mathcal{F}$ satisfies* (M-6). *Conversely if $\mathcal{F} = \{f_k\}$ satisfies the conditions* (8), (M-3), *and* (M-6) *then* (M-4) *and* (M-5) *are valid.*

Proof If $\mathcal{F}$ is a mean then

$$\begin{aligned}
&f_{kl}(x_{11}, \ldots, x_{1l}, x_{21}, \ldots, x_{2l}, \ldots, x_{k1}, \ldots, x_{kl}) \\
&\quad = f_{kl}\big(\overbrace{f_l(x_{11}, \ldots, x_{1l}), \ldots, f_l(x_{11}, \ldots, x_{1l})}^{l}, \\
&\qquad x_{21}, \ldots, x_{2l}, \ldots, x_{k1}, \ldots, x_{kl}\big) && \text{by (M-5)} \\
&\quad \vdots \\
&\quad = f_{kl}\big(\overbrace{f_l(x_{11}, \ldots, x_{1l}), \ldots, f_l(x_{11}, \ldots, x_{1l})}^{l}, \\
&\qquad \ldots, \overbrace{f_l(x_{k1}, \ldots, x_{kl}), \ldots, f_l(x_{k1}, \ldots, x_{kl})}^{l}\big) && \text{by (M-3), (M-5)} \\
&\quad = f_k\big(f_l(x_{11}, \ldots, x_{1l}), \ldots, f_l(x_{k1}, \ldots, x_{kl})\big), && \text{by (M-4)}
\end{aligned}$$

so (M-6) holds. Conversely suppose that $\mathcal{F}$ satisfies (8), (M-3), and (M-6). Put $x_{ij}=x_i$ $(j=1,\dots,l)$ in (M-6) then (M-4) holds by (8). For any $k\in\mathbb{N}$ and l with $1\leq l\leq k$

$$\begin{aligned}
&f_k\big(\overbrace{f_l(x_1,\dots,x_l),\dots,f_l(x_1,\dots,x_l)}^{l},x_{l+1},\dots,x_k\big)\\
&\quad=f_k\big(f_l(x_1,\dots,x_l),\dots,f_l(x_1,\dots,x_l),\\
&\qquad f_l(\overbrace{x_{l+1},\dots,x_{l+1}}^{l}),\dots,f_l(\overbrace{x_k,\dots,x_k}^{l})\big) && \text{by (8)}\\
&\quad=f_{kl}(x_1,\dots,x_l,\dots,x_1,\dots,x_l,\overbrace{x_{l+1},\dots,x_{l+1}}^{l},\\
&\qquad\dots,\overbrace{x_k,\dots,x_k}^{l}) && \text{by (M-6)}\\
&\quad=f_{kl}(\overbrace{x_1,\dots,x_1}^{l},\overbrace{x_2,\dots,x_2}^{l},\dots,\overbrace{x_k,\dots,x_k}^{l}) && \text{by (M-3)}\\
&\quad=f_k(x_1,\dots,x_k), && \text{by (M-4)}
\end{aligned}$$

which implies (M-5).

The order relation of two means $\mathcal{F}=\{f_k\}$ and $\mathcal{G}=\{g_k\}$ is defined in each coordinate, that is, $\mathcal{F}\leq\mathcal{G}$ if

$$f_k(x_1,\dots,x_k)\leq g_k(x_1,\dots,x_k)\quad\big(\forall k\in\mathbb{N},\ \forall(x_1,\dots,x_k)\in\mathbb{R}_+^k\big).$$

Consider two means $\mathcal{F}=\{f_k\}$, $\mathcal{G}=\{g_k\}$ and fix $n\in\mathbb{N}$. For any k with $1\leq k\leq n$ and $\mathbf{x}=(x_1,\dots,x_n)\in\mathbb{R}_+^n$, as (2) and (3), we define

$$\begin{aligned}
u(\mathcal{F},\mathcal{G},\mathbf{x};k)&\equiv f_{{}_nC_k}\big(g_k(x_1,\dots,x_k),\dots,g_k(x_{n-k+1},\dots,x_n)\big),\\
u(\mathcal{G},\mathcal{F},\mathbf{x};k)&\equiv g_{{}_nC_k}\big(f_k(x_1,\dots,x_k),\dots,f_k(x_{n-k+1},\dots,x_n)\big).
\end{aligned}\tag{10}$$

By the definition

$$\begin{aligned}
u(\mathcal{F},\mathcal{G},\mathbf{x};1)&=u(\mathcal{G},\mathcal{F},\mathbf{x};n)=f_n(x_1,\dots,x_n),\\
u(\mathcal{F},\mathcal{G},\mathbf{x};n)&=u(\mathcal{G},\mathcal{F},\mathbf{x};1)=g_n(x_1,\dots,x_n),
\end{aligned}$$

so if $\mathcal{F}\leq\mathcal{G}$ then

$$u(\mathcal{G},\mathcal{F},\mathbf{x};1)\geq u(\mathcal{G},\mathcal{F},\mathbf{x};n),\qquad u(\mathcal{F},\mathcal{G},\mathbf{x};1)\leq u(\mathcal{F},\mathcal{G},\mathbf{x};n).$$

The following is a generalization of Theorem 2.1.

THEOREM 3.2 *Fix* $n \in \mathbb{N}$ *and* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$. *If* $\mathcal{F} \le \mathcal{G}$ *then the refinement* $u(\mathcal{G}, \mathcal{F}, \mathbf{x}; k)$ *is nonincreasing and* $u(\mathcal{F}, \mathcal{G}, \mathbf{x}; k)$ *is nondecreasing with respect to* k $(1 \le k \le n)$, *that is,*

$$u(\mathcal{G}, \mathcal{F}, \mathbf{x}; 1) \ge u(\mathcal{G}, \mathcal{F}, \mathbf{x}; 2) \ge \cdots \ge u(\mathcal{G}, \mathcal{F}, \mathbf{x}; n-1) \ge u(\mathcal{G}, \mathcal{F}, \mathbf{x}; n), \tag{11}$$

$$u(\mathcal{F}, \mathcal{G}, \mathbf{x}; 1) \le u(\mathcal{F}, \mathcal{G}, \mathbf{x}; 2) \le \cdots \le u(\mathcal{F}, \mathcal{G}, \mathbf{x}; n-1) \le u(\mathcal{F}, \mathcal{G}, \mathbf{x}; n). \tag{12}$$

Proof Choose k with $2 \le k \le n$. Since for any $(y_1, \ldots, y_k) \in \mathbb{R}_+^k$

$$\begin{aligned} & f_k\big(f_{k-1}(y_1, \ldots, y_{k-1}), f_{k-1}(y_1, \ldots, y_{k-2}, y_k), \ldots, f_{k-1}(y_2, \ldots, y_k)\big) \\ &\quad = f_{k(k-1)}(y_1, \ldots, y_{k-1}, y_1, \ldots, y_{k-2}, y_k, \ldots, y_2, \ldots, y_k) && \text{by (9)} \\ &\quad = f_{k(k-1)}(\overbrace{y_1, \ldots, y_1}^{k-1}, \ldots, \overbrace{y_k, \ldots, y_k}^{k-1}) && \text{by (M-3)} \\ &\quad = f_k(y_1, \ldots, y_k) && \text{by (M-4),} \end{aligned}$$

we can deduce that

$$\begin{aligned} & u(\mathcal{G}, \mathcal{F}, \mathbf{x}; k) \\ &\quad = g_{{}_nC_k}\big(f_k(x_1, \ldots, x_k), \ldots, f_k(x_{n-k+1}, \ldots, x_n)\big) \\ &\quad = g_{{}_nC_k}\big(f_k\big(f_{k-1}(x_1, \ldots, x_{k-1}), f_{k-1}(x_1, \ldots, x_{k-2}, x_k), \ldots, \\ &\qquad\qquad f_{k-1}(x_2, \ldots, x_k)\big), \ldots, f_k\big(f_{k-1}(x_{n-k+1}, \ldots, x_{n-1}), \ldots, \\ &\qquad\qquad f_{k-1}(x_{n-k+2}, \ldots, x_n)\big)\big). \end{aligned}$$

According to the inequality $\mathcal{F} \le \mathcal{G}$ and (M-2)

$$\begin{aligned} & u(\mathcal{G}, \mathcal{F}, \mathbf{x}; k) \\ &\quad \le g_{{}_nC_k}\big(g_k\big(f_{k-1}(x_1, \ldots, x_{k-1}), f_{k-1}(x_1, \ldots, x_{k-2}, x_k), \ldots, \\ &\qquad\qquad f_{k-1}(x_2, \ldots, x_k)\big), \ldots, g_k\big(f_{k-1}(x_{n-k+1}, \ldots, x_{n-1}), \ldots, \\ &\qquad\qquad f_{k-1}(x_{n-k+2}, \ldots, x_n)\big)\big) \\ &\quad = g_{{}_nC_k \cdot k}\big(f_{k-1}(x_1, \ldots, x_{k-1}), f_{k-1}(x_1, \ldots, x_{k-2}, x_k), \\ &\qquad\qquad \ldots, f_{k-1}(x_2, \ldots, x_k), \ldots, f_{k-1}(x_{n-k+1}, \ldots, x_{n-1}), \\ &\qquad\qquad \ldots, f_{k-1}(x_{n-k+2}, \ldots, x_n)\big) && \text{by (9)} \end{aligned}$$

$$= g_{{}_nC_k \cdot k}\Big(\overbrace{f_{k-1}(x_1,\ldots,x_{k-1}),\ldots,f_{k-1}(x_1,\ldots,x_{k-1})}^{n-k+1},\ldots,$$
$$\overbrace{f_{k-1}(x_{n-k+2},\ldots,x_n),\ldots,f_{k-1}(x_{n-k+2},\ldots,x_n)}^{n-k+1}\Big) \quad \text{by (M-3)}$$
$$= g_{{}_nC_{k-1}\cdot\{n-(k-1)\}}\Big(\overbrace{f_{k-1}(x_1,\ldots,x_{k-1}),\ldots,f_{k-1}(x_1,\ldots,x_{k-1})}^{n-k+1},\ldots,$$
$$\overbrace{f_{k-1}(x_{n-k+2},\ldots,x_n),\ldots,f_{k-1}(x_{n-k+2},\ldots,x_n)}^{n-k+1}\Big)$$
$$= g_{{}_nC_{k-1}}\big(f_{k-1}(x_1,\ldots,x_{k-1}),\ldots,f_{k-1}(x_{n-k+2},\ldots,x_n)\big) \quad \text{by (M-4)}$$
$$= u(\mathcal{G},\mathcal{F},\mathbf{x};k-1).$$

Hence $u(\mathcal{G},\mathcal{F},\mathbf{x};k)$ is nonincreasing and (12) is proved similarly.

Remark For any $n\in\mathbb{N}$ and $t\neq 0$ consider the function M_n^t defined by

$$M_n^t(x_1,\ldots,x_n) \equiv \left(\frac{x_1^t+\cdots+x_n^t}{n}\right)^{1/t} \quad (\forall(x_1,\ldots,x_n)\in\mathbb{R}_+^n).$$

Because $\lim_{t\to 0} M_n^t(x_1,\ldots,x_n) = (x_1\cdots x_n)^{1/n}$, we define

$$M_n^0(x_1,\ldots,x_n) \equiv (x_1\cdots x_n)^{1/n} \quad (\forall(x_1,\ldots,x_n)\in\mathbb{R}_+^n).$$

For a given n and $(x_1,\ldots,x_n)$, $M_n^t(x_1,\ldots,x_n)$ is nondecreasing with respect to t. In particular, M_n^{-1}, M_n^0, and M_n^1 are the harmonic, geometric, and arithmetic mean, respectively. So the functions M_n^t are interpolated in the harmonic, geometric, and arithmetic mean (see [2] for a detail of the function M_n^t). For a fixed t, the sequence $\{M_n^t\}_n$ satisfies the conditions (M-1)–(M-5). So $\mathcal{M}^t = \{M_n^t\}$ is also a mean in our sense.

Fix $n\in\mathbb{N}$, $\mathbf{x}=(x_1,\ldots,x_n)\in\mathbb{R}_+^n$ and choose k with $1\le k\le n$. For any s, $t\in\mathbb{R}$ let us consider

$$u(s,t,\mathbf{x};k) = u(\mathcal{M}^s,\mathcal{M}^t,\mathbf{x};k)$$
$$= M^s_{{}_nC_k}\big(M_k^t(x_1,\ldots,x_k),\ \ldots,M_k^t(x_{n-k+1},\ldots,x_n)\big).$$

If $s \le t$ then $\mathcal{M}^s \le \mathcal{M}^t$, so by Theorem 3.2 we can conclude that

$$u(t, s, \mathbf{x}; 1) \geq u(t, s, \mathbf{x}; 2) \geq \cdots \geq u(t, s, \mathbf{x}; n-1) \geq u(t, s, \mathbf{x}; n),$$
$$u(s, t, \mathbf{x}; 1) \leq u(s, t, \mathbf{x}; 2) \leq \cdots \leq u(s, t, \mathbf{x}; n-1) \leq u(s, t, \mathbf{x}; n).$$

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